DIFFERENTIAL CALCULUS

[FOR B.A. AND B.Sc. STUDENTS]

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Thoroughly Revised
Third Edition

UNIVERSAL PUBLISHING CO.

53, ESPLANADE ROAD, DELHI-6.

[Price : Rs. 6·50 P.

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CHAPTER I

REAL NUMBERS

1.0. **Introduction.** The most important event in the history of Mathematics was the creation of the **infinitesimal calculus** in the seventeenth century by Newton in England and Leibnitz in Germany. It was realised then that to study the relationship between two quantities whose values were inter-dependent, it was necessary to study the infinitesimal or small changes produced in the values of one by infinitesimal changes in those of the other, or, in other words, to study the rate of growth of one quantity relative to the other. The calculus of infinitesimals thus developed is today the most important tool of Mathematics and finds its application in all branches of Science.

Consider, for example, the relation

$$y=x^2$$

between the length x of the side of a square and its area y. If x is measured in centimetres, then y is given by this relation in square centimetres. If we imagine the length x of the side to increase then the area y also increases and we may be interested in knowing the rate at which the area increases as the side increases. Again, let the relationship between the time t elapsed since a train left a station and the distance travelled by the train in this time be given by

$$s=3t^2+5t.$$

We may desire to know the rate at which the distance is being described at any time t, that is, we may want to know the speed of the train at any instant.

These are some of the typical problems with which the infinitesimal calculus is concerned. It is divided into two parts, the **Differential Calculus** and the **Integral Calculus**.

The Differential Calculus is concerned with the study of the rate of growth of one quantity relative to another at some particular point, whereas the Integral Calculus studies the changes in one quantity relative to changes in another over intervals. The two branches are very intimately connected.

1·1. **The Real Number System. Its Evolution.** Since in the Calculus we work with the measures of quantities expressed in numbers, it is necessary to study in detail the structure of the real number system in order to build up the Calculus properly. However this detailed study is better left to later and more advanced courses on the subject and, in this first course, we shall be content with recalling some important facts concerning the real number system. It is not intended to give any rigorous definitions or proofs in this

brief account. Some arguments are given here and there in order to make the subject matter clearer.

The system of real numbers has evolved as a result of successive extension with the **positive integers** or **natural numbers** or **whole numbers**, as these are often called, as the base. The natural numbers were evolved when man first learnt counting. Extensions of the system of natural numbers became necessary when man learnt the arithmetical operations of addition, subtraction, etc It is always possible to add one positive integer to another and the result is a positive integer. However, it is not always possible to subtract one positive integer from another if positive integers are the only type of numbers available. For example, it is not possible to subtract 5 from 3 if we have only the positive integers. In order to make subtraction always possible, it became necessary to introduce a new type of numbers called the **negative integers**, and these and the **signless integer zero** were added to the natural numbers to form the **extended system of integers**. It may be observed that it is only comparatively recently in the history of mankind that the negative numbers have been introduced Similarly, to make it always possible to divide one number by another, division by zero being excepted, the fractions, both positive and negative, were added to the integers to form the system of **rational numbers** that is, *the system of all numbers of the form p/q where p and q are integers with* $q \neq 0$. Experience convinced Mathematicians long ago that permitting division by zero would lead to all sorts of absurd results and so division by zero is ruled out.

The result of adding or multiplying together two rationals, subtracting one rational from another or dividing one by another not equal to zero, is always a rational number. Hence, the four fundamental operations of Arithmetic, *viz*, addition, subtraction, multiplication and division performed on the rationals, however often, give rise to rationals only. Therefore, if Mathematics were limited to these four operations only, there would be no need to extend the system of rational numbers But the process of root extraction necessitates a further extension of the number system. For example, consider the application of the ordinary method of extraction of square roots to the integer 2. To, however, many places of decimals the process is carried, it never comes to an end, We get the succession of numbers

$$1,\ 1{\cdot}4,\ 1{\cdot}41,\ 1{\cdot}414,\ 1{\cdot}4142,\ 1{\cdot}41421, \ldots\ldots$$

in which the square of any number is closer to 2 than that of the one preceding. Each of these decimals can be converted into a fraction of the form p/q, and is thus a rational number. Since all these rational numbers are different and none of them has its square *exactly* equal to 2, we soon begin to be convinced that there is no rational number whose square is equal to 2. This can, in fact, be proved quite easily by proper mathematical argument. For, if possible let there be a rational number p/q whose square is 2. We

assume that any common factor that the integers p and q have has been removed already and that p and q have no common factor. Then

$$(p/q)^2=2 \quad \text{or} \quad p^2=2q^2. \tag{1}$$

$\therefore$ p^2 is even and hence p must be even because the square of every odd integer is odd and that of every even integer is even. Let $p=2m$, where m is an integer. Substituting in (1), we get

$$4m^2=2q^2 \quad \text{or} \quad q^2=2m^2.$$

Hence q^2 is even and, therefore, q also is even. Thus p and q are both even and therefore have 2 as a common factor contrary to the hypothesis that p and q have no common factor. Hence our supposition that there is a rational number whose square is 2 leads to a contradiction. Therefore there is no rational number whose square is 2.

On the other hand. the above succession of numbers suggests that there should exist a number lying between 1 and 2, 1·4 and 1·5, 1·41 and 1·42, 1·414 and 1·415,............, thus successively narrowing the interval inside which it must lie, whose square is exactly equal to 2. We then say that there does exist such a number and denote it by $\sqrt{2}$ just as we would denote the square root of a number like 4 or 9 by using the symbol '$\sqrt{\ }$'. Since this number is not rational it is called **irrational**. Numbers of this kind which cannot be expressed as rationals p/q but for which a systematic approximation to any desired degree can always be obtained in the rational system are called **irrational**. Surds are not the only kind of irrational numbers. These are in fact a very special kind of irrational numbers which can be obtained as roots of polynomial equations. There are many more irrational numbers. The number π, the ratio of the circumference of a circle to its diameter, is an irrational number. Another important irrational number is e which is the sum* of the infinite series

$$1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots\ldots$$

This number is the base of natural logarithms and will occur very often in this course.

The need for the extension of the rational number system is felt also when we consider the decimal representation of numbers. We know that when any vulgar fraction p/q, *i.e.*, any rational number, is converted into a decimal fraction, it can be written as a terminating or recurring decimal or *vice versa*. For example,

$$\frac{3}{5}=0{\cdot}6,\ \frac{15}{8}=1{\cdot}875,\ \frac{11}{3}=3{\cdot}666\ldots\ldots=3{\cdot}\dot{6},\ \frac{1}{7}=0{\cdot}\dot{1}4285\dot{7},\ \text{etc}$$

But terminating or recurring decimals do not exhaust all the

*The word sum is not used here in the ordinary sense of an arithmetical sum, for the process of adding together an infinite number of terms would never come to an end. The word is used in an extended sense to be explained later.

decimals. There are many more decimals which neither terminate nor recur. For example, consider

$$0{\cdot}101001000100001\ldots\ldots,$$

$$0{\cdot}10010000100\ldots\ldots,$$

where in the first we have a succession of ones and zeros after the decimal point, the number of zeros increasing by one each time, and in the second, the nth digit after the decimal point is a one or zero according as n is or is not a perfect square. These are both non-terminating, non-recurring decimals. The student will be able no doubt to think of many more such examples where the digits do not necessarily follow in some prescribed manner as above. All the non-terminating, non-recurring decimals correspond to the **irrational numbers**. We have

$$\sqrt{2}=1{\cdot}41421\ldots\ldots,\ \pi=3{\cdot}14159\ldots\ldots,\ e=2{\cdot}71828\ldots\ldots$$

All rational and irrational numbers put together are called the **real numbers**. Thus all integers, positive, negative or zero, all rational fractions, and all irrational numbers are real numbers. Henceforth, we shall take the word number to mean real number unless the context makes it clear otherwise.

1·11. Arithmetical operations on the Reals. With the introduction of negative numbers, the process of subtraction becomes a special case of addition. For, if $(-b)$ be the **negative** of a number b, then the operation of subtracting the number b from a number a can be defined as adding $(-b)$ to a. Thus

$$a-b=a+(-b)$$

Similarly, if $(1/b)$ be the reciprocal of a non-zero number b, then the operation of dividing a number a by the number b can be defined as multiplying a by $1/b$. Thus

$$\frac{a}{b}=a\times\frac{1}{b}.$$

Hence the operations of subtraction and division need no longer be regarded as distinct from those of addition and multiplication respectively and we may say that the system of real numbers is endowed essentially with two operations, addition and multiplication.

We now recall the laws obeyed by the operation of addition and multiplication. We cannot play a game of cards or any other game without any knowing all its rules. The same is true of playing or working with numbers. The students has learnt all the rules for addition and multiplication but has never thought consciously about them or enumerated them. Our object is to draw the pointed attention of the student to the fundamental laws of addition and multiplication of reals.

First of all, it is known that the result of adding or multiplying two real numbers is again a real number, that is, addition and multiplication are always possible and lead to unique answers. We express this property of real numbers by saying that *the real number system*

is closed *for the operations of addition and multiplication.* It may be observed that this property is possessed also by the system of all the rational numbers but it is not possessed by the system of all the negative numbers for, in the case of the latter, the product of two negative numbers is not a negative number.

Next, let a and b be any two real numbers and consider the two expressions $a+b$ and $b+a$ The student is so much used to regarding these as equal or the same that he may fail to appreciate that these represent two differ operations. Tne first represents the result of adding b to a and the second that of adding a to b. Perhaps the logical difference between the two operations is appreciated better by taking examples where the result is not the same, thus 'A hit B' and 'B hit A' are seen to be different evidently. Similarly, one will see at once that $a-b$ and $b-a$ are not the same. We thus see that observing that $a+b=b+a$ is a significant property of the real numbers in relation to the operation of addition and is not mere hair-splitting. This property is expressed by saying that *addition among the reals is* **commutative** *or obeys the* **commutative law**. The same is true of the operation of multiplication and we have $a \cdot b=b \cdot a$ always. It is important to realize that changing the order of things in any operation may change the final result.

We next observe that *addition and multiplication both obey the* **associative law**. If a, b, c are any three numbers, whose sum is to be found, this law permits us to associate the middle number b with either the first or the third. That is, we may first add b to a and then add c to the sum thus obtained, or we may add c to b and then add this sum to a, the two final sums are the same. In symbols, $(a+b)+c=a+(b+c)$, the brackets being placed to indicate what sums are to be formed first on either side. The associative law permits us to write the sum of three numbers a, b, c as simply $a+b+c$. Similar remarks hold for multiplication and we may write

$$(a \cdot b) \cdot c=a \cdot (b \cdot c)=abc.$$

The operations of addition and multiplication together obey the **distributive law**, that is, if a, b, c are any three reals, then me can distribute the produot $a(b+c)$ as the sum $ab+ac$, *i.e.*,

$$a(b+c)=ab+ac.$$

Finally, we recall that 0 and 1 possess the property that

$$a+0=0+a=a \quad \text{and} \quad a \cdot 1=1 \cdot a=a.$$

For this reason, 0 is called the **additive identity** and 1 the **multiplicative identity**.

Thus, putting the above together, we say that the system of real numbers possesses the following properties, a, b, c are any real numbers.

I. *The system of real numbers is* **closed** *for the operations of addition and multiplication.*

$a+b$ and ab are both real numbers.

II. *The* **commutative law** *is true for addition and multiplication.*

$$a+b=b+a \quad \text{and} \quad ab=ba.$$

III. *The* **associative law** *is true for addition and multiplication.*

$$(a+b)+c=a+(b+c) \quad \text{and} \quad (ab)c=a(bc).$$

IV. *The* **distributive law** *is true for addition and multiplication together.*

$$a(b+c)=ab+ac.$$

V. 0 *and* 1 *are the* **additive** *and* **multiplicative identities.**

$$a+0=a \quad \text{and} \quad a\,.\,1=a.$$

VI. *Corresponding to every real number a, there is a real number $-a$ (the* **negative** *of a) such that $a+(-a)=0$.*

VII. *Corresponding to every real number $a \neq 0$, there is a real number $1/a$ (the* **reciprocal** *of a) such that $a\,.\,\dfrac{1}{a}=1$.*

VIII. *If $ab=0$, then either $a=0$ or $b=0$.* This is equivalent to the **cancellation law** : *If $ab=ac$ and $a\neq 0$, then $b=c$.*

It is easy to verify that all the above properties are also possessed by addition and multiplication in the system of rational numbers alone, whereas they are not all possessed by the system of integers alone or the system of positive integers alone.

1·12. Relations of magnitude between the reals. We know that if a, b, c are any three reals, then :

(*i*) Either $a=b$ or $a>b$ or $a<b$, *i.e.*, only one of these relations is true.

(*ii*) If $a<b$ and $b<c$, then $a<c$. Similarly, if $a>b$ and $b>c$, then $a>c$.

(*iii*) If $a<b$, then $a+c<b+c$. Similarly, if $a>b$, then $a+c>b+c$.

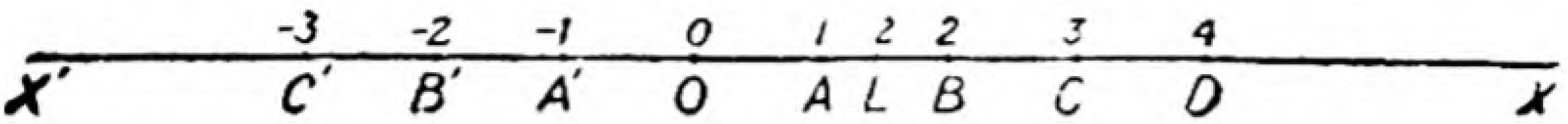
On account of these properties *the real number system is said to be* **ordered.**

(*iv*) **Axiom of Archimedes**. If a, b are any two positive numbers, then a positive integer n can be found such that $na<b$.

We also observe that if a is **positive** then $a>0$ while, if a is **negative**, then $a<0$. The integer 0 has no sign and separates the positive and negative real numbers.

If may be remarked that the above rules are true for relations of magnitude not only among all the reals but also for rationals alone or for the system of integers alone. Thus the system of integers and the system of rationals are also ordered systems.

1·2. Representation of numbers by points on a straight line. Let $X'OX$ be a straight line drawn across the paper from left to right, extending indefinitely on either side, with two points O and A marked on it such that A is to the right of O. We associate the number zero with O and call O the **origin** on the line.



There are two distinct senses along the, $X'OX$ and XOX'. We can regard one of these as positive and the other negative. Then all distances measured in the positive sense can be regarded as positive and those in the opposite direction as negative. We regard the sense $X'OX$, *i.e.*, from left to right, as positive. Since the line $X'OX$ has been provided with a positive direction along it, we call it a **directed line**. We associate the number 1 with A or, in, other words, choose our scale of measurement of distances along $X'OX$ to be such that $OA=1$. Thus the line $X'OX$ is provided with a positive direction along it, an origin O and the unit of length OA.

If we mark points B, C,...on the line to the right of A in succession such that each of AB, BC,......is equal in length to OA, then

$$OB=2OA=2, \ OC=3\ OA=3,......$$

and we associate the number 2 with B, 3 with C, and so on. We may also say that the points O, A, B, C,...correspond to the numbers 0, 1, 2, 3,...or that *the points O, A, B, C, etc. represent the numbers* 0, 1, 2, 3, *etc. respectively on the line $X'OX$*. Similarly, if points A', B', C', etc. are marked in succession to the left of O such that each of the lengths OA', $A'B'$, $B'C'$, etc. is equal to OA, then

$$OA'=-OA=-1, \ OB'=-2OA=-2, \ OC'=-3OA=-3,...,$$

and we may say that A' represents the number -1, B' represents -2, C' represents -3, and so on. Thus all integers, positive, negative or zero are represented by points on the line.

To represent the rational number p/q, where the integer q may always be taken to be positive since

$$\frac{p}{-q}=\frac{-p}{q} \text{ and } \frac{-p}{-q}=\frac{p}{q},$$

let the point P represent the integer p. P will be to the right or left of O according as p is positive or negative. If we divide OP into q equal parts, then the point of division nearest O will represent p/q. Thus every rational number is represented by a unique point on a directed line when the origin and the unit of length are fixed. Naturally, the question arises : "Will *every* point on the line represent a rational number ?" The answer is "no".

The points representing all the rational numbers including all the integers by no means exhaust all the points on the line. If we mark a point L on the line to the right of O such that OL is equal in length to the diagonal of a square whose side is of length unity,

then L cannot represent a rational number. For, by Pythagoras' theorem, $OL=\sqrt{2}$ and we have proved earlier that $\sqrt{2}$ is not a rational number. In fact, according to our method of representing numbers by points on the line, L represents the irrational number $\sqrt{2}$. This shows that points on the line can be chosen to represent irrational numbers also. We are, therefore led to assume that every irrational number can also be represented by a unique point on the line $X'OX$. This means that we assume that every real number is represented by some one point on $X'OX$. We go a step further and assume that conversely every point on the line $X'OX$ represents some one real number. This is the

Dedekind-Cantor Axiom : *To every real number corresponds a unique point on a directed line and to every point on a directed line corresponds a unique real number.*

This may also be expressed roughly as : there are as many points on a line as there are real numbers. And, in techinal language : *there is a one—one* (1—1) *correspondence between the real numbers and the points of a directed line* On account of this 1—1 correspondence we may take a number and the point representing it as synonymous and speak of the 'number a' as the 'point a' and *vice versa,* and refer to the line $X'OX$ as the **real line.**

We may remark that if the points P, Q represent two real numbers a, b respectively, then P lies to the right or left of Q according as $a>b$ or $a<b$. This is because we have chosen the sense from left to right as the positive direction along $X'OX$.

1·3 **Introduction of reals as points of a line** We have seen above that not all points of the directed line represent rational numbers. We can use this fact to extend the system of rational numbers by the introduction of irrational numbers as numbers corresponding to such points of the line which do not represent rational numbers. In other words, we assume the Dedekind-Cantor axiom and postulate, that *to each point of the directed line corresponds a unique real number which is* **rational** *if it corresponds to a point representing a rational number and is* **irrational** *otherwise* This method of introducing the irrational numbers may appear to some to be simple but suffers from the defect that it makes the extension of the number system depend on a geometrical argument. It may be observed that *the development of the Calculus is in no way dependent on the graphical representation of real numbers by points of a straight line*. However, the graphical representation is a great help in a clear understanding of the subject and we shall make use of it quite often.

1·4. **The real numbers form a dense system.** If we consider the system of integers alone, then we can speak of two consecutive integers because for every integer n, there is a definite one, *viz.* $n+1$, succeeding it and a definite one, $n-1$, preceding it Hence we can talk of the integer next to n, either greater than n or less than n. If, however, we consider the system of all the rational numbers, then we cannot talk of two consecutive rational numbers or of a rational

number next to another either greater than it or smaller than it. In other words, if a, b are any two distinct rational numbers with say $a<b$, then there is always at least one rational number c such that $c>a$ but $<b$. For, if such a number c did not exist we would be able to say that b is the rational number next to a and $>a$. The existence of such a number c is easily established. Take $c=\frac{1}{2}(a+b)$, the A.M. between a and b. Then evidently $c>a$ and $<b$. In fact, it follows that there lies not only one rational number between a and b but an infinite number. For, if there lies at least one rational number c between any two rationals a, b with $a<c<b$, then by the same hypothesis there lies a rational d between a and c with $a<d<c$, and a rational e between a and d with $a<e<d$, and so on indefinitely. Thus we have an unending succession of numbers $c, d, e, \ldots\ldots\ldots\ldots$ all lying between a and b. We, therefore, see that the statements '*Between any two rational numbers there lies at least one other rational number*' and '*Between any two rational numbers there lie an infinity of other rational numbers*', are equivalent. On account of this property *the rational numbers are said to form a* **dense** *system*. If we visualize all the rational numbers as points on the directed line then the property of denseness means that there are an infinity of such points on every patch or segment of the line, however small in length the segment be, provided the end points of the segment represent rational numbers.

The system of real numbers is also **dense** *everywhere, i.e.*, between any two reals there lie an infinity of other reals. An appreciation of this property follows from the consideration that the rational numbers alone form a dense system and the real number system is obtained from the rational number system by the addition of irrational numbers. We shall not attempt to prove this property.

1 5. Continuity. The Arithmetic Continuum. We have seen in the last paragraph that the rationals and the reals both form dense systems, and it would appear that the reals form a denser system. The question arises, "How much denser ?" The answer is that the reals form a completely dense system in the sense that the real number system cannot be extended to include new numbers in the way in which the rational number system is extended by the addition of irrational numbers, the rationals and irrationals together obeying all the laws which the rationals alone obey and the two systems merging together into a single ordered system.

If the existence of rational numbers be assumed, the simplest way of introducing or defining irrational numbers is **Dedekind's method of sections.** *If* **all** *the rational numbers be divided into two classes L and R according to same principle such that* (*i*) *there is at least one rational in each class and* (*ii*) *every member of L class is smaller than every member of R class, then such a division is called a* **section** *or* **partition** *or* **cut** *in the field of rational numbers*. It follows at once from (*ii*) that if a rational number a belongs to the L class, then every rational number $<a$ also belongs to the L class while if a

belongs to the R class, then every rational number $>a$ belongs to the R class. Now, it can be shown that all sections of the rationals fall into two categories. First, sections in which there is a rational number, say a, such that every rational number $<a$ belongs to the L class while every rational $>a$ belongs to the R class, the rational a may belong to L or to R, and second, sections in which there is no such rational number. In other words, in the first category of sections, there always is a rational number which separates the two classes or serves as the frontier or boundary between them, while in the second category there is no such rational number which can serve as the frontier between the two classes. Dedekind made this fact the basis for the introduction of irrational numbers. In essence, Dedekind's method amounts to this. *We postulate that in the second category of sections also there is always a number which separates the two classes but that this number is not rational, we call it irrational.*

The geometrical representation of the rationals as points of the directed line makes things clearer. The word section or cut brings to the mind at once the action of cutting the line at some point which may or may not represent a rational number.

Dedekind proved that if we consider sections in the field of all the real numbers, *i.e.*, rationals and irrationals put together, then there is no second category of sections, that is, there is always a real number which separates the two classes and is such that every real number less than it belongs to the L class and every real number greater than it to the R class. This result, called **Dedekind's theorem**, shows that the real number system cannot be extended by the introduction of new numbers in the way in which the rational number system is extended. Or, in a rough way of speaking, it shows that there are no gaps in the real number system of the kind that exist between the rationals. We may thus say that *the real numbers form a* **continuous** *system, Dedekind's theorem being the expression of this property of* **continuity**. Preferrably we should regard the statement 'the real numbers form a continuous system' as just another way of saying that the system possesses the property expressed by Dedekind's theorem. In view of this property the real number system is also called the **Arithmetic Continuum**.

By virtue of the Dedekind-Cantor axiom, the system of points of the directed line also possesses the property of continuity and is therefore called the **Geometric Continuum** or the **Linear Continuum.** The axiom itself may be called the **Dedekind-Cantor axiom of the continuity of the straight line**.

It may be observed that the continuity of the real number system or of the real line is the mathematical idealization or perfection of our physical notion of continuity. It appears according to the Atomic Theory of Matter that no physical substance is continuous in the arithmetical sense. A straight line drawn on a piece of paper would inevitably be seen to consist of a number of dots if the picture were magnified a sufficient number of times. The concept of continuity as

outlined above means that however many times the picture is magnified the points forming the line must still appear as a connected whole, that is, in going from one point of the line to another it should not be necessary to step out of the medium on which the line is drawn.

1 6. Other number systems. It has been remarked above that the real number system cannot be extended in the way in which the rational number system is extended. This does not mean that the real number system cannot be extended at all. The student has already learnt of the **complex numbers**, numbers of the form $a+ib$, $i=\sqrt{(-1)}$, which arise when we attempt to solve quadratic equations of the type $x^2+1=0$ or $x^2+x+3=0$, or more generally, quadratics of the type $Ax^2+Bx+C=0$, in which $B^2-4AC<0$. Complex numbers include as special cases (*i*) the real numbers, obtained by taking $b=0$, and (*ii*) the pure imaginary numbers, obtained by taking $a=0$. However, no relations of magnitude can be defined for the complex numbers as such and therefore the system of complex numbers cannot be ordered like the system of reals.

It is best to regard the complex numbers $a+ib$ as ordered pairs of real numbers (a, b) so that (a, b) and (b, a) are different complex numbers if $a\neq b$. This at once shows a suitable way of representing complex numbers geometrically, *viz.*, by points in a plane, the point of coordinates (a, b) representing the complex number $a+ib$.

There can be many other extensions of the concept of number but we shall not mention these here as we shall be dealing entirely with real numbers in this book.

1·7. Lastly, we may mention some important attributes of real numbers which enable them to be used in various ways.

First, the positive integers can be used for **counting**, as when we say that there are 20 boys in the class or there are 50,000 books in this library. Second, the positive integers are used for **ordering** or **arranging** or **enumerating** the objects in a collection, as when arranging the result of an examination we say that so and so stands *first*, so and so *second*, and so on, or when writing an A.P. or G.P. we speak of the *first*, *second*, *third*,........., *nth* terms, etc. Third, the positive integers are used for **labelling**, as when we assign roll numbers to the students in a class to distinguish one from the other. We could also arrange the class by roll numbers but arranging is not intended when we assign roll numbers. Fourth, the positive integers are also used for **measuring** or **comparing integral multiples** of a given magnitude or quantity, as when we speak of 5 kilos of sugar or 10 litres of milk.

Other classes of real numbers such as all the rationals or all the reals cannot be used for purposes of counting or arranging objects in a collection as first, second third, etc. but these can be used for labelling objects or for measuring or comparing magnitudes. When we associate numbers with points of a line we are already labelling the points of the line and we can refer to various points of the line as the point 1, 2, 3/2, $\sqrt{5}$, etc. Also by using all the real numbers we can

measure not only integral multiples of a given magnitude, but any fractional or irrational multiples as well It should, however, be remembered that the physical process of measurement is only an approximate one and measures only fractional or rational multiples of the unit.

1·8. Aggregates or Sets. The concept of an 'aggregate' or 'set' or 'class' or 'collection' of objects is fundamental to Mathematics. However, there is no completely satisfactory definition of this concept, and it is best to assume that we have an intuitive notion of this concept. A definition is

Any collection of objects which is made according to rules which determine without ambiguity whether any given object belongs to the collection or not is called an **aggregate** *or* **set**.

The objects may be numbers, or points, or any other things including aggregates themselves, *i.e.*, we may even think of aggregates of aggregates. Here we shall be concerned only with aggregates of real numbers or, for illustrative purposes, with aggregates of points which represent the numbers graphically. If the collection contains only a finite number of objects, then the aggregate is called ***finite***, otherwise it is called ***infinite***. We shall be concerned mainly with infinite aggregates. The following are some examples of aggregates of numbers.

1. All the rational numbers $\geqslant 0$ and $\leqslant 1$.
2. All the real numbers >0 and <1.
3. All the numbers $\frac{1}{n}$, where $n=1, 2, 3, \ldots\ldots\ldots$, *i.e.*, the numbers $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots$
4. The odd numbers from 1 to 100, *i.e*, the numbers

$$1, 3, 5, \ldots\ldots, 97, 99.$$

5. All the numbers of the type $2n-1$, where n is any positive integer, *i.e.*, the numbers $1, 3, 5, 7, \ldots\ldots, 2n-1, \ldots\ldots$
6. All the numbers of the type $\frac{n-1}{n}$, where n is any positive integer, *i.e.*, the numbers $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\ldots$
7. All the numbers $(-1)^n n$, where $n=1, 2, 3, 4, \ldots\ldots$, *i.e.*, the numbers $-1, 2, -3, 4, \ldots\ldots$

With the exception of No. 4, all these are examples of infinite aggregates. No. 4 is a finite aggregate containing only 50 numbers.

When the members of an aggregate are given by a mathematical expression as in Nos. 3, 5, 6, 7 above, it is advisable to write down a number of members of the aggregate to get a clear idea of the spread of the aggregate or of any special features of the aggregate. The

geometrical representation given in Sec. 1·2 is very helpful in this matter and corresponding to each set of real numbers we have a set of points on the directed line and *vice versa*. Thus six members,

$$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6},$$

of set No. 6, given by $n=1, 2, 3, 4, 5, 6$, are represented by points

on the line in the attached figure. Even with six numbers shown in the figure, we begin to notice that the members of the aggregate are crowding near the point 1.

Sets are usually denoted by the first letter S of the word set. **The set of all** the real numbers, *i e.*, the Arithmetic Continuum, is usually denoted by **R**.

1·81. Subsets. If we have two sets S_1 and S_2 such that every member of S_1 is also a member of S_2, then we say that S_1 is a **sub-set** of the set S_2. If, in addition, there is at least one member of S_2 which is not a member of S_1, then S_1 is called a **proper subset** of S_2. Thus the set of all the rational numbers between 0 and 1 is a proper subset of the set of all the real numbers between 0 and 1.

1·82. Intervals. If a and b are any two real numbers with $a<b$, then the set of all real numbers x such that $a \leqslant x \leqslant b$ is called the **closed interval** $[a, b]$. If the end value a is excluded, *i.e.*, the set consists of all values x such that $a<x \leqslant b$, then, the interval is said to be **open at the left-hand end-point** and is written $(a, b]$. Similarly, if the end value b is excluded, then the interval is said to **be open at the right-hand end-point** and is written as $[a, b)$. If, however, both a and b be excluded, then the interval is said to be **open** at both ends or simply open and is written as (a, b).

If ϵ be a small positive number, then the interval $[a-\epsilon, a+\epsilon]$ is called **an ϵ-neighbourhood of a**.

Corresponding to intervals and neighbourhoods of numbers, we have intervals and neighbourhoods of points on the real line formed by the corresponding aggregates of points on the line.

1·83. Bounds of an aggregate. *An aggregate is said to be* **bounded above** or **bounded on the right** *if there exists a number K such that every number of the aggregate is less than or equal to K.* The number K itself is called **an upper bound** for the aggegate. Evidently, if K is an upper bound for an aggregate, then every number $>k$ is also an upper bound for the aggregate. If no such number K can be found, that is, if *whatever* number K is chosen, however large, some member of the aggregate $>K$, then the aggregate is said to be **not bounded above** or **not bounded on the right**. For **example**, aggregate Nos. 1, 2, 3, 6 of Sec. 1·8 are all bounded above, the number 2 being an upper bound for each, while Nos. 5, 7 are not bounded above.

Similarly, *an aggregate* is **bounded below** or **bounded on the left** *if there exists a number k such that every member of the aggregate is greater than or equal to k*. The number k is called **a lower bound** of the aggregate. If k is a lower bound for an aggregate, then every number $<k$ is also a lower bound for the aggregate. If no such number k can be found, that is, however large a negative number k is taken, there is some member of the aggregate $<k$, then the aggregate is said to be **not bounded below** or **not bounded on the left**. All the aggregates in Sec 1·8 with the exception of No. 7 are bounded below, the number -1 being a lower bound for each of them. No. 7 is not bounded below.

An aggregate is said to be **bounded** if it is bounded both above and below, *i e.*, on both sides. Aggregate Nos. 1, 2 3, 4, 6 of Sec. 1 8 are all bounded.

Let S be a set bounded above, then it possesses an infinite number of upper bounds K which themselves form a set, say S_1. Evidently, S_1 is not bounded above. For if K is an upper bound of S, then every number $>K$ is an upper bound of S and is, therefore, a member of S_1 and thus S_1 contains *all* numbers $>K$. On the other hand S_1 is bounded below for every member of S_1 is greater than or equal to every member of S. We are interested in knowing whether S_1 has a smallest number. If so, then this smallest member of S_1 is *the* smallest of all the lower bounds of S. *We assume that S_1 has a smallest member B.* This can be proved with the help of Dedekind's theorem mentioned in Sec. 1·5. *The number B is called the* **least upper bound** *or the* **exact upper bound** *or simply* **the upper bound** *of the set S.*

Similaly, if a set S is bounded below, the set S_2 of its lower bounds possesses a greatest member b which is called the **greatest lower bound** or the **exact lower bound** or simply **the lower bound** of S.

It is easy to verify that 1 is the upper bound of aggregates Nos. 1, 2, 3, 6 of Sec. 1 8, while 99 is the upper bound of aggregate No. 4. The lower bounds of the aggregates 1, 2, 3, 4, 5, 6 are 0, 0, 0, 1, 1, 0 repectively.

Remarks : 1. It should be observed that it is not necessary that an infinite aggregate should have a greatest or a smallest member even if it is bounded. Thus the aggregate of all the real numbers>0 and <1 has the upper bound 1 and the lower bound 0 but it has neither a greatest number nor a least. In case an aggregate possesses a greatest number, then it is its upper bound and we say that the upper bound belongs to the aggregate or that the upper bound is attained Similarly, if an aggregate possesses a smallest number, then it is its lower bound and we say that the lower bound belongs to the aggregate or that the lower bound is attained. On the other hand, a finite aggregate always has a greatest and a smallest member and these are its upper and lower bounds respectively.

2. Evidently, $B \geqslant b$.

1 9. Absolute value. *The* **absolute value** *or* **modulus** *or* **numerical value** *of a real number a is defined to be a, $-a$ or 0 according as a is positive, negative or zero.* The absolute value is denoted by $|a|$ and is read as 'mod a'. For example,

$$|2|=2, |-2|=2, |0|=0.$$

If the number a is represented on a line $X'OX$, then $|a|$ is the distance of the point a from the origin. It should be observed that the absolute value or the modulus of a number is always positive, unless the number is zero. The following are immdediate consequences of the definition.

Theorem 1. *If a, b are any two real numbers, then*

$$(i)\ |ab| = |a| \times |b|. \quad (ii)\ \left|\frac{a}{b}\right| = \frac{|a|}{|b|},$$

where in the latter $b \neq 0$.

For example,

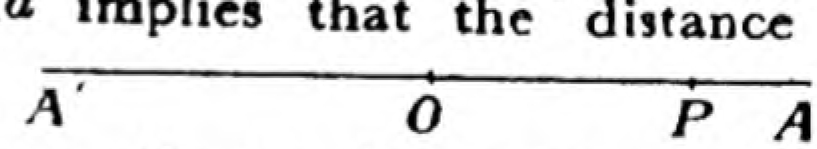
$$|(-3)(4)| = |-12| = 12, \text{ and } |-3| \times |4| = 3\times4=12,$$

$$\left|\frac{-3}{4}\right| = \left|-\frac{3}{4}\right| = \frac{3}{4}, \text{ and } \frac{|-3|}{|4|} = \frac{3}{4}.$$

Theorem 2. *If $|x| < a$, then $-a < x < a$.*

It should be noticed that when we write $|x| < a$, it automatically implies that $a > 0$ for, as observed above, $|x|$ is always positive. We illustrate this and the following theorems geometrically.

Let A, A' and P represent the numbers $a, -a$ and x respectively, then $A'O = OA = a$ and $OP = x$ $|x| < a$ implies that the distance $OP < a$. Hence if x is positive, P lies between O and A, while if x is negative P lies between A' and O. Hence P must lie to the right of A' and left of A, *i.e.*,



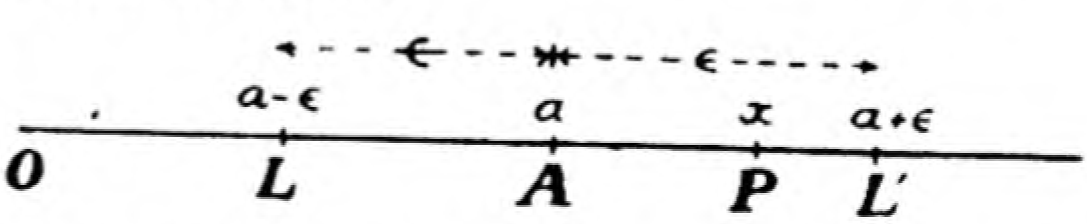
$$-a < x < a.$$

Theorem 3. *If $|x - a| < \epsilon$, then $a - \epsilon < x < a + \epsilon$.*

Let the points A and P represent the numbers a and x respectively then $OA = a$, $OP = x$, and therefore

$$AP = OP - OA = x - a,$$

and therefore $|x - a|$ *represents the numerical value of the distance AP.*



If we measure a distance ϵ to the left and right of A, we get two points L and L' which represent $a-\epsilon$ and $a+\epsilon$ respectively. Since $|x-a| < \epsilon$ implies that the distance AP is less than AL and AL' numerically, it follows that P lies between L and L', *i.e.*,

$$a - \epsilon < x < a + \epsilon$$

Cor. *If $|x-a| < \epsilon$, then $x - \epsilon < a < x + \epsilon$.*

For, by theorem 3, $a - \epsilon < x < a + \epsilon$. From the left half of the inequality, we get $a < x + \epsilon$, by adding ϵ to both sides, and from

the right half we get, similarly, $x-\epsilon<a$. Combining these two, we get $x-\epsilon<a<x+\epsilon$.

Theroem 4. *For all real values of a and b,* $|a+b| \leqslant |a|+|b|$.

The equality holds when a and b are of the same sign, otherwise the inequality holds.

We have to consider three cases :

(*i*) If a, b are both positive, then $a+b$ is positive, and

$$|a+b|=a+b=|a|+|b|.$$

(*ii*) If a, b are both negative, then $a+b$ is negative, and

$$|a+b|=-(a+b)=-a-b=|a|+|b|.$$

(*iii*) If one of a, b is negative, then, to be definite, let a be positive and b negative. Let next $a+b$ be positive, then

$$|a+b|=a+b=a-(-b)=|a|-|b|<|a|+|b|.$$

If, however, $a+b$ is negative, then

$$|a+b|=-(a+b)=-a-b=-|a|+|b|<|a|+|b|.$$

We can give a similar proof for the case when a is negative and b is positive.

The theorem can be extended to the sum of any number of real quantities. It may be stated in words, 'the modulus of a sum is less than or equal to the sum of the moduli'. This inequality is used very frequently and the student is advised to construct numerical examples for all the cases and verify the truth of the inequality for every case.

Cor. 1. $|a-b| \geqslant |a|-|b|$.

This can be proved by writing $a=a-b+b$ and applying Th. 4.

Cor. 2. *If x, y, a are real numbers such that* $|x-a|<\epsilon$, $|y-a|<\epsilon$, *then* $|x-y|<2\epsilon$.

For, we can write

$$\begin{aligned} |x-y| &= |(x-a)-(y-a)| \\ &\leqslant |x-a|+|y-a| \qquad \text{by Th. 4} \\ &< \epsilon+\epsilon=2\epsilon. \end{aligned}$$

CHAPTER II

FUNCTIONS

2·1. Constants and variables. A quantity which retains the same value throughout a problem is called a **constant** whereas a quantity which can assume different values at different places in the problem is called a **variable**. Constants are denoted usually by the first few letters of the alphabet, *i.e.*, by a, b, c, etc., while variables are generally denoted by the last few letters such as x, y, z, etc.

The set of values which a variable can assume is called its **domain of variation.**

2·2. Function. Def. (*i*) *When, by some rule or rules, the value of a real variable y is uniquely determined by that which another real variable x receives, we say that y is a* **function** *of x.*

x is called the **independent variable** and y is called the **dependent variable.** The independent variable is also sometimes called the *argument.*

The set of values of x for which y is **defined,** *i.e.*, for which y has a definite, finite value, is called the **domain** of the function.

The set of all the values that y takes corresponding to the values of x in the domain is called the **range** of the function.

The volume V of a cube of side x is given by $V=x^3$. Since V depends upon x, we say V is a function of x.

It may be noted that according to our definition a *unique* value of y must correspond to *each* value of x while the *same* value of y may correspond to *several* values of x

2·21. Notation. A function of x is generally denoted by the symbol $f(x)$ [read as 'f of x' or 'f at x']. We can also use other letters besides f in the functional symbol outside the brackets and denote a function as $g(x)$, $h(x)$, $\phi(x)$, $\psi(x)$, $y(x)$, etc.

The symbol $f(x)$ does not stand for the product of f and x. It is one composite symbol. We shall, however, sometimes speak of a function f when there is no ambiguity about the argument of f.

The value of f at $x=a$ is denoted by $f(a)$. Thus if $f(x)=x^2+1$, then $f(1)=1^2+1=2$, $f(-\frac{1}{2})=(-\frac{1}{2})^2+1=\frac{5}{4}$, etc.

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*Strictly speaking, f is the rule which establishes a correspondence between elements of the domain and elements of the range of the function and $f(x)$ is the *value of the fuction at x*. For example, let f stand for 'fatherhood' *i.e.*, f is the rule which associates with each person his/her father. Then we write

f(Akbar) = Humayun,
f(Jawaharlal) = Motilal

and so on. We speak of f (Akbar) as the value of f at 'Akbar'.

2·22. Real functions. Functions whose domains and ranges are sets of real numbers are called *real functions* or *real-valued functions of a real variable.*

A function is said to be defined on the whole of the real line if its domain is the set **R** of real numbers. It is said to be defined over a segment of the real line if its domain is an interval.

2·3. The continuous real variable. Let x be a variable. If, in its variation, x takes on every real value lying between every two of its values, then we say that x is a **continuous real variable.** For example, if x can take up all the real values which are $\geqslant 0$ and $\leqslant 1$, then x is a continuous real variable and its domain of variation is the closed interval $[0, 1]$. If, on the other hand, x could assume only the rational values lying between 0 and 1, then it would not be a continuous real variable, because it fails to assume irrational values. In this work, we shall be primarily concerned with functions of a continuous real variable.

We consider a few examples to illustrate the idea of a function.

Ex. 1. $f(x)=x^2$.

Since x^2 has a unique positive value for every real x, the domain of $f(x)$ is the set of all the reals and the range is the set of all reals $\geqslant 0$.

Ex. 2. $f(x)={}^xP_x$, *i.e., the number of permutations of x different things taken all at a time.*

Evidently the function is defined only for positive integral values of x for which ${}^xP_x=x\,!$ Also, by convention, we take ${}^0P_0=0\,!=1$. Hence

$$f(0)=1, f(1)={}^1P_1=1, f(2)={}^2P_2=2\,!=2, f(3)=3\,!=6,\ldots\ldots\ldots$$

Hence the domain of the function is the set of all integers $\geqslant 0$ and the range is the set of integers 1, 2, 6, 24, 120,......, *i.e.*, a subset of the set of positive integers.

Ex. 3. $f(x)=\surd(a^2-x^2)$.

$f(x)$ has a unique real value for values of x lying between $-a$ and a; for other real values of x, $f(x)$ does not have a real value and therefore is not defined. Also, for values of x between $-a$ and a, $f(x)$ takes values lying between 0 and a. Thus the domain of x is the set of real numbers given by $-a\leqslant x\leqslant a$, *i.e.*, the interval $[-a, a]$, and the range is the interval $[0, a]$.

Ex. 4. $f(x)$, *where* $[f(x)]^2=a^2-x^2$.

Here $f(x)$ has a *unique* real value for $x=+a$ or $-a$ only. Therefore, according to our definition, the domain of $f(x)$ consists of two real numbers $-a$ and a and the range consists of the single

number 0. However, if we take square roots, we get

$$f(x)=+\sqrt{(a^2-x^2)} \quad \text{or} \quad -\sqrt{(a^2-x^2)},$$

and it seems profitable to regard $f(x)$ as defined above to be a combination of two distinct functions

$$f(x)=+\sqrt{(a^2-x^2)} \text{ and } f(x)=-\sqrt{(a^2-x^2)}$$

the first of which is the function of Ex. 3. For the second, the domain is the same as for the first, *viz*, the interval $[-a, a]$ while the range is the interval $[-a, 0]$.

In the older terminology, it is not required that y or $f(x)$ should have a *unique* value for each x, a function being allowed to be *multiple valued*. In this terminology the function defined by

$$[f(x)]^2=a^2-x^2$$

will be called a two-valued function, the two values $f(x)=+\sqrt{(a^2-x^2)}$ and $f(x)=-\sqrt{(a^2-x^2)}$, being called the two *branches* of the function.

Ex. 5. $f(x)=2x+1$ *for* $x\leqslant 0$ *and* $=x+4$ *for* $x>0$.

$f(x)$ has a unique real value for every real x. Hence the domain of the function is the set of all the reals, *i.e.*, the whole real line. Also $f(x)$ can assume every value $\leqslant 1$ for $x\leqslant 0$ and every value >4 for $x>0$, while $f(x)$ cannot assume any value x such that $1<x\leqslant 4$. Hence the range of the function is the whole of the real line with the exception of the set of values $1<x\leqslant 4$.

It should be observed that a function may be given by different expressions over different parts of its domain.

Ex. 6. $f(x)$=*height of citizen* x *of India.*

Let us suppose that there are 400,000,000 citizens in India. Then these can be labelled as 1, 2, 3,......, 400,000,000. The height of each citizen is *uniquely* defined and, even allowing for abnormally tall persons, may be supposed to be less than 8 ft. in all cases. Hence the height of citizen x of India is a function whose domain is the set of integers 1 to 4 million and range is a set of positive real numbers all less than 8 assuming that all heights are measured in feet.

This is an example of a function where the functional relationship cannot be expressed by a mathematical formula. Naturally, we can perform the various mathematical operations better on functions which are expressed by mathematical formulae and we shall be concerned entirely with such functions in the sequel.

Ex. 7. *Find the values of* x *for which the function* $f(x)=1/(\cos x)$ *is not defined.*

In general, a function is not defined for such values of x which make $f(x)$ imaginary or which make the denominator of some expression in $f(x)$ vanish, *i.e.*, which involve division by zero which operation is not allowed. The present function is not defined for such values of x for which $\cos x=0$, *i.e.*, for $x=\frac{1}{2}(2n+1)\pi$, where n is any integer, positive, negative or zero.

EXAMPLES I

1. If $f(x)=x^2+2x-5$, find $f(2)$, $f(-1)$, $f(0)$, $f(1/x)$.
2. If $f(x)=\sin x$, find $f(\frac{1}{2}\pi)$, $f(\frac{1}{3}\pi)$, $f(0)$.
3. If $f(x)=\tan x$, show that $f(\frac{1}{4}\pi-x)=\dfrac{1-f(x)}{1+f(x)}$.
4. If $f(x)=\log_a x$, show that $f(mn)=f(m)+f(n)$.
5. If $\phi(x)=(1-x^2)/(1+x^2)$, prove that $\phi(\tan\theta)=\cos 2\theta$.
6. If $f(x)=x^3$, find $\{f(a+h)-f(a)\}/h$.
7. If $f(x)=\tan x$, find $\{f(\frac{1}{4}\pi+h)-f(\frac{1}{4}\pi)\}/h$.
8. Show that if $y=f(x)=(ax+b)/(cx-a)$, then $x=f(y)$.
9. Find for what values of x have the following functions no definite value?

(*i*) $\dfrac{1}{x-1}$ (*ii*) $\dfrac{1}{x^2-5x+6}$ (*iii*) $\dfrac{1}{\sin x}$

(*iv*) $\cos\dfrac{1}{x}$ (*v*) $\dfrac{x^2-4}{x-2}$ (*vi*) $\sin^2\left(\dfrac{\pi}{x-1}\right)$.

10. Find the domains and ranges of the following functions:

(*i*) $f(x)=\sqrt{(4-x^2)}$. (*ii*) $f(x)=\sqrt{(x^2-4)}$.

(*iii*) $f(x)=1/\sqrt{(4-x^2)}$. (*iv*) $f(x)=\dfrac{x}{(x-1)(x-2)}$.

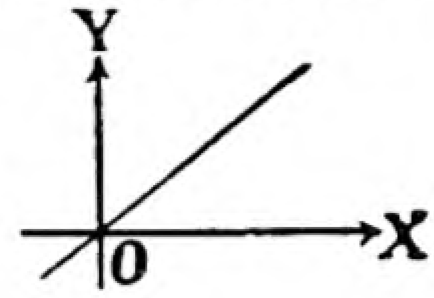
2·4. **Graphs of functions.** Let $y=f(x)$ be a function defined over the domain D. Let x be an element of D and $f(x)=y$ the corresponding value of the function. We construct a geometrical representation of the functional relation between x and y by using the method of coordinates with which the student must be already familiar. Take two perpendicular lines OX and OY in the plane XOY and plot the point $P[x, f(x)]$ or, what is the same thing, the point $P(x, y)$. Then *the totality of all the points P as x covers the domain D, i.e., takes all values in the domain D, is called the* **graph of the function.**

The graph of a function represents it geometrically. When x varies continuously, the graph is usually a continuous curve but it is not necessary that it must be always so. It may consist of disjointed pieces or arcs or even of isolated points. The graph of $y=f(x)$ may be spoken of as the curve $y=f(x)$.

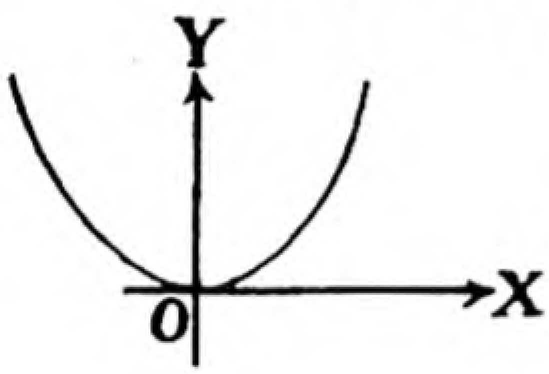
We illustrate the graphical representation of functions by some examples.

Ex. 1. $f(x)=x$, *or* $y=x$.

The domain of the function consists of all the real numbers. For each point $\{x, f(x)\}$ on the graph, $y=x$, *i.e.*, the ordinate is equal to the abscissa. The graph is the whole straight line which bisects the angle XOY in the first and fourth quadrants.



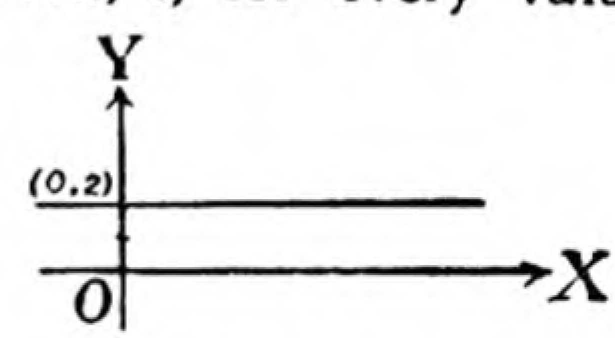
Ex. 2. $F(x)=x^2$, *or* $y=x^2$.



The domain is again the set o all the real numbers, while the range is the set of real numbers $\geqslant 0$. The points (x, y) lie on a curve which passes through the origin and opens out in both the first and second quadrants because $y=0$ for $x=0$, and y increases indefinitely as x increases whether positively or negatively. We know from coordinate geometry that the graph is a *parabola* touching the x-axis at the origin and having its axis along the y-axis.

Ex. 3. $g(x)=2$.

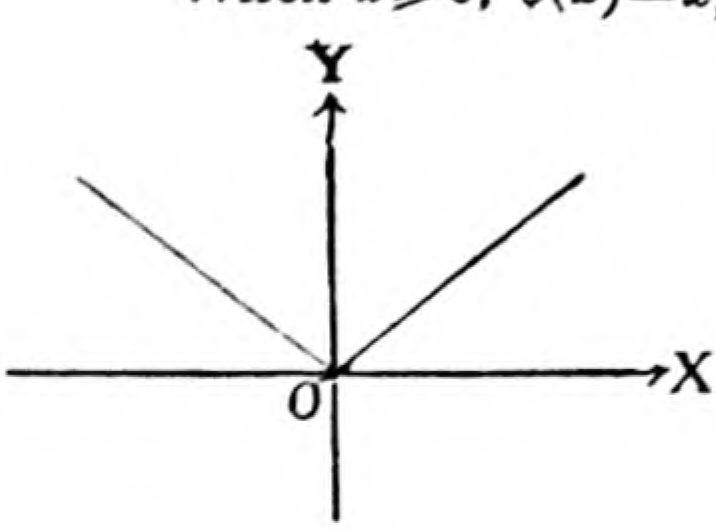
The value of the function is the same, *viz.*, 2, for every value of x. Hence the domain of the function is the set of all real numbers and the range is the solitary number 2. The graph of the function is a straight line parallel to the x-axis and cutting the y-axis at the point (0, 2).



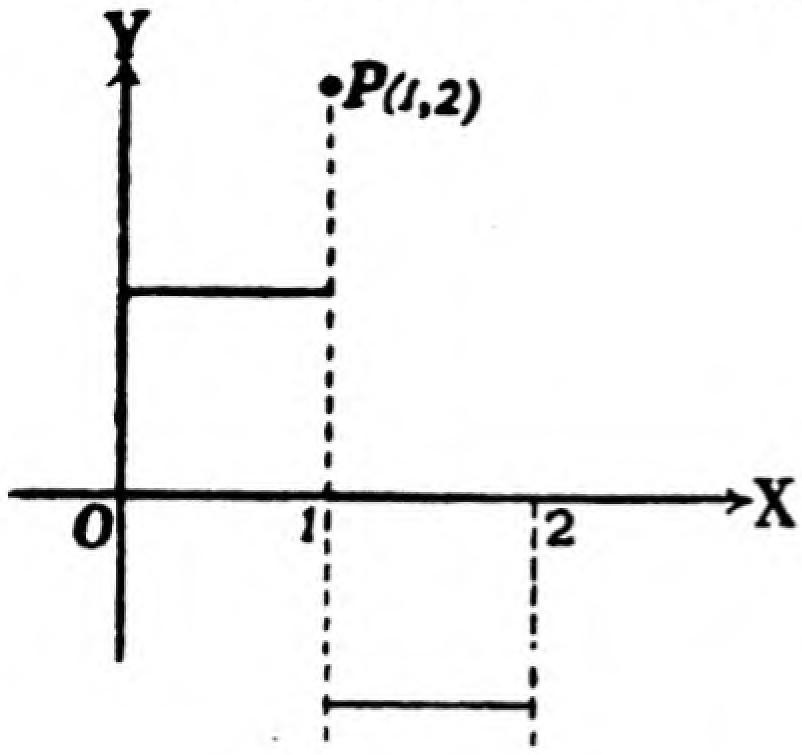
Such a function is called a **constant function** or simply a **constant**. The general representation of such a function is $f(x)=C$, where C is any constant.

Ex. 4. $\varphi(x)=|x|$.

When $x\geqslant 0$, $\varphi(x)=x$, and when $x<0$, $\varphi(x)=-x$. Hence the domain of $\varphi(x)$ consists of all real numbers and the range of all real numbers $\geqslant 0$. The graph consists of the parts of the two straight lines $y=x$ and $y=-x$ lying in the first and second quadrants respectively meeting at the origin which is a part of the graph.



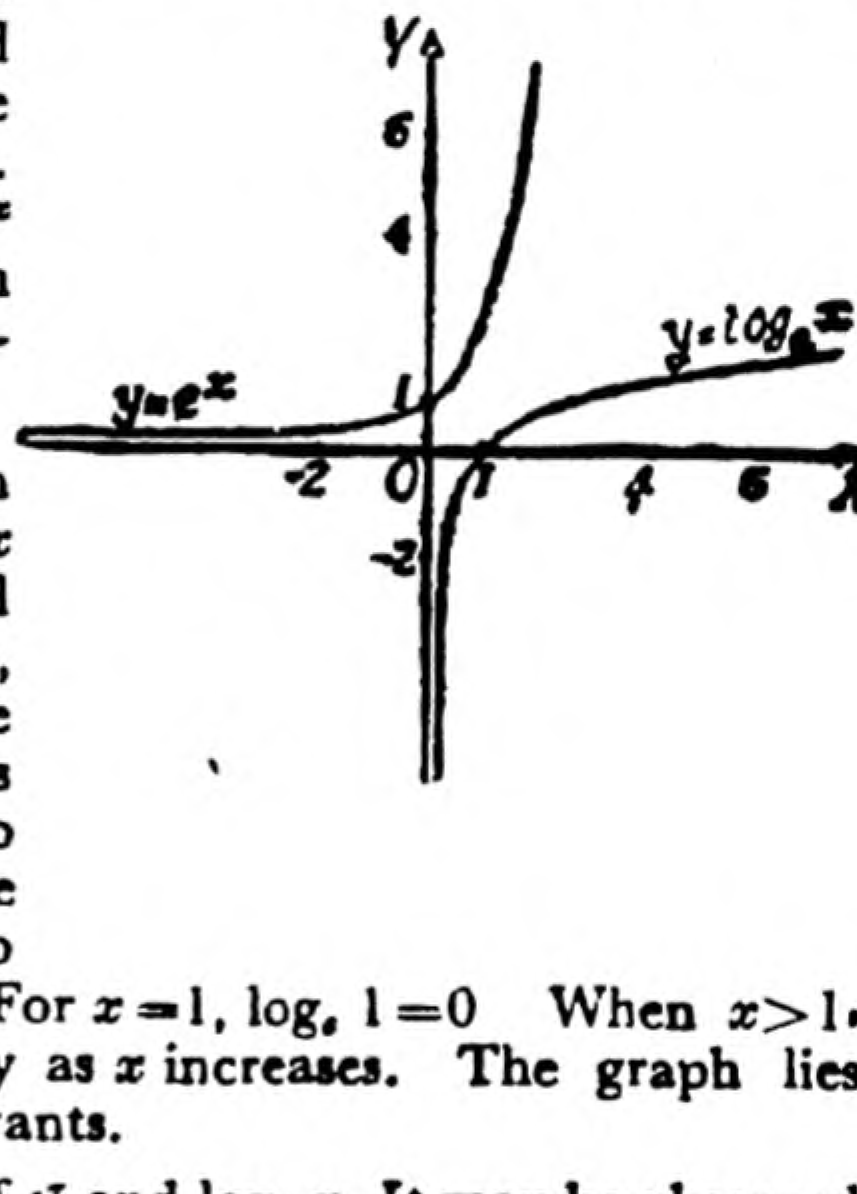
Ex. 5.
$$\begin{aligned} f(x) &= 1 \quad \textit{for } 0<x<1 \\ &= 2 \quad \textit{for } x=1 \\ &= -1 \ \textit{for } 1<x<2. \end{aligned}$$



The graph consists of (*i*) the segment of the line $y=1$ lying between the points $(0, 1)$ and $(1, 1)$ excluding these points, (*ii*) the isolated point $P(1, 2)$, and (*iii*) the segment of the line $y=-1$ between the points $(1, -1)$ and $(2, -1)$ excluding these points. The domain of the function is the open interval $(0, 2)$ and the range consists of the three numbers 1, 2 and -1. The graph is discontinuous at $x=1$ in the ordinary sense of the word discontinuous. The technical sense will be explained in a later chapter.

Ex. 6. (*i*) $f(x)=e^x$, (*ii*) $f(x)=\log_e x$, *where e is the base of the natural logarithms.*

(*i*) This function is defined and is positive for all real values of x and the graph lies entirely in the first and second quadrants. When x takes increasing large negative values, e^x becomes smaller and smaller and the graph lies very close to the x-axis in the second quadrant. For increasing positive values of x, e^x increases very rapidly and the graph rises very sharply in the first quadrant.



(*ii*) By definition, when $y=\log_e x$, $x=e^y$, which shows that x cannot be negative for any real value of y. The function $\log_e x$ is, therefore, defined for positive value of x only. For $0<x<1$, $\log_e x$ is negative and for values of x close to zero, $\log_e x$ is very large but negative and so the graph lies very close to the y-axis in the fourth quadrant. For $x=1$, $\log_e 1=0$ When $x>1$, $\log_e x$ is positive and increases slowly as x increases. The graph lies entirely in the fourth and first quadrants.

The figure shows the graphs of e^x and $\log_e x$. It may be observed that the graphs of 10^x and $\log_{10} x$ are similar in shape to those of e^x and $\log_e x$ respectively.

2·5. **Equality of functions.** *Two functions f and g are said to be equal if they have the same domain and for every x, $f(x)=g(x)$. Thus two functions are equal if they are identical.*

The functions x^2 and $(x-1)(x+1)+1$ are obviously identical but the functions $f(x)=x^2$ and $g(x)=x^3/x$ are *not* identical. While the domain of f is the whole of the real line that of g is the whole of

the real line except zero. The two functions, therefore, do not have the same domain.

2·6. Inverse functions. Let $y=f(x)$ be a function, D its domain and E its range. If the function is such that to each value of x corresponds a distinct value of y, *i.e.* to no two values of x corresponds the same value of y, then the correspondence between x and y is $1-1$ and to each value of y in E corresponds a unique value of x in D. Hence we may regard x as a function of y, the domain of this function being E and the range D. This function is called the **inverse** of $f(x)$ and the symbol f^{-1} is often used to denote the inverse of f. Thus if $y=f(x)$ has an inverse function, we may denote it by $x=f^{-1}(y)$. Notice that the notation $f^{-1}(y)$ *does not* mean $1/f(y)$. For example, $y=e^x$, $x=\log_e y$ are a pair of inverse functions.

When the same value of y corresponds to several values of x, the definition of an inverse function is not so simple. For example, $y=\sin x$ has the same value for x and $n\pi+(-1)^n x$ where n is any integer. The definitions of the inverse trigonometric functions are given in Ch. IV.

If we wish to use the same notation for the independent and dependent variables both for a function and its inverse, then we can say that $y=e^x$ and $y=\log_e x$ are a pair of inverse functions, and similarly for other pairs of inverse functions. It may be observed that since $y=\log_e x$ implies $x=e^y$, the graph of $y=\log_e x$ may be obtained from that of $y=e^x$ by merely interchanging x and y in the latter or, what comes to the same thing, by interchanging the x- and the y-axis in the latter.

2·7. Explicit and implicit functions. If $y=f(x)$, then y is said to be defined **explicitly** by this relation, or is said to be an **explicit** function of x. All the functions considered so far have been explicit functions. However, very often the relationship between x and y is given by an equation of the form

$$\varphi(x, y)=0,$$

where y is not solved for x. We then say that y is defined **implicitly** by this relation or by φ. For example, the relation

$$\varphi(x, y)=7xy+5x+3y+9=0$$

defines y as an **implicit** function of x. We can solve this equation for y in terms of x getting

$$y=-\frac{5x+9}{7x+3},$$

which defines the same function explicitly. It is not generally possible to solve $\varphi(x, y)=0$ for y in terms of x and get an explicit representation for the function defined implicitly by φ.

Very often, a relation of the type $\varphi(x, y)=0$ does not give a single value of y for each value of x. In such cases, $\varphi(x, y)=0$ must

be regarded as defining several functions implicitly. Consider the relation,

$$\varphi(x, y)=x^2+y^2-a^2=0.$$

Here two values of y, $+\sqrt{(a^2-x^2)}$ and $-\sqrt{(a^2-x^2)}$, correspond to each value of x in the interval $-a<x<a$ except at $x=\pm a$ where the two values coincide. Hence we say that this relation defines implicitly the two functions whose explicit expressions are

$$y=+\sqrt{(a^2-x^2)},\ y=-\sqrt{(a^2-x^2)}.$$

Here, the domains of the two functions are the same. In general, this need not be so.

2·8. **Parametric representation of a function.** Very often the functional relationship between two quantities x and y can be expressed through the medium of a third quantity t, say, so that

$$x=g(t),\quad y=h(t),\ \text{and}\ \ y=f(x). \tag{1}$$

where g and h are defined over a common domain A, and the ranges of g and h are the domain and range respectively of the function f. In other words, if the three relations (1) are regarded as equations in the ordinary way and we eliminate t from the first two, the result of elimination is $y=f(x)$, or, what is the same thing, if the values of x and y from the first two are substituted in $y=f(x)$, then this equation is satisfied identically. The relations $x=g(t)$, $y=h(t)$ are then called the **parametric representation** of the function $y=f(x)$. In geometrical language, the equations $x=g(t)$, $y=f(t)$ are called the **parametric equations** of the curve $y=f(x)$. t is called the **parameter**.

For example,

$$x=a\cos t,\quad y=a\sin t,\quad 0\leqslant t\leqslant\pi,$$

is a parametric representation for the function

$$y=\sqrt{(a^2-x^2)},\quad -a\leqslant x\leqslant a.$$

If we take $\pi\leqslant t\leqslant 2\pi$, the same equations give the parametric representation of the function $y=-\sqrt{(a^2-x^2)}$, $-a\leqslant x\leqslant a$.

It may be observed that although in general it may not be easy to find a parametric representation for a given function, the same function can have any number of such representations.

For example,

$$x=a\frac{1-t^2}{1+t^2},\quad y=\frac{2at}{1+t^2},\quad 0\leqslant t$$

is a parametric representation of the same function as above.

Parametric representations often serve to simplify calculations with functions. When parametric representations are used, the distinction between independent and dependent variables does not remain significant and the two have an equality of status, so to say. Parametric representations are particularly useful where functions are defined implicitly.

2·9. Classification of functions.

(1) **Polynomials**. Let

$$f(x)=a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n,$$

where n is a positive integer and $a_0, a_1, \ldots\ldots\ldots, a_n$ are constants with $a_0 \neq 0$, then $f(x)$ is called a **polynomial** in x. The exponent n of the highest power of x is called the **degree** of the polynomial. Thus the functions

$$f(x)=2x^2-3x+5 \quad \text{and} \quad g(x)=5x^7-6x^4+2x^2-4$$

are polynomials in x of degrees 2 and 7 respectively.

A polynomial is defined for all real values of x. Hence the domain of every polynomial is the set of all real values.

(2) **Rational functions.** A function

$$R(x)=\frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials in x, is called a **rational function of x.** Examples are

$$R(x)=\frac{x^2-7}{3x^3+x-9}; \quad S(x)=\frac{5x^4-2x^3+7x^2-9}{x^2+3x^2+4x+5}.$$

A rational function is defined for all real values of x except those which make the denominator vanish. Hence the domain of a rational function is the set of all real numbers except those which make the denominator vanish.

(3) **Irrational functions.** Functions of the type

$$\sqrt{(2x^2+5x+7)}, \quad \frac{(x+3)^{\frac{1}{3}}-(2x+5)^{\frac{2}{7}}}{(2x^3+7x^2+11x-1)^{\frac{1}{4}}}, \text{ etc.,}$$

which involve radicals are called **irrational** functions.

(4) **Algebraic functions.** The three types listed above are examples of algebraic functions. A function is said to be algebraic if it is evolved through the processes of addition, subtraction, multiplication and division together with the process of root extraction of polynomials.

(5) **Transcendental functions.** *A function which is not algebraic is called* **transcendental.**

The simplest transcendental functions are the elementary trigonometric functions such as $\sin x$, $\cos x$, etc. and their inverses. Two other elementary transcendental functions are the exponential function e^x and the logarithmic function $\log_e x$.

EXAMPLES II

Draw the graphs of the following functions :—

1. $f(x)=x+1$.

2. $f(x)=x^3$.

3. $f(x)=1/x$. 4. $f(x)=\sin x$.

5. $f(x)=x$ when $0 \leqslant x \leqslant 3$, $=2x-3$ when $3<x\leqslant 5$, and $=7$ when $x>5$.

6. $f(x)=[x]$, $-3\leqslant x\leqslant 3$.

[The symbol $[x]$ stands for the integral part of x or, in other words, the largest integer $\leqslant x$. For example, if $x=3{\cdot}5$, $[x]=3$; if $x=-2{\cdot}7$, $[x]=-3$; and if $x=2$, $[x]=2$.]

7. Verify if the following pairs of functions are equal or not.

(*i*) $f(x)=x+1$, $g(x)=(x^2-1)/(x-1)$.

(*ii*) $f(x)=2\,|\,x\,|$, $g(x)=|\,x-1\,|+|\,x+1\,|$.

8. Obtain explicit expressions for the functions defined implicitly by the relation $y^2-2y+x^2=0$ and give the domain of each.

9. *A function $f(x)$ is called* **even** *if $f(-x)=f(x)$ for all x and* **odd** *if $f(-x)=-f(x)$ for all x.* Determine whether the following functions are even or odd.

(*i*) $y=\cos x$. (*ii*) $y=e^x-e^{-x}+\sin x$.

(*iii*) $y=x^4+3x^2+5x+7$.

10. Show that every function $f(x)$ can be expressed as the sum of an even and an odd function.

CHAPTER III

LIMIT AND CONTINUITY

3·1. The phrases 'x tends to a' and 'x tends to ∞'.

(*i*) Let x be a real variable which passes *successively* through an infinity of values according to any unambiguous law. *If the successive values of x approach a definite number a in such a way that the numerical value of the difference $x-a$ becomes and remains smaller than* **every** *given positive number* ϵ, *however small, then we say that x* **tends to a** *or* **x has the limit a'** and we write

$$x \to a, \text{ or Lt } x = a, \text{ or lim } x = a.$$

The definition implies that for *every* such ϵ there must be a stage in the succession of values of x such that all the values of x after this stage lie inside the interval $(a-\epsilon,\ a+\epsilon)$. However, the definition *does not* imply that x must take the value a. This point is illustrated in the examples below. In Ex. 1—3, Lt $x=1$ but 1 is not a value of x. In Ex. 4, Lt $x=1$ and 1 is a value of x. In Ex. 5, Lt $x=0$ and 0 is not a value of x.

Ex. 1. Let x take successively the values

$$\frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \ldots\ldots\ldots\ldots,\ \frac{n-1}{n},\ \ldots\ldots\ldots,$$

then $x \to 1$, for the numerical value of the difference between x and 1 is successively

$$\frac{1}{2},\ \frac{1}{3},\ \frac{1}{4}, \ldots\ldots\ldots, \frac{1}{n},\ \ldots\ldots\ldots,$$

which becomes smaller and smaller and can evidently be made smaller than any given positive numbe ϵ, however small, for *all* values of n after some stage, *viz.* for $n > 1/\epsilon$.

Ex. 2. Next, let x take successively the values

$$\frac{3}{2},\ \frac{4}{3},\ \frac{5}{4},\ \ldots\ldots\ldots,\ \frac{n+1}{n},\ \ldots\ldots\ldots$$

then again $x \to 1$, for the numerical value of the difference between x and 1 is successively again

$$\frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots\ldots\ldots,\ \frac{1}{n}, \ldots\ldots\ldots$$

and is less than ϵ for *all* $n > 1/\epsilon$.

Ex. 3. Finally, let x take successively the values

$$\frac{3}{2},\ \frac{1}{2},\ \frac{4}{3},\ \frac{2}{3},\ \frac{5}{4},\ \frac{3}{4},\ \ldots\ldots,\ \frac{n+1}{n},\ \frac{n-1}{n}, \ldots\ldots,$$

then again $x \to 1$, for the difference between x and 1 is alternately

$\frac{1}{n}$ and $-\frac{1}{n}$ and once again the numerical value of the difference $x-1$, *i.e.*, $|x-1|$, can be made less than ϵ for *all* values of x after a suitable stage.

Ex. 4. Let x vary continuously in the interval $0 \leqslant x \leqslant 1$ and take increasing values, then $x \to 1$.

Ex. 5. Let x vary continuously in the interval $0 < x \leqslant 1$ and take decreasing values, then Lt $x=0$.

Let x tend to a. It may so happen that all the values taken by x are $\leqslant a$. We then say that **x tends to a from below** or **from the left** and write

$$x \to a-0 \text{ or simply } x \to a-.$$

In Ex. 1 and 4 above all the values taken by x are less than 1 and therefore we may write $x \to 1-0$.

If, on the other hand, all the values taken by x are $\geqslant a$, then we say that **x tends to a from above** or **from the right** and we write

$$x \to a+0 \text{ or simply } x \to a+.$$

This is illustrated by Ex. 2 and 5 above. In Ex. 2, all the values of x are greater than 1 and we may write $x \to 1+0$. In Ex. 5 all the values are >0 and we may write $x \to 0+0$ or simply $x \to 0+$.

(*ii*) Let next x be a real variable which takes on successively values which ultimately become and remain greater than every real positive number, however large, then we say that **x tends to plus infinity** and we write $x \to -\infty$. In other words, for *every* positive number G, there must be a stage in the succession of values of x after which *all* values of $x > G$.

If, on the other hand, the successive values assumed by x become and remain smaller than every negative number, then we say that **x tends to minus infinity** and we write $x \to -\infty$.

Notes 1. It may be pointed out once for all that there is no such real number as $+\infty$ or $-\infty$. When we say that an aggregate is infinite, we only mean that the number of members is not finite, or, in other words, whatever large number G be thought of, the number of members in the aggregate is greater than G. Since $+\infty$ or $-\infty$ is not a number we cannot work with these symbols as with ordinary numbers. Phrases involving these symbols are to have the meanings that are assigned to them by definition.

2. In defining the phrases 'x tends to a' and 'x tends to ∞', the variable x is not required to be a continuous variable necessarily. It may be, as in Ex. 4, 5 above, or it may not be, as in Ex. 1, 2, 3. We shall be concerned more often with the continuous real variable in the sequel.

3·2. **Limit of a function**. Let $f(x)$ be a function defined in some interval containing the point a, but may or may not be defined at $x=a$ itself. We consider the behaviour of $f(x)$ as $x \to a$. It

may so happen that the values of $f(x)$ become closer and closer to a number l as $x \to a$, or, in other words, the numerical value of the difference $f(x) - l$ can be made smaller than any pre-assigned positive number ϵ, however small, by taking x sufficiently close to a. In such circumstances, we say that $f(x)$ **approaches** or **converges** or **tends to the limit** l as $x \to a$ and we write

$$\operatorname*{Lt}_{x \to a} f(x) = l \text{ or } f(x) \to l \text{ as } x \to a.$$

In many cases of practical importance, $f(x)$ is defined at $x=a$ and $l=f(x)$, but it is worth emphasizing that this is not necessary for the existence of a limit. As pointed out in the last section, when $x \to a$, a is not necessarily a value of x. Similarly, when $f(x) \to l$, l is not necessarily a value of $f(x)$. We consider a few examples before giving a formal definition of a limit

Ex. 1. Let $f(x)=2x+1$ and $x \to 1$. $f(x)$ is defined for all real x including $x=1$. It is easy to see that as x approaches the value 1, $f(x)$ becomes closer and closer to 3 which is the value of $f(x)$ at $x=1$. The difference between $f(x)$ and 3, *viz*. $2x+1-3=2(x-1)$, can be made numerically as small as we please by taking x sufficiently close to 1. In fact, if we want this difference to be less than 0·01 in absolute value, we need take x to be such that $|2(x-1)| < 0.01$ *i.e.*, $|x-1| < 0.005$; if we want $|f(x)-3| < 0.0001$, we need take $|x-1| < 0.00005$, and so on. In general, if ϵ is any given positive number, however small, then

$$|f(x)-3| = 2\,|x-1| < \epsilon \text{ if } |x-1| < \tfrac{1}{2}\epsilon.$$

Hence we can say that $f(x) \to 3$ as $x \to 1$.

Ex. 2. Let $f(x)=x^2$ and $x \to 3$.

Here again, $f(x)$ is defined for all real x, including $x=3$. It is again easy to see that as x approaches 3, $f(x)$ approaches 9 which is the value of $f(x)$ at $x=3$. We list below the values of x and $f(x)$ as $x \to 3$ both from below and above.

x	2·9	2·99	2·999	2·9999	$\to$	$3-0$
$f(x)$	8·41	8·9401	8·994001	8·99940001	$\to$	9
x	3·1	3·01	3·001	3·0001	$\to$	$3+0$
$f(x)$	9·61	9·0601	9·006001	9·00060001	$\to$	9

We observe that the values of $f(x)$ become closer and closer to 9 as $x \to 3$ whether from below or above and we feel convinced that $f(x) \to 9$ as $x \to 3$. However, the concept of limit of a function requires that we must be able to show that, *given any positive number ϵ, however small, we must have*

$$|f(x)-9| = |x^2-9| < \epsilon,$$

for all x sufficiently close to 3, *i.e.*, for every ϵ, we should be able to find a positive number $\delta=\delta(\epsilon)$ such that the preceding inequality holds for all x which lie in a δ-neighbourhood of a. The notation $\delta=\delta(\epsilon)$ is used to emphasize that, in general, δ depends on ϵ and is a function of ϵ.

In the preceding example, $\delta = \frac{1}{2}\epsilon$. We proceed to find such a δ for the present example. Let $x = 3 + h$, so that we must have

$$|(3+h)^2 - 9| < \epsilon, \text{ i.e., } |h^2 + 6h| < \epsilon$$

or

$$-\epsilon < h^2 + 6h < \epsilon. \tag{1}$$

For the right-hand inequality to be true, $h^2 + 6h - \epsilon < 0$, and therefore by the theory of quadratic expressions, h must lie between the two roots of the quadratic $h^2 + 6h - \epsilon = 0$.* Hence

$$-3 - \sqrt{(9+\epsilon)} < h < -3 + \sqrt{(9+\epsilon)} \tag{2}$$

For the left-hand inequality (1) to be true, $h^2 + 6h + \epsilon > 0$, and therefore h must be greater than the greater of the two roots of $h^2 + 6h + \epsilon = 0$, the roots being real since ϵ is small. Hence, we must have

$$h > -3 + \sqrt{(9-\epsilon)}. \tag{3}$$

Combining (2) and (3), we must have

$$-3 + \sqrt{(9-\epsilon)} < h < -3 + \sqrt{(9+\epsilon)}$$

or,

$$-3 + \sqrt{(9-\epsilon)} < x - 3 < -3 + \sqrt{(9+\epsilon)}$$

i.e.,

$$-\{3 - \sqrt{(9-\epsilon)}\} < x - 3 < -3 + \sqrt{(9+\epsilon)} \tag{4}$$

If we take δ to be the smaller of $3 - \sqrt{(9-\epsilon)}$ and $-3 + \sqrt{(9+\epsilon)}$, then (4) is satisfied, if

$$-\delta < x - 3 < \delta, \quad \text{i.e.,} \quad |x - 3| < \delta,$$

Hence, we have

$$|f(x) - 9| < \epsilon \text{ for } |x - 3| < \delta.$$

Hence $f(x) \to 9$ as $x \to 3$.

We have purposely carried out this investigation for finding the value of $\delta = \delta(\epsilon)$, corresponding to any given ϵ, to show that this investigation is not simple even for very simple problems like the present one. In practice, we investigate the behaviour of a few standard functions and discuss that of more complicated ones by the application of theorems about the limit of a sum, etc. to be given below.

Ex. 3. Let $f(x) = (x^2 - 4)/(x - 2)$ and $x \to 2$. This function is defined for all real values of x except $x = 2$ for which $f(x)$ assumes the meaningless form 0/0.

As remarked earlier, in the limit $x \to 2$ we are not concerned with the value $x = 2$. For other values of x, however close to 2 but not equal to 2, $x - 2$ is not zero and we can divide out by $x - 2$ getting

$$f(x) = x + 2 \quad \text{for} \quad x \neq 2.$$

Now as $x \to 2$, it is clear that $f(x) \to 4$.

In the present example Lt $f(x) = 4$ when $x \to 2$ but $f(x)$ is not defined at $x = 2$.

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*The expression $ax^2 + bx + c$ has always the same sign as a except when the roots of $ax^2 + bx + c = 0$ are real and x lies between the roots. In the present case $a(-1)$ is positive.

The last example is typical of the limits that have to be calculated in the Differential Calculus. Here x^2-4 is the increase in the value of x^2 and $x-2$ the increase in the value of x as x changes from the value 2 to x. Thus $(x^2-4)/(x-2)$ is the average rate of increase of the function $y=x^2$ in the interval $[2, x]$. When this interval becomes smaller and smaller, *i.e.*, when $x \to 2$, the average rate of increase tends to the limit 4. This limit can be taken as a suitable definition of the **rate of change** or **rate of increase** of $y=x^2$ at the point $x=2$.

Instead of the value $x=2$, we can take any general value $x=a$ and then

$$\operatorname*{Lt}_{x \to a} \frac{x^2-a^2}{x-a},$$

which can be shown to be equal to $2a$, will give us the rate of increase of the function $y=x^2$ at $x=a$. For the function $y=x^2$, we can substitute any other function $y=f(x)$ and get the ratio $\{f(x)-f(a)\}/(x-a)$ as the average rate of increase of $f(\ )$ in the interval $[a, x]$. *The limit of this ratio as $x \to a$ will then give us the* **rate of change of f(x) at x=a.** The rate of change is called the **derivative** or **differential coefficient** and we shall discuss the derivative and the methods of its evaluation in Ch. IV.

The last two paragraphs will have convinced the reader that the Differential Calculus is founded on the theory of limits and a thorough study of this theory is essential to the development of the Calculus.

We now give a formal definition.

Def. *A function $f(x)$ is said* **to tend to the limit l** as x *tends to a, if corresponding to any positive number ϵ, however small, we can find a positive number δ, such that*

$$|f(x)-l| < \epsilon$$

for every x such that $0<|x-a|<\delta$ or for every x in the two intervals $a-\delta<x<a$ and $a<x<a+\delta$. The value $x=a$ is left out of consideration, for in the existence or otherwise of a limit of a function when $x \to a$, we are not concerned with the value of the function at $x=a$. This value may or may not be equal to the limit l. In fact, $f(x)$ need not even be defined at $x=a$ as in the third example.

3·21. Right-handed and left-handed limits. The above definition of the limit implies that $f(x)$ approaches the same limit l irrespective of the manner in which x approaches a, whether from above or below. However, $f(x)$ may tend to a limit l when x tends to a from above or from the right only, *i.e.*, when $x \to a+0$. We then call this the **right-handed limit** of $f(x)$ at $x=a$ and denote it by

$$\operatorname*{Lt}_{x \to a+0} f(x) \quad \text{or} \quad f(a+0).$$

Similarly, $f(x)$ may tend to a limit as $x \to a-0$. We then call it the **left-handed limit** of $f(x)$ at $x=a$ and denote it by

$$\operatorname*{Lt}_{x \to a-0} f(x) \quad \text{or} \quad f(a-0).$$

It follows from the remark made at the beginning of this section that if $\underset{x\to a}{\text{Lt}}\ f(x)=l$, then both the right-handed and left-handed limits exist and each is equal to l. Conversely, if both the right-handed and the left-handed limits exist and are equal, then $\underset{x\to a}{\text{Lt}}\ f(x)$ exists and is equal to them. Hence a necessary and sufficient condition that $f(x)\to$ a unique limit as $x\to a$ is that

$$\underset{x\to a-0}{\text{Lt}}\ f(x)=\underset{x\to a+0}{\text{Lt}}\ f(x)$$

Ex. 1. *Prove that* $\underset{x\to 0}{\text{Lt}}\ x^n=0$, *where n is a positive integer.*

At first let $x\to 0$ through positive values, then since x is approaching 0, we may take $x<1$. Then

$$|x^n-0|=x^n\leqslant x<\varepsilon \text{ if } x<\delta=\varepsilon.$$

Hence, given any $\varepsilon>0$, we can find a δ (in this case it is equal to ε itself) such that $x^n<\varepsilon$ for all $x<\delta$. Hence

$$\underset{x\to 0+}{\text{Lt}}\ x^n=0.$$

If x be negative, let $x=-y$, then y is +ive, and

$$|x^n-0|=|(-1)^n y^n|=y^n\leqslant y<\varepsilon$$

for $y<\varepsilon$. Hence $|x^n|<\varepsilon$ for $-\varepsilon<x<0$. Hence,

$$\underset{x\to 0-}{\text{Lt}}\ x^n=0.$$

Since the left-handed and the right-handed limits are equal, $x^n\to$ the unique limit 0 as $x\to 0$.

Ex. 2. *Discuss the behaviour, when $x\to 3$, of the function*

$$f(x)=2x+1 \quad \textit{when } x<3$$

and

$$=3x+2 \quad \textit{when } x>3.$$

Here

$$f(3-0)=\underset{x\to 3-0}{\text{Lt}}\ (2x+1)=7,$$

while

$$f(3+0)=\underset{x\to 3+0}{\text{Lt}}\ (3x+2)=11.$$

Hence the left-handed and the right-handed limits are not equal and therefore $f(x)$ does not tend to a unique limit as $x\to 3$, *i.e.*, Lt $f(x)$ does not exist as $x\to 3$.

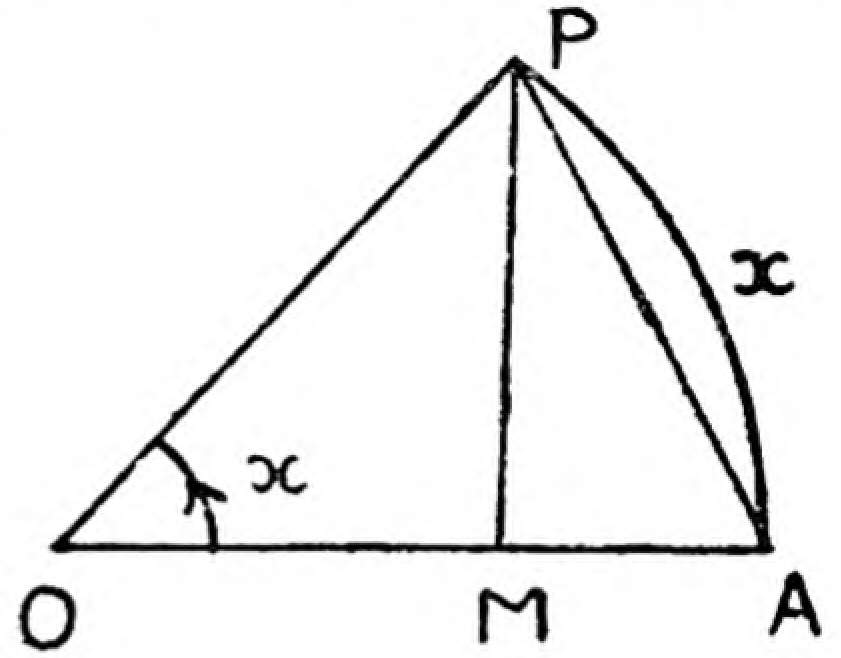
In this example $f(x)$ is not defined at $x=3$.

Ex. 3. *Prove that* $\underset{x\to 0}{\text{Lt}}\ \sin x=0$ *and* $\underset{x\to 0}{\text{Lt}}\ \cos x=1$.

(i) Let at first x be positive. Since $x\to 0$ we may take it to be an acute angle. Let O be the centre of a circle of unit radius and $\angle AOP=x$ radians. If $PM\perp OA$, then

$$\sin x=\frac{MP}{OP}=MP. \qquad [\because\ OP=1.]$$

Further, $MP <$ chord $AP <$ arc AP, and arc $AP = x$, since the radius of the circle is unity. Thus



$$\sin x = MP < \text{arc } AP = x$$

and therefore as $x \to 0$, $\sin x \to 0$. In symbols, $|\sin x - 0| = \sin x < \varepsilon$ if $x < \varepsilon$.

Hence $\underset{x \to 0+}{\text{Lt}} \sin x = 0$.

When x is negative, let $x = -y$, then y is positive, and $y \to 0+$ as $x \to 0-$. Hence

$\sin x = \sin(-y) = -\sin y \to 0$ as $y \to 0+$

and therefore $\underset{x \to 0-}{\text{Lt}} \sin x = 0$.

Since the right-handed and left-handed limits are both zero, it follows that Lt $\sin x = 0$ when $x \to 0$.

(*ii*) For $x > 0$, $\cos x = \dfrac{OM}{OP} = OM$. As $x \to 0$, arc $AP \to 0$ and therefore $M \to A$ and $OM \to OA = 1$. Hence $\cos x \to 1$ as $x \to 0$. In symbols,

$$| \cos x - 1 | = | 1 - \cos x | = | OA - OM |$$

$$= MA < \text{chord } AP < \text{arc } AP = x.$$

$$\therefore \qquad | \cos x - 1 | < \varepsilon \ \text{ if } x < \varepsilon.$$

Hence $\cos x \to 1$ as $x \to 0+$.

Next, since $\cos(-x) = \cos x$, it follows that $\cos x \to 1$ when $x \to 0-$.

Since the right-handed and left-handed limits are both equal to 1, it follows that Lt $\cos x = 1$ when $x \to 0$.

3·22. Consider next the behaviour of a function $f(x)$ which increases continually as x approaches a. If the values of $f(x)$ become and remain greater than any positive number, however large, for all values of x sufficiently close to a, we say that $f(x)$ increases beyond limit and that it tends to plus infinity. A formal definition is :

Def. 2. *A function $f(x)$ is said* **to tend to plus infinity** *or* **diverge to plus infinity,** *if corresponding to any positive number G, however large, we can find a positive number δ, such that*

$$f(x) > G$$

for all x such that $0 < | x - a | < \delta$.

In symbols, we write $f(x) \to +\infty$ as $x \to a$ or $\underset{x \to a}{\text{Lt}} f(x) = +\infty$.

On the other hand, as $x \to a$, $f(x)$ may decrease beyond all limit, *i.e.*, may grow large negatively beyond any limit, then we have the

Def. 3. *A function $f(x)$ is said* **to tend to minus infinity** *or* **diverge to minus infinity** *if, corresponding to any positive number G, however large, we can find a positive number δ, such that*

$$f(x) < -G$$

for all x such that $0 < |x-a| < \delta$.

We write $f(x) \to -\infty$ as $x \to a$ or $\underset{x \to a}{\text{Lt}}\ f(x) = -\infty$.

Finally, we have the

Def. 4. *If $f(x)$ does not tend to a unique finite limit or tend to plus or minus infinity as $x \to a$, then it is said to oscillate,* **finitely** *if $f(x)$ remains bounded in the neighbourhood of $x=a$, otherwise* **infinitely.**

Ex. 4. *Find the limit of $f(x)=1/x$ when $x \to 0$.*

Let at first $x \to 0$ through positive values. Then as x becomes smaller and smaller, $1/x$ becomes larger and larger and crosses all bounds. In fact, if G be any positive number, however large,

$$1/x > G \text{ for all } x < 1/G = \delta.$$

Hence, whatever number G is given, we can find a positive δ, such that $f(x)=1/x > G$ for $0 < x < \delta = 1/G$. Hence

$$\underset{x \to 0+}{\text{Lt}}\ 1/x = +\infty.$$

Similarly, we can show that

$$\underset{x \to 0-}{\text{Lt}}\ 1/x = -\infty.$$

Since the right-handed and the left-handed limits are different, it follows that $1/x$ does not tend to any unique limit, finite or infinite, hen $x \to 0$. As $1/x$ does not tend to any unique limit and is also not ided in any neighbourhood of zero, therefore, $1/x$ oscillates ly as $x \to 0$. $f(x)$ is not defined at $x=0$.

3. **Limit of a function as $x \to +\infty$ or $x \to -\infty$.** We next r the case when $x \to \infty$ and give the following definitions.

Def. 5. *A function $f(x)$ is said* **to tend to the limit** *l as x nds to infinity, if corresponding to any given positive number ϵ, however mall, we can find a positive number N, such that*

$$|f(x)-l| < \epsilon \textit{ for all } x > N.$$

In symbols, we write $f(x) \to l$ as $x \to +\infty$ or

$$\underset{x \to +\infty}{\text{Lt}}\ f(x) = l$$

Def. 6. *A function $f(x)$ is said* **to tend to plus infinity** *as x tends to infinity, if corresponding to any given positive number G, however large, we can find a positive number N, such that*

$$f(x) > G \textit{ for all } x > N.$$

In symbols, we write

$$\underset{x \to +\infty}{\text{Lt}}\ f(x) = +\infty \text{ or we say } f(x) \to +\infty \text{ as } x \to +\infty.$$

In a similar way we can give definitions for the cases when $f(x) \to -\infty$ and $f(x)$ oscillates when $x \to +\infty$ and also for the various cases when $x \to -\infty$.

The following can be proved quite easily :

(*i*) $\underset{x \to \infty}{\text{Lt}} \dfrac{1}{x} = 0.$ (*ii*) $\underset{x \to \infty}{\text{Lt}} \dfrac{1}{x^n} = 0$, where n is a positive integer.

3·4. Theorems on limits. As remarked earlier, most limits are evaluated by the application of certain theorems on the behaviour of the sum, difference, etc. of two functions. We state these without proof

If $f(x)$ and $g(x)$ both tend to finite limits as $x \to a$ or as $x \to +$ or $-\infty$, then

Th. I. $\text{Lt}\,\{f(x)+g(x)\} = \text{Lt}\,f(x) + \text{Lt}\,g(x).$

Th. II. $\text{Lt}\,\{f(x)-g(x)\} = \text{Lt}\,f(x) - \text{Lt}\,g(x).$

Th. III. $\text{Lt}\,\{f(x)\,.\,g(x)\} = \text{Lt}\,f(x)\,.\,\text{Lt}\,g(x).$

Th. IV. $\text{Lt}\,\dfrac{f(x)}{g(x)} = \dfrac{\text{Lt}\,f(x)}{\text{Lt}\,g(x)}$, *provided* $\text{Lt}\,g(x) \neq 0.$

One of the functions f or g may be a constant C ; we then have the following particular cases of these theorems.

$$\text{Lt}\,\{f(x) \pm C\} = \text{Lt}\,f(x) \pm C,$$
$$\text{Lt}\,\{Cf(x)\} = C\,\text{Lt}\,f(x),$$
$$\text{Lt}\,\frac{C}{f(x)} = \frac{C}{\text{Lt}\,f(x)}, \text{ provided } \text{Lt}\,f(x) \neq 0.$$

Extensions. Theorems I and III can be extended to the sum and product, respectively, of a *finite* number of functions. Thus

$$\text{Lt}\,\{f(x)+g(x)+\ldots\ldots \text{to } n \text{ terms}\}$$
$$= \text{Lt}\,f(x) + \text{Lt}\,g(x) + \ldots\ldots \text{to } n \text{ terms}$$

and $\text{Lt}\,\{f(x)\,.\,g(x)\ldots\ldots \text{to } n \text{ factors}\}$

$$= \text{Lt}\,f(x)\,.\,\text{Lt}\,g(x)\,.\,\ldots\ldots \text{to } n \text{ factors}.$$

Note on Th. IV. If $f(x) \to l \neq 0$ and $g(x) \to 0$, then $f(x)/g(x) \to +\infty$ or $-\infty$ or oscillates infinitely. If $l = 0$, then $\text{Lt}\,f(x)/\text{Lt}\,g(x)$ takes the form $0/0$ which is meaningless. However, the quotient $f(x)/g(x)$ may tend to a limit in this case. As has already been pointed out, the Differential Calculus is based on the existence of such a limit.

3·41. Further Theorems on Limits. We also state, without proof, the following theorems which are more or less intuitively obvious,

(*i*) *If $f(x)$ and $\varphi(x)$ both tend to the same limit l and $\psi(x)$ be a function such that $\psi(x)$ lies always between $f(x)$ and $\varphi(x)$, then $\psi(x)$ also tends to l.*

(*ii*) *If* $\text{Lt} f(x)=u>0$, *then*

$$\text{Lt} \{f(x)\}^{\lambda}=\{\text{Lt} f(x)\}^{\lambda}=u^{\lambda}.$$

(*iii*) *If* $\text{Lt} f(x)=u$, *then*

$$\text{Lt}\, a^{f(x)}=a^{\text{Lt} f(x)}=a^{u}.$$

(*iv*) *If* $\text{Lt} f(x)=u>0$, *then*

$$\text{Lt} \log f(x)=\log \text{Lt} f(x)=\log u.$$

3·42. **Passing to the limits.** When there is any relation between two or more functions of x which is true for all values of x in the neighbourhood of a value a and the various functions tend to definite limits as $x\to a$, then, in general, we can replace the functions by their limits in that relation. This process is called **passing to limits.**

We can also, in general, pass to the limits when $x\to\infty$ or $x\to-\infty$ under similar circumstances.

3·43. **Evaluation of limits.** The evaluation of the limit of $f(x)$ as $x\to a$ by listing the values of $f(x)$ and then observing their tendency is not only a hopelessly cumbersome procedure but also wholly impractical in most cases. Since most functions are formed from the elementary functions by the operations of addition, multiplication, division, etc., the application of the theorems on limits given in sections 3·4, 3·41 will in such cases reduce the calculation of the limits of given functions to those of the elementary functions. This makes the thorough study of the behaviour of the elementary functions of the utmost importance.

When we have to calculate Lt $f(x)$ as $x\to a$, it is frequently helpful to put $x=a+h$, thus introducing the new variable h. It is evident that as $x\to a$, $h\to 0$ and therefore

$$\underset{x\to a}{\text{Lt}}\ f(x)=\underset{h\to 0}{\text{Lt}}\ f(a+h).$$

The following examples illustrate the above techniques.

Ex. 1. *Evaluate* $\underset{x\to 4}{\text{Lt}}\ \dfrac{x^2-3x}{x^3+1}$.

$$\underset{x\to 4}{\text{Lt}}\,(x^2-3x)=\underset{x\to 4}{\text{Lt}}\ x^2-\underset{x\to 4}{\text{Lt}}\ 3x=\underset{x\to 4}{\text{Lt}}\ (x.x)-3\underset{x\to 4}{\text{Lt}}\ x$$
$$=4.4-3.4=4$$

and $\underset{x\to 4}{\text{Lt}}\,(x^3+1)=\underset{x\to 4}{\text{Lt}}\,(x.x.x)+1=4.4.4+1=65.$

$$\text{Hence}\quad \underset{x\to 4}{\text{Lt}}\ \frac{x^2-3x}{x^3+1}=\frac{\text{Lt}\,(x^2-3x)}{\text{Lt}\,(x^3+1)}=\frac{4}{65}.$$

Ex. 2. *Evaluate* $\underset{x\to a}{\text{Lt}}\ x^n$, *where n is a positive integer.*

$$\underset{x\to a}{\text{Lt}}\ x^n=\underset{x\to a}{\text{Lt}}\ (x.x.x\ldots\ldots\text{to } n \text{ factors})$$
$$=\underset{x\to a}{\text{Lt}}\ x\,.\,\underset{x\to a}{\text{Lt}}\ x\ldots\ldots\text{to } n \text{ factors.}$$
$$=a.a.a\ldots\ldots\text{to } n \text{ factors}=a^n.$$

Ex. 3. *If* $f(x)=p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\dots+p_n$, *a polynomial, show that* $f(x)\to f(a)$ *as* $x\to a$.

$$\begin{aligned}\operatorname*{Lt}_{x\to a} f(x) &= \operatorname*{Lt}_{x\to a}(p_0x^n+p_1x^{n-1}+\dots\dots+p_n)\\ &= \operatorname{Lt}(p_0x^n)+\operatorname{Lt}(p_1x^{n-1})+\dots\dots+p_n\\ &= p_0a^n+p_1a^{n-1}+\dots\dots+p_n \qquad \text{[by Ex. 2]}\\ &= f(a).\end{aligned}$$

Ex. 4. *If* $f(x)$ *and* $g(x)$ *be polynomials and* $g(a)\neq 0$, *prove that*

$$\operatorname*{Lt}_{x\to a}\frac{f(x)}{g(x)}=\frac{f(a)}{g(a)}.$$

The result follows from Ex. 3 and Theorem IV., Art 3·4.

Ex. 5. *Evaluate* $\operatorname*{Lt}_{x\to 2}\dfrac{x^3-8}{x^2-4}$.

Here Theorem IV of § 3·4 is not applicable as the numerator and denominator both tend to zero as $x\to 2$.

Put $x=2+h$, then $h\to 0$ as $x\to 2$. We observe that in the limiting process h is not to take the value zero and therefore we can divide by h. Then

$$\begin{aligned}\operatorname*{Lt}_{x\to 2}\frac{x^3-8}{x^2-4} &= \operatorname*{Lt}_{h\to 0}\frac{(2+h)^3-8}{(2+h)^2-4}=\operatorname*{Lt}_{h\to 0}\frac{12h+6h^2+h^3}{4h+h^2}\\ &= \operatorname*{Lt}_{h\to 0}\frac{12+6h+h^2}{4+h}=\frac{\operatorname{Lt}(12+6h+h^2)}{\operatorname{Lt}(h+4)}\\ &= \frac{12+6\operatorname{Lt}h+\operatorname{Lt}h^2}{4+\operatorname{Lt}h}=\frac{12+0+0}{4+0}=3.\end{aligned}$$

Ex. 6. *Find* $\operatorname*{Lt}_{x\to\infty}\dfrac{x^3+5x^2+3x+2}{2x^3+7x^2+4x-3}$.

Dividing the numerator and the denominator of the given fraction by x^3, the required limit

$$=\operatorname*{Lt}_{x\to\infty}\frac{1+\dfrac{5}{x}+\dfrac{3}{x^2}+\dfrac{2}{x^3}}{2+\dfrac{7}{x}+\dfrac{4}{x^2}-\dfrac{3}{x^3}}=\frac{1+0+0+0}{2+0+0+0}=\frac{1}{2},$$

since $\dfrac{1}{x}$, $\dfrac{1}{x^2}$, $\dfrac{1}{x^3}$ all $\to 0$ as $x\to\infty$.

Ex. 7. *Evaluate* $\operatorname*{Lt}_{x\to 0}\dfrac{\sqrt{(1+x)}-1}{x}$.

Th. IV of § 3·4 is not applicable as it would lead to the meaningless form 0/0. We proceed by rationalising the numerator

This device is of frequent use when such radicals are involved. Then

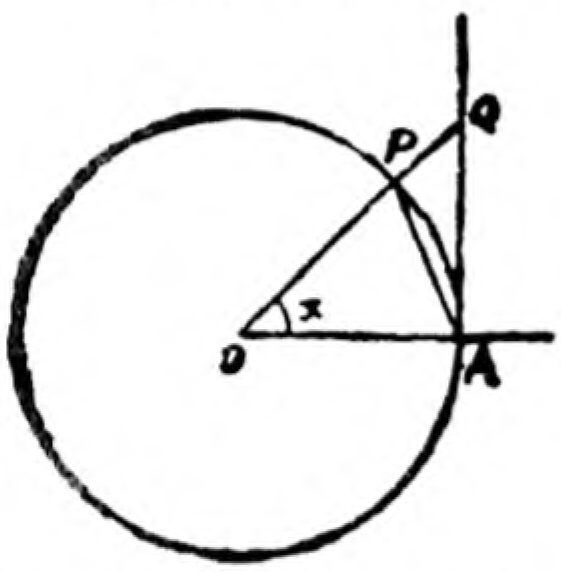
$$\underset{x\to 0}{\text{Lt}}\frac{\sqrt{(1+x)}-1}{x}=\underset{x\to 0}{\text{Lt}}\frac{\sqrt{(1+x)}-1}{x}\times\frac{\sqrt{(1+x)}+1}{\sqrt{(1+x)}+1}$$

$$=\underset{x\to 0}{\text{Lt}}\frac{1}{\sqrt{(1+x)}+1}=\underset{x\to 0}{\text{Lt}}\frac{1}{\sqrt{(1+x)}+1}=\frac{1}{1+1}=\frac{1}{2}.$$

3·5. **Some Important Limits.** We now obtain the values of some important limits.

I. *To prove that* $\underset{x\to 0}{\text{Lt}}\dfrac{\sin x}{x}=1$, *$x$ being expressed in radians.*

Let at first x be positive. Since $x\to 0$, we may assume that $0<x<\frac{1}{2}\pi$. Draw a circle of unit radius and let O be its centre. Take two points A and P on the circle such that $\angle AOP=x$ radians. Draw $AQ\perp OA$ to meet OP produced in Q. Now



area of $\triangle OAP<$area of sector OAP

$<$area of $\triangle\ OAQ$

or $\frac{1}{2}.OA.OP.\sin AOP<\frac{1}{2}OA^2.\angle AOP$

$<\frac{1}{2}OA.OA\tan AOP.$

i.e., $\qquad \frac{1}{2}\sin x<\frac{1}{2}x<\frac{1}{2}\tan x,$

i.e. $\qquad \sin x<x<\tan x,$

or $\qquad 1<\dfrac{x}{\sin x}<\dfrac{1}{\cos x},$

or $\qquad 1>\dfrac{\sin x}{x}>\cos x.$

Since $\underset{x\to 0}{\text{Lt}}\cos x=1$, $\dfrac{\sin x}{x}$ always lies between 1 and a quantity which tends to 1 as $x\to 1$. Hence $\dfrac{\sin x}{x}$ also tends to 1, and we have

$$\underset{x\to 0+}{\text{Lt}}\frac{\sin x}{x}=1.$$

Next, let x be negative and $=-y$, then

$$\underset{x\to 0-}{\text{Lt}}\frac{\sin x}{x}=\underset{y\to 0}{\text{Lt}}\frac{\sin(-y)}{-y}=\underset{y\to 0+}{\text{Lt}}\frac{\sin y}{y}=1$$

Since the right-handed and the left-handed limits are equal, therefore,

$$\underset{x\to 0}{\text{Lt}}\frac{\sin x}{x}=1.$$

Cor. 1. $\underset{x\to 0}{\text{Lt}} \dfrac{\tan x}{x}=1$, for we can write

$$\frac{\tan x}{x}=\frac{\sin x}{x}\cdot\frac{1}{\cos x}$$

and each of the factors on the right-hand side tends to 1.

Cor. 2. $\underset{x\to 0}{\text{Lt}} \dfrac{x}{\sin x}=1.$

Cor. 3. $\underset{x\to 0}{\text{Lt}} \dfrac{x}{\tan x}=1.$

Since $\dfrac{x}{\sin x}=1/\dfrac{\sin x}{x}$ and $\dfrac{x}{\tan x}=1/\dfrac{\tan x}{x}$, these two corollaries follow immediately by the application of **Th. IV** of § 3·4.

II. $\underset{x\to a}{Lt} \dfrac{x^n-a^n}{x-a}=na^{n-1}$, *where n is an integer* ($\neq 0$).

Case (*i*). *Let n be a positive integer.*

Since $x\neq a$, we can divide by $x-a$, and then

$$\begin{aligned}\underset{x\to a}{\text{Lt}}\left(\frac{x^n-a^n}{x-a}\right) &= \underset{x\to a}{\text{Lt}}\ (x^{n-1}+x^{n-2}a+x^{n-3}a^2+\ldots\ldots+a^{n-1})\\ &= \underset{x\to a}{\text{Lt}}\ x^{n-1}+\underset{x\to a}{\text{Lt}}\ x^{n-2}a+\underset{x\to a}{\text{Lt}}\ x^{n-3}a^2+\ldots\ldots+a^{n-1}.\\ &=a^{n-1}+a^{n-1}+a^{n-1}+\ldots\ldots\text{to } n \text{ terms.}\\ &=na^{n-1}.\end{aligned}$$

Case (*ii*). *Let n be a negative integer.*

Let $n=-m$, where m is a positive integer.

$$\begin{aligned}\text{Then}\quad \underset{x\to a}{\text{Lt}}\left(\frac{x^n-a^n}{x-a}\right) &= \underset{x\to a}{\text{Lt}}\left(\frac{x^{-m}-a^{-m}}{x-a}\right)\\ &= \underset{x\to a}{\text{Lt}}\left\{\frac{a^m-x^m}{x-a}\cdot\frac{1}{x^m a^m}\right\}\\ &=-\frac{1}{a^m}.\underset{x\to a}{\text{Lt}}\frac{x^m-a^m}{x-a}.\underset{x\to a}{\text{Lt}}\frac{1}{x^m}\\ &=-\frac{1}{a^m}.ma^{m-1}.\frac{1}{a^m}.\\ &=-m.a^{-m-1}=na^{n-1}.\end{aligned}$$

We assume the result when n is not an integer.

Cor. $\underset{x\to 1}{\text{Lt}} \dfrac{x^n-1}{x-1}=n.$

III. $\underset{n\to\infty}{\text{Lt}}\left(1+\dfrac{1}{n}\right)^n$ *when $n\to\infty$ through positive integral values.*

By the Binomial theorem for a positive integral index, we have

$$\left(1+\frac{1}{n}\right)^n=1+n.\frac{1}{n}+\frac{n(n-1)}{2\,!}\left(\frac{1}{n}\right)^2+\frac{n(n-1)(n-2)}{3\,!}\left(\frac{1}{n}\right)^3$$
$$+\ldots\ldots+\frac{n(n-1)\ldots\ldots2.1}{n\,!}\left(\frac{1}{n}\right)^n$$
$$=1+\frac{1}{1\,!}+\frac{1}{2\,!}\left(1-\frac{1}{n}\right)+\frac{1}{3\,!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)$$
$$+\ldots\ldots+\frac{1}{n\,!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\ldots\left(1-\frac{n-1}{n}\right)$$

As n increases, the number of terms on the right increases. Also as n increases, each of the factors $\left(1-\dfrac{1}{n}\right)$, $\left(1-\dfrac{2}{n}\right)$,...... increases so that the value of each term in the expansion on the right also increases. Hence $\left(1+\dfrac{1}{n}\right)^n$ increases with n.

Again, since each of $\left(1-\dfrac{1}{n}\right)$, $\left(1-\dfrac{2}{n}\right)$, ..., $\left(1-\dfrac{n-1}{n}\right)$ is positive and less than 1, we have

$$\left(1+\frac{1}{n}\right)^n<1+\frac{1}{1\,!}+\frac{1}{2\,!}+\frac{1}{3\,!}+\ldots\ldots+\frac{1}{n\,!}$$
$$<1+1+\frac{1}{2}+\frac{1}{2^2}+\ldots\ldots+\frac{1}{2^{n-1}}$$
$$<1+1+\frac{1}{2}+\frac{1}{2^2}+\ldots\ldots\infty$$
$$=1+\frac{1}{1-\frac{1}{2}}=3.$$

We, therefore, conclude that $\left(1+\dfrac{1}{n}\right)^n$ increases with n but always remains less than 3. It is obviously >2. Under these conditions*, $\left(1+\dfrac{1}{n}\right)^n$ tends to a limit which lies between 2 and 3.

We denote this limit by e and write

$$\underset{n\to\infty}{\text{Lt}}\left(1+\frac{1}{n}\right)^n=e.$$

Correct to five places of decimals, $e=2{\cdot}71828$.

*If a quantity continually increases but always remains less than a fixed number K, then it tends to a limit $l\leqslant K$.

Cor. 1. $\underset{x\to\infty}{\mathrm{Lt}}\left(1+\frac{1}{x}\right)^x=e$ *when* $x\to\infty$ *through the set of positive reals.*

For any positive x, we can find a positive integer n such that

$$n\leqslant x<n+1.$$

$$\therefore \qquad \frac{1}{n}\geqslant\frac{1}{x}>\frac{1}{n+1},$$

or

$$1+\frac{1}{n}\geqslant 1+\frac{1}{x}>1+\frac{1}{n+1}.$$

Hence $\left(1+\frac{1}{n}\right)^{n+1}>\left(1+\frac{1}{x}\right)^x>\left(1+\frac{1}{n+1}\right)^n.$

Let $x\to\infty$. Then n and $n+1$ both $\to\infty$ through positive integral values.

$$\therefore \quad \underset{n\to\infty}{\mathrm{Lt}}\left\{\left(1+\frac{1}{n}\right)^n\cdot\left(1+\frac{1}{n}\right)\right\}\geqslant\underset{x\to\infty}{\mathrm{Lt}}\left(1+\frac{1}{x}\right)^x$$

$$\geqslant\underset{n\to\infty}{\mathrm{Lt}}\left\{\left(1+\frac{1}{n+1}\right)^{n+1}\div\left(1+\frac{1}{n+1}\right)\right\}$$

Now $\underset{n\to\infty}{\mathrm{Lt}}\left\{\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)\right\}$

$$=\underset{n\to\infty}{\mathrm{Lt}}\left(1+\frac{1}{n}\right)^n.\underset{n\to\infty}{\mathrm{Lt}}\left(1+\frac{1}{n}\right)=e.1=e,$$

and

$$\underset{n\to\infty}{\mathrm{Lt}}\left\{\left(1+\frac{1}{n+1}\right)^{n+1}\div\left(1+\frac{1}{n+1}\right)\right\}$$

$$=\underset{n\to\infty}{\mathrm{Lt}}\left(1+\frac{1}{n+1}\right)^{n+1}\div\underset{n\to\infty}{\mathrm{Lt}}\left(1+\frac{1}{n+1}\right)=e\div 1=e.$$

$$\therefore \qquad \underset{x\to\infty}{\mathrm{Lt}}\left(1+\frac{1}{x}\right)^x=e.$$

Cor. 2. *To deduce that* $\underset{x\to-\infty}{Lt}\left(1+\frac{1}{x}\right)^x=e.$

Putting $x=-y$, we get

$$\left(1+\frac{1}{x}\right)^x=\left(1-\frac{1}{y}\right)^{-y}=\left(1+\frac{1}{y-1}\right)^y=\left(1+\frac{1}{z}\right)\left(1+\frac{1}{z}\right)^z$$

where $y-1=z$. When $x\to-\infty$, $y\to+\infty$ and $\therefore$ $z\to\infty$. Hence

$$\underset{x\to-\infty}{\mathrm{Lt}}\left(1+\frac{1}{x}\right)^x=\underset{z\to\infty}{\mathrm{Lt}}\left(1+\frac{1}{z}\right)\left(1+\frac{1}{z}\right)^z$$

$$=1\times\underset{z\to\infty}{\mathrm{Lt}}\left(1+\frac{1}{z}\right)^z=e.$$

Cor. 3. *To prove that* $\underset{z\to 0}{\text{Lt}}\ (1+z)^{1/z}=e.$

Putting $x=1/z$ in Cor. 1, we see that $\underset{z\to 0+}{\text{Lt}}\ (1+z)^{1/z}=e$, while putting $x=1/z$ in Cor. 2, we see that $\underset{z\to 0-}{\text{Lt}}\ (1+z)^{1/z}=e.$ Since both the right-handed and the left-handed limits are equal to e, the required result follows.

Note. The number e is taken as the base of the natural system of logarithms. We shall be dealing with this system in the calculus. The base e will be generally omitted so that by $\log x$ will be meant $\log_e x$.

IV. *To prove that* $\underset{x\to 0}{\text{Lt}}\ \dfrac{\log(1+x)}{x}=1.$

$$\underset{x\to 0}{\text{Lt}}\ \frac{\log(1+x)}{x}=\underset{x\to 0}{\text{Lt}}\ \log(1+x)^{1/x}=\log\left\{\underset{x\to 0}{\text{Lt}}\ (1+x)^{1/x}\right\}$$
$$=\log e=1.$$

V. *To prove that* $\underset{x\to 0}{\text{Lt}}\ \dfrac{a^x-1}{x}=\log a.$

Put $a^x-1=u$, then as $x\to 0$, $a^x\to 1$ and $\therefore$ $u\to 0$. Also $a^x=1+u$ and so $x=\log_a(1+u)$. Hence

$$\underset{x\to 0}{\text{Lt}}\ \frac{a^x-1}{x}=\underset{u\to 0}{\text{Lt}}\ \frac{u}{\log_a(1+u)}=\underset{u\to 0}{\text{Lt}}\ \frac{u\log_e a}{\log_e(1+u)}$$
$$=\frac{\log a}{\underset{u\to 0}{\text{Lt}}\ \dfrac{\log(1+u)}{u}}=\frac{\log a}{1}=\log a.$$

Cor. 1. If we take $a=e$, and write $x=h$, then we get

$$\underset{h\to 0}{\text{Lt}}\ \frac{e^h-1}{h}=1.$$

Ex. 1. *Evaluate* $\underset{x\to 0}{\text{Lt}}\ \dfrac{\sin 5x}{\sin 3x}.$

The required limit

$$=\underset{x\to 0}{\text{Lt}}\left(\frac{\sin 5x}{5x}\cdot\frac{3x}{\sin 3x}\cdot\frac{5}{3}\right)=\underset{x\to 0}{\text{Lt}}\ \frac{\sin 5x}{5x}\cdot\underset{x\to 0}{\text{Lt}}\ \frac{3x}{\sin 3x}\cdot\frac{5}{3}$$
$$=1.1.\ \frac{5}{3}=\frac{5}{3}.$$

It may be remarked that since x is not actually to take the value zero in taking limit as $x\to 0$, we can divide and multiply by x as above.

Ex. 2. *Evaluate* $\underset{x\to 0}{\mathrm{Lt}} \dfrac{\tan x - \sin x}{x^3}$.

$$\underset{x\to 0}{\mathrm{Lt}} \frac{\tan x-\sin x}{x^3} = \underset{x\to 0}{\mathrm{Lt}} \frac{\sin x}{\cos x} \cdot \frac{1-\cos x}{x^3}$$

$$= \underset{x\to 0}{\mathrm{Lt}} \frac{\sin x}{\cos x} \cdot \frac{2\sin^2 \frac{1}{2}x}{x^3} = \underset{x\to 0}{\mathrm{Lt}} \frac{1}{2} \cdot \frac{\sin x}{x} \cdot \frac{1}{\cos x} \cdot \left(\frac{\sin \frac{1}{2}x}{\frac{1}{2}x}\right)^2$$

$$= \tfrac{1}{2} \times 1 \times 1 \times (1)^2 = \tfrac{1}{2}.$$

Ex. 3. $\underset{x\to \frac{1}{2}\pi}{\mathrm{Lt}} \left(\dfrac{\cos 5x}{\cos 3x}\right) = \underset{h\to 0}{\mathrm{Lt}} \dfrac{\cos 5\left(\frac{1}{2}\pi+h\right)}{\cos 3\left(\frac{1}{2}\pi+h\right)}$ $\left[x = \dfrac{1}{2}\pi + h\right.$

$$= \underset{h\to 0}{\mathrm{Lt}} \frac{\cos\left(\frac{5}{2}\pi+5h\right)}{\cos\left(\frac{3}{2}\pi+3h\right)} = \underset{h\to 0}{\mathrm{Lt}} \left(\frac{-\sin 5h}{\sin 3h}\right)$$

$$= -\underset{h\to 0}{\mathrm{Lt}} \left(\frac{\sin 5h}{5h} \cdot \frac{3h}{\sin 3h} \cdot \frac{5}{3}\right)$$

$$= -\left(\underset{h\to 0}{\mathrm{Lt}} \frac{\sin 5h}{5h} \cdot \underset{h\to 0}{\mathrm{Lt}} \frac{3h}{\sin 3h} \cdot \frac{5}{3}\right)$$

$$= -(1.1.\tfrac{5}{3}) = -\tfrac{5}{3}.$$

Ex. 4. *Find the limit of* $(x^{16}-1)/(x^5-1)$ *as* $x\to 1$.

$$\underset{x\to 1}{\mathrm{Lt}} \frac{x^{16}-1}{x^5-1} = \underset{x\to 1}{\mathrm{Lt}} \left(\frac{x^{16}-1}{x-1} \div \frac{x^5-1}{x-1}\right)$$

$$= \underset{x\to 1}{\mathrm{Lt}} \frac{x^{16}-1}{x-1} \div \underset{x\to 1}{\mathrm{Lt}} \frac{x^5-1}{x-1}$$

$$= 16 \div 5 = \tfrac{16}{5}.$$

Ex. 5. *Find the limit of* $\dfrac{e^{\sin x}-1}{x}$ *as* $x\to 0$.

$$\underset{x\to 0}{\mathrm{Lt}} \frac{e^{\sin x}-1}{x} = \underset{x\to 0}{\mathrm{Lt}} \left(\frac{e^{\sin x}-1}{\sin x} \cdot \frac{\sin x}{x}\right) = 1 \times 1 = 1.$$

To show that $\underset{x\to 0}{\mathrm{Lt}} \dfrac{e^{\sin x}-1}{\sin x} = 1$, put $\sin x = h$, then $h\to 0$ as $x\to 0$

Hence,

$$\underset{x\to 0}{\mathrm{Lt}} \frac{e^{\sin x}-1}{\sin x} = \underset{h\to 0}{\mathrm{Lt}} \frac{e^h-1}{h} = 1$$

In general, if $f(x)\to 0$ as $x\to 0$, we have

$$\underset{x\to 0}{\mathrm{Lt}} \frac{e^{f(x)}-1}{f(x)} = 1.$$

EXAMPLES III

Evaluate the following :

1. (*i*) $\underset{x\to 2}{\mathrm{Lt}}\ (x^3+x^2-2)$ (*ii*) $\underset{x\to -1}{\mathrm{Lt}}\ (x^2+5x-6)$.

2. (*i*) $\underset{x\to 3}{\mathrm{Lt}}\ \dfrac{x^2-9}{x-3}$. (*ii*) $\underset{x\to 1}{\mathrm{Lt}}\ \dfrac{x^5-1}{x^3-1}$.

(*iii*) $\underset{x\to a}{\mathrm{Lt}}\ \dfrac{x^m-a^m}{x^n-a^n}$, m, n being positive integers.

(*iv*) $\underset{x\to 0}{\mathrm{Lt}}\ \dfrac{a+bx+cx^2+\ldots\ldots+lx^n}{a_1+b_1x+c_1x^2+\ldots\ldots+k_1x^m}$.

(*v*) $\underset{x\to 1}{\mathrm{Lt}}\ \dfrac{x^7-2x^5+1}{x^3-3x^2+2}$.

3. (*i*) $\underset{x\to \infty}{\mathrm{Lt}}\ \dfrac{x-1}{x^4}$. (*ii*) $\underset{x\to \infty}{\mathrm{Lt}}\ \dfrac{ax^n+bx^{n-1}+\ldots\ldots+k}{a_1x^m+b_1x^{m-1}+\ldots\ldots+l_1}$.

4. $\underset{x\to \infty}{\mathrm{Lt}}\ \dfrac{x}{\sqrt{(4x^2+1)}-1}$.

5. $\underset{x\to 0}{\mathrm{Lt}}\ \dfrac{1}{\sqrt{(4x^2+1)}-1-x}$. 6. $\underset{x\to 0}{\mathrm{Lt}}\ \dfrac{a^x-1}{b^x-1}$.

7. Prove that

$$\underset{x\to 0}{\mathrm{Lt}}\ \frac{\sqrt{(1+x)}-\sqrt{(1-x)}}{x}=1 \text{ and } \underset{x\to 0}{\mathrm{Lt}}\ \frac{\sqrt{(1+x+x^2)}-1}{x}=\tfrac{1}{2}.$$

8. Evaluate the following limits :

(*i*) $\underset{h\to 0}{\mathrm{Lt}}\ \dfrac{\sin(x+h)-\sin x}{h}$. (*ii*) $\underset{h\to 0}{\mathrm{Lt}}\ \dfrac{\sec(x+h)-\sec x}{h}$.

9. Find the limits of the following when $x\to 0$.

(*i*) $\dfrac{\sin 2x}{\sin 5x}$. (*ii*) $\left(\dfrac{\sin ax}{\sin bx}\right)^n$. (*iii*) $\dfrac{\sin 3x\cos 2x}{\sin 2x}$.

10. Prove that when $x\to 0$, the expressions

$$\frac{1-\cos x}{x^2},\quad \frac{1-\cos x}{\sin^2 x},\ \text{and}\ \frac{\operatorname{cosec} x-\cot x}{x}$$

all tend to the same limit $\frac{1}{2}$.

11. Prove that : (*i*) $\underset{x\to 0}{\mathrm{Lt}}\ \dfrac{\sin^{-1} x}{x}=1$. (*i*) $\underset{x\to 0}{\mathrm{Lt}}\ \dfrac{\tan^{-1} x}{x}=1$.

12. If α is measured in degrees, prove that

$$\underset{\alpha\to 0}{\mathrm{Lt}}\ \frac{\sin\alpha}{\alpha}=\frac{\pi}{180}.$$

13. Prove that $\underset{\alpha\to 0}{\mathrm{Lt}}\ \dfrac{\sin\alpha-\tan\alpha}{\sin^3\alpha}=-\frac{1}{2}$.

14. Prove that $\sin(1/x)$ does not tend to a limit as $x \to 0$.

[**Hint**. Let $x \to 0$ through the set of values $1/\pi$, $1/2\pi$, $1/3\pi$,...... and again through the set $2/\pi$, $2/5\pi$, $2/9\pi$,......].

15. Prove that $\cos(1/x)$ and $\tan(1/x)$ do not tend to any limit as $x \to 0$.

8·6. **Continuity**. While defining the limit of a function f as $x \to a$, it was pointed out that f may or may not be defined at $x=a$. From the solved examples in the preceding sections it is clear that even when f is defined at $x=a$, the value $f(a)$ need not be equal to the limit l. If f is defined at $x=a$ and $f(a)=l$, f is said to be continuous at a.

Def. *A function $f(x)$ is said to be* **continuous** *at the point a if*

$$\underset{x \to a}{\text{Lt}}\, f(x) = f(a).$$

If we write $x=a+h$, so that $h \to 0$ as $x \to a$, then this condition becomes

$$\underset{h \to 0}{\text{Lt}}\, f(a+h) = f(a).$$

The definition implies that (*i*) $f(x)$ is defined in a neighbourhood of a including the point a itself, (*ii*) $f(x)$ approaches a limit as $x \to a$, and (*iii*) this limit is equal to the value of the function at $x=a$.

A function which is not continuous at a point is said to be **discontinuous** at the point. It is evident that a function $f(x)$ can become discontinuous at a point $x=a$ for one or more of the following reasons.

(*i*) $f(x)$ is not defined at a, *i.e.*, the value a does not belong to the domain of f.

(*ii*) $f(x)$ does dot tend to a unique finite limit as $x \to a$.

(*iii*) $\underset{x \to a}{\text{Lt}}\, f(x)$ and $f(a)$ both exist but are unequal.

Since $f(x)$ has a unique limit as $x \to a$ if, and only if, the left-handed and the right-handed limits both exist and are equal, we may also say that *a function $f(x)$ is* **continuous** *at $x=a$ if, and only if,*

$$f(a-0)=f(a)=f(a+0).$$

This form of the definition is often more useful.

If only $f(a-0)=f(a)$ while either $f(a+0)$ does not exist or is not equal to $f(a)$, then we say that $f(x)$ is **continuous to the left of a**. If, on the other hand, $f(a)=f(a+0)$ while $f(a-0)$ either does not exist or is not equal to $f(a)$, then we say that $f(x)$ is **continuous to the right of a**.

A function $f(x)$ is said to be **continuous in the closed interval [a, b]** if it is continuous for every value of x lying between a and b, continuous to the right of a and to the left of b. A function is said to be **continuous in the open interval (a, b)** if it is continuous at every point of (a, b). A function which is discontinuous even

at a single point of an interval is said to be **discontinuous in the interval.**

It should be observed that if $f(x)=c$, a constant, then $f(x)$ is continuous for all values of x. For, whatever number a be,

$$\operatorname*{Lt}_{x\to a} f(x)=\operatorname*{Lt}_{x\to a} c=c=f(a),$$

since $f(x)$ has the same value c for every x.

Ex. 1. *If $f(x)=x^2+x$ prove that $f(x)$ is continuous at $x=2$.*

Here $\operatorname*{Lt}_{x\to 2} f(x)=\operatorname*{Lt}_{x\to 2} (x^2+x)=4+2=6$. Also $f(2)=2^2+2=6$.

$$\therefore \qquad \operatorname*{Lt}_{x\to 2} f(x)=f(2).$$

Hence f is continuous at 2.

Ex. 2. *If $f(x)=\dfrac{x^2-9}{x-3}$ when $x\neq 3$, and $f(3)=16$ prove that $f(x)$ is discontinuous at $x=3$.*

Here $f(3)=16$. If we write $x=3+h$, so that $h\to 0$ as $x\to 3$, then

$$\operatorname*{Lt}_{x\to 3} \frac{x^2-9}{x-3}=\operatorname*{Lt}_{h\to 0}\frac{(3+h)^2-9}{h}=\operatorname*{Lt}_{h\to 0}(6+h)=6\neq f(3).$$

Hence $f(x)$ is discontinuous at $x=3$.

Ex. 3. *If $f(x)=\dfrac{x^2-9}{x-3}$ when $x\neq 3$, and $f(3)=6$, prove that $f(x)$ is continuous at $x=3$.*

Here $f(x)=6$. Hence by Ex. 2,

$$\operatorname*{Lt}_{x\to 3} f(x) = \text{Lt } 6=f(3).$$

Hence $f(x)$ is continuous at $x=3$.

Ex. 4. *Show that* sin x *and* cos x *are continuous for all values of x.*

The function $f(x)=\sin x$ is defined for all real values of x. If $x=a$ be any such value, then as $h\to 0$,

$$\begin{aligned}\text{Lt } f(a+h) &= \text{Lt } \sin(a+h)\\ &= \text{Lt } (\sin a\cos h+\cos a\sin h)\\ &= \sin a \text{ Lt } \cos h+\cos a \text{ Lt } \sin h\\ &= \sin a\,.\,1+\cos a\,.\,0\\ &= \sin a=f(a).\end{aligned}$$

$\therefore$ $f(x)$, *i.e.*, $\sin x$ is continuous for all values of x.

The proof for $\cos x$ follows on the same lines.

Since $\tan x=(\sin x)/(\cos x)$ and $\sec x=1/(\cos x)$ it, follows that these two are continuous functions of x for all real values of x except those which make $\cos x$ vanish, *i.e.*, with the exception of the values $x=\frac{1}{2}(2n+1)\pi$. Similarly, $\cot x$ and $\operatorname{cosec} x$ are continuous for all real values of x except $x=n\pi$.

Ex. 5. *Examine $f(x)$ for continuity at the origin if $f(x)=\sin(1/x)$ when $x\neq 0$, and $f(0)=1$.*

Here $f(0)=1$. If we let $x\to 0$ through the set of values

$$\frac{1}{\pi},\ \frac{1}{2\pi},\ \frac{1}{3\pi},\ldots\ldots$$

then $f(x)=0$ for every value of this set and therefore has the limit 0. On the other hand, if $x\to 0$ through the set of values

$$\frac{2}{\pi},\ \frac{2}{5\pi},\ \frac{2}{9\pi},\ldots\ldots,$$

then $f(x)=1$ for every value of this set and therefore tends to the limit 1. Hence $f(x)$ does not tend to any unique limit. (In fact, it takes every value lying between -1 and 1 infinitely often as close to the origin as we please.) Hence $f(x)$ is not continuous at the origin.

Ex. 6. *Discuss the continuity of the function $f(x)=|x|$ at the origin.*

By definition, $f(x)=x$ if $x\geqslant 0$, and $-x$ if $x<0$.

Here $f(0)=0$ and

$$\operatorname*{Lt}_{x\to 0+} f(x)=\operatorname*{Lt}_{x\to 0+} x=0,\qquad \operatorname*{Lt}_{x\to 0-} f(x)=\operatorname*{Lt}_{x\to 0-}(-x)=0.$$

Hence $f(0)=f(0+)=f(0-)$ and therefore $f(x)$ is continuous at $x=0$.

Ex. 7. *If $f(x)=e^{\frac{1}{x-2}}$ when $x\neq 2$, and $f(2)=1$, show that f is discontinuous at $x=2$.*

We have $f(2)=1$, Also

$$\operatorname*{Lt}_{x\to 2-} f(x)=\operatorname*{Lt}_{h\to 0-} f(2+h)=\operatorname*{Lt}_{h\to 0-} e^{\frac{1}{h}}=0,$$

for as $h\to 0$, $1/h\to -\infty$ and therefore $e^{1/h}\to 0$.

Again $$\operatorname*{Lt}_{x\to 2+} f(x)=\operatorname*{Lt}_{h\to 0+} f(2+h)=\operatorname*{Lt}_{h\to 0+} e^{\frac{1}{h}}=\infty,$$

for as $h\to 0+$, $1/h\to +\infty$ and therefore $e^{1/h}\to +\infty$. Since

$$f(2-0)\neq f(2)\neq f(2+0),$$

the function is discontinuous at $x=2$.

3·61. Theorems on continuous functions. From the corresponding theorems on limits we obtain at once the following theorems on the continuity of the sum, product, etc., of a finite number of functions.

I. *The algebraic sum as well as the product of a finite number of functions which are all continuous at $x=a$ is itself continuous at $x=a$.*

II. *The quotient of two functions which are continuous at $x=a$ is continuous at $x=a$ provided the denominator does not vanish at $x=a$.*

In particular, if f is continuous at $x=a$ and $f(a)\neq 0$, then the reciprocal function g, *i.e.*, $g(x)=1/f(x)$ is also continuous at $x=a$. Also, if f is continuous in the closed interval $[a, b]$ and vanishes nowhere in $[a, b]$, then g is also continuous in the closed interval $[a, b]$.

Ex. 1. *Show that a polynomial is continuous for all values of x.*

At first, let $f(x)=x^n$, where n is a positive integer, then

$$f(a)=a^n \quad \text{and} \quad \underset{x\to a}{\text{Lt}}\ x^n=a^n,$$

whatever real value a may be. Hence x^n is continuous for every value of x. Let c be any constant, then cx^n is also continuous for all values of x.

Next, let $f(x)=a_0x^n+a_1x^{n-1}+\ldots\ldots+a_n$, a polynomial of the nth degree in x. This is the sum of $n+1$ terms of the type cx^n, each of which is continuous for all values of x. Hence $f(x)$, *i.e.*, any polynomial in x, is continuous for all values of x.

Ex. 2. *Show that every rational function of x is continuous for all values of x except those which make the denominator vanish.*

Let $f(x)=\dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials in x, then $f(x)$ is defined for all values of x except those which make $Q(x)$ vanish. Also $P(x)$ and $Q(x)$ are continuous for all real values of x, hence their quotient $f(x)$ is continuous for all real values of x except those which make the denominator $Q(x)$ vanish.

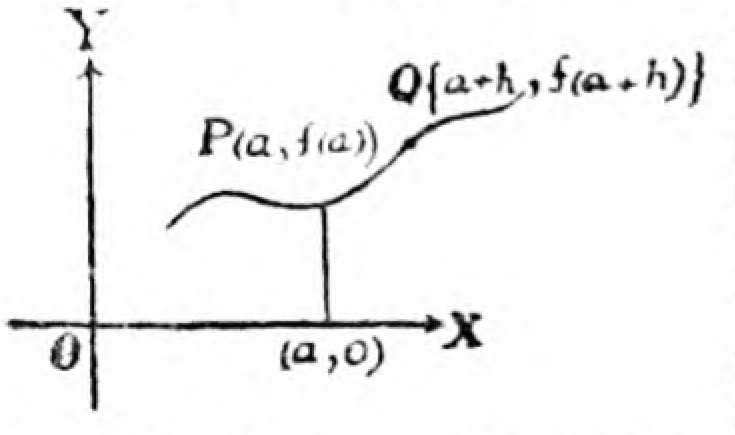
3·62. If we insert the definition of limit into that of continuity, the definition of continuity can be stated as :

A function $f(x)$ is **continuous** *at a point $x=a$ if given $\epsilon>0$, however small, there exists a positive number δ such that*

$$|f(x)-f(a)|<\epsilon \textit{ for all } x \textit{ such that } |x-a|<\delta.$$

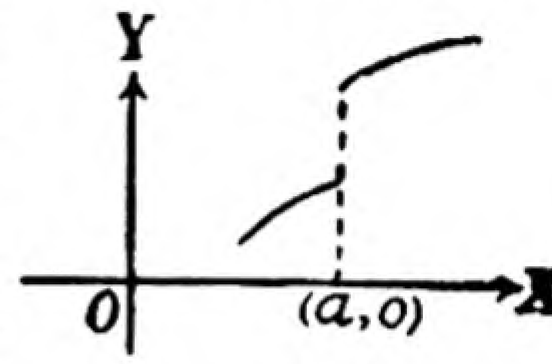
In other words, if $\epsilon>0$ is given, we can find a $\delta>0$, generally a function of ϵ, scuh that whenever x differs from a by less than δ, $f(x)$ differs from $f(a)$ by less than ϵ. This implies that the change in the value of a continuous function $f(x)$, corresponding to a small change in the value of its argument x, is small.

3·7 **Graphical meaning of continuity.** Let the function f be continuous at $x=a$. Let $P\{a, f(a)\}$ and $Q\{a+h, f(a+h)\}$ be points on the graph of the function. Since f is continuous at $x=a$, $f(x)\to f(a)$ as $x\to a$ or, in other words, $f(a+h)\to f(a)$ as $h\to 0$. This means that as $h\to 0$, the abscissa and the ordinate of the point Q tend to the abscissa and the ordinate respectively of the point P. In other words, as $h\to 0$, the point $Q\to P$ on the graph of the function. Thus the graph of the function $y=f(x)$ is without a gap or break at P.



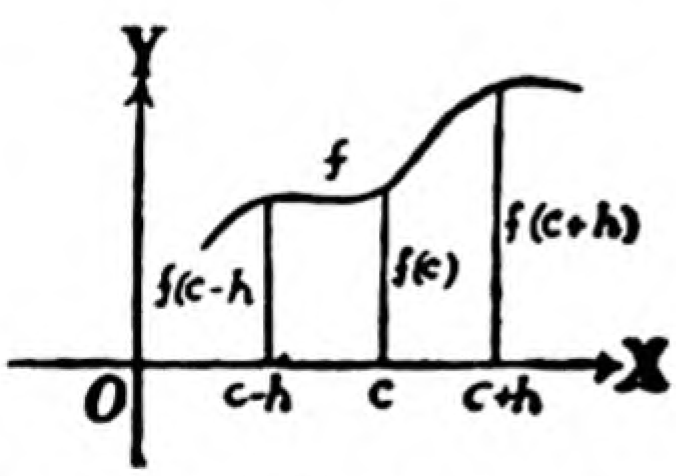
If a function is continuous throughout an interval $[a, b]$ the graph of the function in this interval is without any gaps or breaks. In rough language, if the point of a pencil is placed at one end of the graph, we can move the pencil on the graph to the other end of the graph without ever having to lift the pencil off the paper. Further, if a line is drawn across the graph, it will pass through at least one point on the graph.

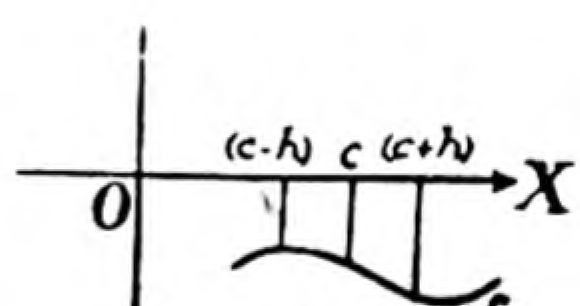
If a function is discontinuous at a point $x=a$, then there is necessarily a gap or a break in its graph at the point corresponding to $x=a$. If the function is not defined at $x=a$, then there is a gap in the graph as there is no point on the graph corresponding to $x=a$. In case the point of a pencil is moved on the graph of the function then, at the point of discontinuity, the point of the pencil will have to be lifted off the paper and will jump from one part of the curve to the other.



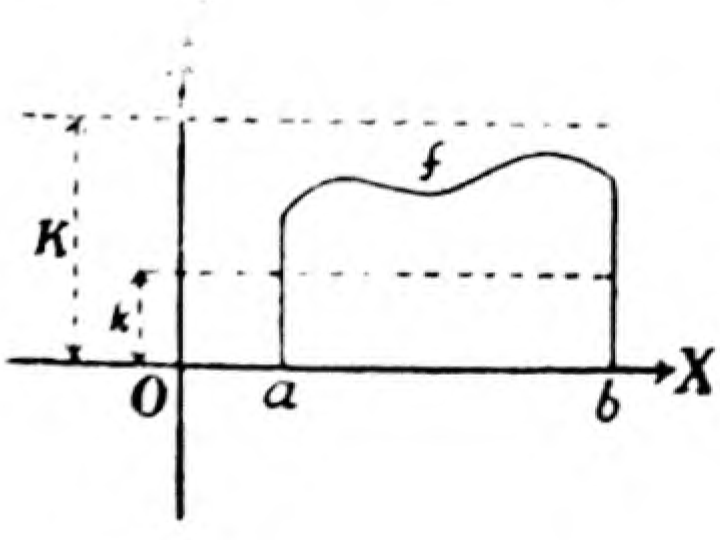
3·8. Properties of continuous functions. We now state without proof some important properties of continuous functions. The reader will be able to appreciate their truth with the help of the observations made in Sec. 3·62 and the graphical illustrations given below :

I. If $f(x)$ is continuous at $x=c$ and $f(c)>0$, then, for all sufficiently small values of h, $f(c+h)$ and $f(c-h)$ are both >0. In other words, if $f(x)$ is continuous and positive at $x=c$, then a neighbourhood of c can be found throughout which $f(x)$ is positive.





Similarly, if $f(x)$ is continuous at $x=c$ and $f(c)$ is negative, then $f(c+h)$ and $f(c-h)$ are both negative for all sufficiently small values of h.



II. If $f(x)$ is continuous in a closed interval $[a, b]$, then the range of $f(x)$ is bounded. In other words, if $f(x)$ is continuous in $[a, b]$ then we can find numbers k and K such that

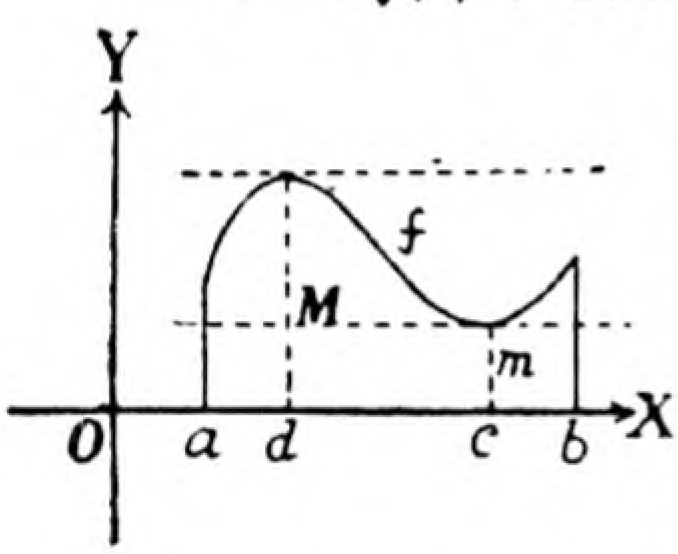
$$k<f(x)<K \text{ for all } x \in [a, b].$$

This property may not be true if the domain of $f(x)$ is not a closed interval or if $f(x)$ is discontinuous even at a single point in this domain. For example, if $f(x)=1/x$, then f is continuous in the open interval [0, 1]. Its range consists of all real numbers $\geqslant 1$ and evidently no number K can be found such that $1/x<K$ for all x in $0<x\leqslant 1$. Again, consider the function $f(x)$ defined in $[-1, 1]$ as follows : $f(x)=1/x$ when $x\neq 0, f(0)=1$. Then f is defined in $[-1, 1]$ and is continuous at every point in this interval except at $x=0$. Evidently no fixed numbers k and K can be found such that

$$k<\frac{1}{x}<K \text{ for all } x \text{ in } [-1, 1].$$

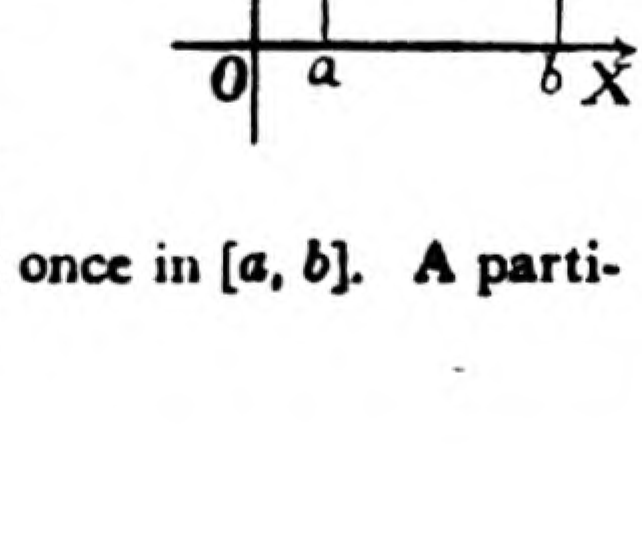
The properties of continuous functions given in III—V below are also essentially properties of functions continuous over closed intervals. These also may not be true if either the domain of $f(x)$ is not a closed interval or if $f(x)$ fails to be continuous even at a single point of the interval.

III If $f(x)$ is continuous over a closed interval $[a, b]$, then there are points in $[a, b]$ where $f(x)$ has its greatest and least values.



It should be observed that this statement asserts that not only is the range of $f(x)$ a bounded aggregate but also that this aggregate has a greatest and a smallest member. In the attached figure, M and m are the greatest and least values of $f(x)$ in the interval $[a, b]$ and c, d are points of this interval such that $f(c)=m, f(d)=M$.

IV. If $f(x)$ is continuous over the closed interval $[a, b]$ and m, M its least and greatest values over $[a, b]$ then $f(x)$ takes every value between m and M at least once in $[a, b]$. In other words, the range of $f(x)$ is the closed interval $[m, M]$.

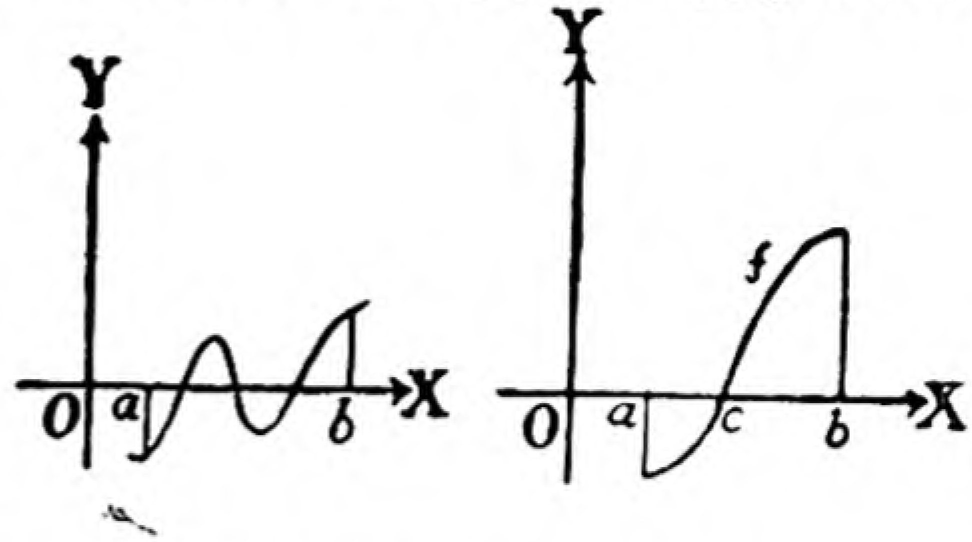


This implies that any line $y=\lambda$ parallel to the x-axis and lying between the lines $y=m$ and $y=M$ cuts the graph of $f(x)$ in at least one point.

An immediate consequence of this result is that if A is a value lying between $f(a)$ and $f(b)$, then $f(x)$ takes this value at least once in $[a, b]$. A particular case of this is the property that follows.

V. If $f(x)$ is continuous over $[a, b]$ and $f(a)$ and $f(b)$ are of opposite signs, then there is at least one point c in (a, b) such that $f(c)=0$

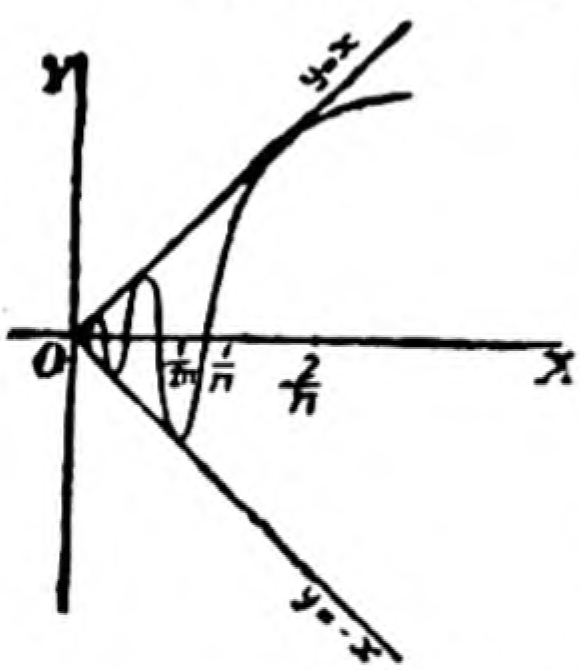
If $f(a)$ and $f(b)$ are of opposite signs, then the extremities $\{a, f(a)\}$ and $\{b, f(b)\}$ of the graph lie on opposite sides of the x-axis. The theorem asserts merely what is geometrically obvious that,



in going from one side of the x-axis to the other, a continuous graph must cut the x-axis at least once.

Ex. 1. *Let $f(x)=x \sin(1/x)$ when $x \neq 0$ and $f(0)=0$. Draw the graph of $f(x)$ and show that $f(x)$ is continuous at $x=0$.*

As $\sin(1/x)$ always lies between -1 and 1, the function $x \sin(1/x)$ always lies between $-x$ and x, and, therefore, the graph lies between the lines $y=-x$ and $y=x$. As x approaches 0, the function oscillates up and down an infinite number of times the magnitude of these oscillations tending to zero as x tends to zero. Also at $x=0$, $f(x)=0$. The part of the graph for positive values of x is shown in the attached figure. The part for negative values of x is similar and can be easily drawn. Evidently there is no break in the graph at $x=0$.



As $\underset{x \to 0+}{\mathrm{Lt}} f(x) = \underset{x \to 0-}{\mathrm{Lt}} f(x) = 0 = f(0)$, the function is continuous at $x=0$.

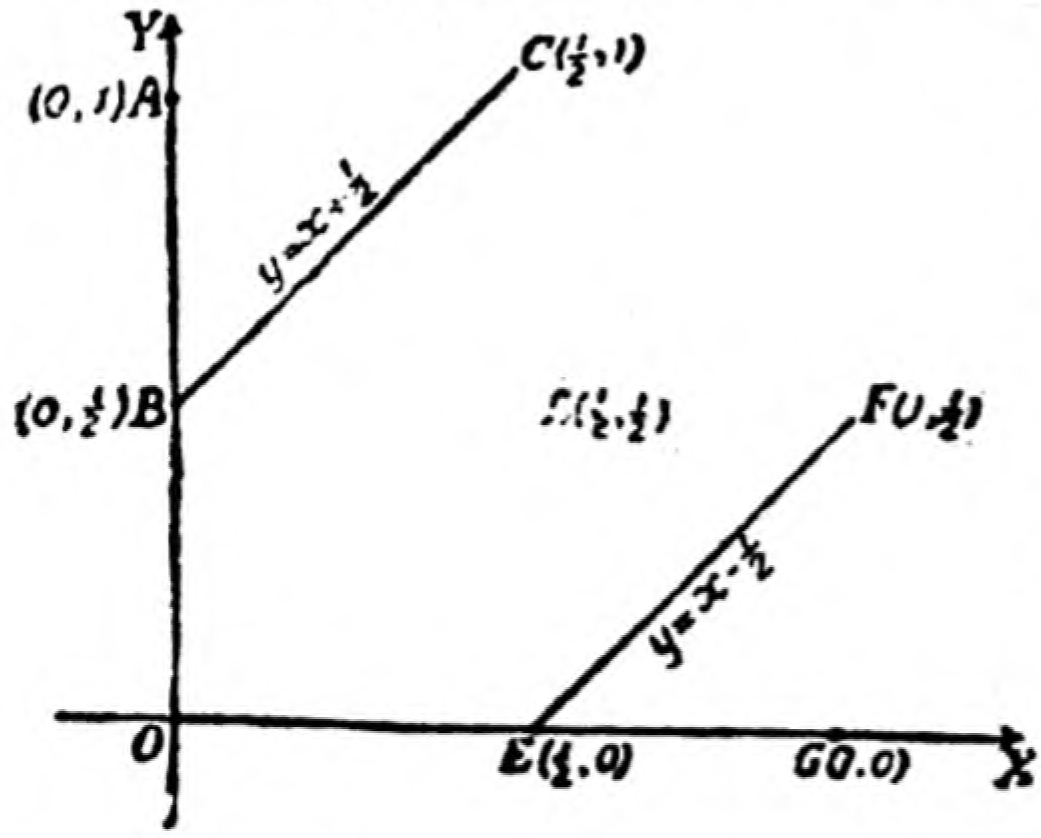
***Ex. 2.** *A function $f(x)$ is defined as follows : $f(0)=1$, $f(x)=x+\frac{1}{2}$ for $0<x<\frac{1}{2}$, $f(\frac{1}{2})=\frac{1}{2}$, $f(x)=x-\frac{1}{2}$ for $\frac{1}{2}<x<1$ and $f(1)=0$. Show that $f(x)$ is discontinuous at $x=0$, $\frac{1}{2}$ and 1 in the interval $[0, 1]$ and yet $f(x)$ assumes every value between $f(0)$ and $f(1)$ at least once in the interval $[0, 1]$.*

Here $f(0)=1$ and $\underset{x \to 0+}{\mathrm{Lt}} f(x) = \underset{x \to 0+}{\mathrm{Lt}} (x+\frac{1}{2}) = \frac{1}{2}$ and these two are unequal. Hence the function is discontinuous at $x=0$.

At $x=\frac{1}{2}$, $f(\frac{1}{2})=\frac{1}{2}$, $\underset{x \to \frac{1}{2}-0}{\mathrm{Lt}} f(x) = \underset{x \to \frac{1}{2}-0}{\mathrm{Lt}} (x+\frac{1}{2}) = 1$ and $\underset{x \to \frac{1}{2}+0}{\mathrm{Lt}} f(x) = \underset{x \to \frac{1}{2}+0}{\mathrm{Lt}} (x-\frac{1}{2}) = 0$. All these three are unequal ; therefore the function is discontinuous at $x=\frac{1}{2}$.

At $x=1$, $f(1)=0$, $\underset{x \to 1-0}{\mathrm{Lt}} f(x) = \underset{x \to 1-0}{\mathrm{Lt}} (x-\frac{1}{2}) = \frac{1}{2}$. Hence there is a discontinuity at $x=1$ also.

If the graph of the function be drawn, it consists of the three isolated points $A(0, 1)$, $D(\frac{1}{2}, \frac{1}{2})$ and $G(1, 0)$ and portions of the two straight lines BC and EF with the exception of their extremities B, C, E, and F. The graph is discontinuous at the points $x=0$, $\frac{1}{2}$ and 1. This is a general property of discontinuous functions. At a point of discontinuity of a function there is a break in the graph of the function.



From the graph it is obvious that $f(x)$ assumes every value between $f(0)$ and $f(1)$, *i.e.*, between 1 and 0 at least once in the interval $[0, 1]$, in fact, only once.

***Ex. 3.** *Show that* $f(x)=\dfrac{e^{1/x}-1}{e^{1/x}+1}$ *is discontinuous at* $x=0$.

(*Panjab, 1943*)

When x tends to zero through positive values, *i.e.*, $x\to 0+$, then $1/x\to+\infty$ and therefore $e^{1/x}\to\infty$ and $e^{-1/x}\to 0$. Again, when $x\to 0-$, $1/x\to-\infty$ and therefore $e^{1/x}\to 0$. Hence

$$\underset{x\to 0+}{\text{Lt}}\ \frac{e^{1/x}-1}{e^{1/x}+1}=\underset{x\to 0+}{\text{Lt}}\ \frac{1-e^{-1/x}}{1+e^{-1/x}}=\frac{1}{1}=1$$

and
$$\underset{x\to 0-}{\text{Lt}}\ \frac{e^{1/x}-1}{e^{1/x}+1}=\frac{-1}{1}=-1.$$

Also $f(x)$ is not defined for $x=0$. Hence $f(x)$ is discontinuous at $x=0$.

It may be added that even if the function be defined at $x=0$ by assigning to it any value, it could not be made continuous at $x=0$ for the simple reason that the limit from the right of $x=0$ is not equal to the limit from the left.

EXAMPLES IV

1. Show that x^2+3x+5 is continuous for all values of x.

2. Show that $1/x$ is continuous for all real values of x except $x=0$.

3. Examine, whether or not, $e^{-1/x}$ is continuous at $x=0$.

(*Calcutta, 1951*)

4. Show that the function f where

$f(x)=x^2$ when $x\neq 1$, and $=2$ when $x=1$,

is discontinuous at $x=1$. (*Panjab, 1956*)

5. If $f(x)=0$ when $x\leqslant 0$, $f(x)=1$ when $0<x\leqslant 1$, and $f(x)=2$ when $x>1$, show that f is discontinuous at two points

6. A function f is defined thus : $f(x)=x$ when $0\leqslant x<1$, $f(x)=2$ when $x=1$ and $f(x)=x+1$ when $1<x\leqslant 2$. Show that it is discontinuous at $x=1$.

7. Let $f(x)=x \sin\dfrac{1}{x}$ when $x\neq 0$ and $f(0)=0$; show that f is continuous at 0.

8. Show that the following functions are discontinuous at the origin :

(*i*) $f(x)=\cos(1/x)$ when $x\neq 0$ and $f(0)=0$.

(*ii*) $f(x)=x\sin(1/x)$ when $x\neq 0$ and $f(0)=2$.

(*iii*) $f(x)=1/(1+e^{1/x})$.

9. Examine the following functions for continuity at $x=2$;

(*i*) $f(x)=(x^2-4)/(x-2)$.

(*ii*) $f(x)=(x^2-4)/(x-2)$ for $x\neq 2$,
$=0$ for $x=2$.

(*iii*) $f(x)=(x^2-4)/(x-2)$ for $x\neq 2$,
$=4$ for $x=2$.

(*iv*) $f(x)=e^{1/(x-2)}$ for $x\neq 2$ and $=0$ for $x=2$.

(*v*) $f(x)=e^{-1/(x-2)^2}$ for $x\neq 2$ and $=0$ for $x=2$.

10. Find the points of discontinuity of the following functions :

(*i*) $\dfrac{x}{(x+1)(x-2)}$. (*ii*) $\dfrac{x^2}{x^2-9}$.

(*iii*) $\dfrac{x^2+x}{x^3-x}$. (*iv*) $\dfrac{\sin x}{1-2\cos x}$.

CHAPTER IV

THE DERIVATIVE

4·1. The derivative. Let $f(x)$ be a function defined over an interval I, open or closed. Let x be a given point in I. As x changes from x to $x+h$, another point in I, f changes from $f(x)$ to $f(x+h)$. Thus corresponding to a change h in the argument, the change in $f(x)$ is $f(x+h)-f(x)$. Form the *difference quotient*

$$\frac{f(x+h)-f(x)}{h}.$$

The limit of this quotient as $h \to 0$, provided it exists, is called the **derivative** of $f(x)$ at x and is denoted by $f'(x)$.

Def. 1. *The derivative of $f(x)$ at x is defined as*

$$\operatorname*{Lt}_{h\to 0} \frac{f(x+h)-f(x)}{h},$$

provided the limit exists, and is denoted by $f'(x)$.

Note that h may tend to zero through positive or negative values. If the limit exists when $h \to 0$ through positive values only, then it is called the **right-handed derivative,** while if the limit exists when $h \to 0$ through negative values only, then it is called the **left-handed derivative.** Evidently, $f(x)$ *has a unique derivative at x if, and only if, both the left-handed and the right-handed derivatives exist and are equal.*

Def. 2. $f(x)$ *is said to be* **derivable** *or* **differentiable** *at x if $f'(x)$ exists.*

The process of finding $f'(x)$ is called **differentiation.**

If we set $y=f(x)$ and denote the change in the argument x by δx instead of h, we may denote the corresponding change in $f(x)$, *i.e.* y, by δy. Then the *incremental ratio*, as the difference quotient is sometimes called, takes the form $\dfrac{\delta y}{\delta x}$. The limiting value of this quotient when $\delta x \to 0$ is then denoted by the symbol $\dfrac{dy}{dx}$ instead of $f'(x)$.

Other symbols which are generally used to denote the derivative are

$$y',\ y_1,\ D_y,\ \frac{df}{dx},\ \frac{d}{dx}[f(x)],\ D_x f, \text{ etc.}$$

Ex. 1. *Find the derivative of f when $f(x)=c$, a constant.*

Since $f(x)=c$, a constant, therefore $f(x+h)=c$, and so

$$\operatorname*{Lt}_{h\to 0} \frac{f(x+h)-f(x)}{h} = \operatorname*{Lt}_{h\to 0} \frac{c-c}{h} = \operatorname*{Lt}_{h\to 0} \left(\frac{0}{h}\right) = 0.$$

Hence *the derivative of a constant is equal to zero.*

Ex. 2. *If $f(x)=x$, find $f'(x)$.*

Since $f(x)=x$, $f(x+h)=x+h$ and therefore

$$f'(x)=\underset{h\to 0}{\text{Lt}}\frac{x+h-x}{h}=\underset{h\to 0}{\text{Lt}}\left(\frac{h}{h}\right)=1.$$

Hence *the derivative of x is equal to* 1.

Ex. 3. *Differentiate x^5 from definition.*

Let $f(x)=x^5$; then $f(x+h)=(x+h)^5$ and so

$$f'(x)=\underset{h\to 0}{\text{Lt}}\frac{(x+h)^5-x^5}{h}=\underset{h\to 0}{\text{Lt}}(5x^4+10x^3h+10x^2h^2+5h^3x+h^4)$$

$=5x^4$, since the limit of every other term is zero.

Hence the derivative of x^5 is $5x^4$.

Ex. 4. *Differentiate* $\sin x$ *from definition.*

Let $f(x)=\sin x$. Then

$$f'(x)=\underset{h\to 0}{\text{Lt}}\frac{f(x+h)-f(x)}{h}$$

$$=\underset{h\to 0}{\text{Lt}}\frac{\sin(x+h)-\sin x}{h}$$

$$=\underset{h\to 0}{\text{Lt}}\frac{2\sin\frac{1}{2}h\cos\frac{1}{2}(2x+h)}{h}$$

$$=\underset{h\to 0}{\text{Lt}}\frac{\sin\frac{1}{2}h}{\frac{1}{2}h}\ \underset{h\to 0}{\text{Lt}}\cos\tfrac{1}{2}(2x+h)$$

$$=1.\cos x=\cos x.$$

$$\therefore\quad \frac{\mathbf{d}}{\mathbf{dx}}(\mathbf{\sin x})=\mathbf{\cos x}.$$

Ex. 5. *If $f(x)=\sqrt{(x^2+a^2)}$, find $f'(x)$.*

Since $f(x)=\sqrt{(x^2+a^2)}$, $f(x+h)=\sqrt{[(x+h)^2+a^2]}$.

$$\therefore\quad f'(x)=\underset{h\to 0}{\text{Lt}}\frac{f(x+h)-f(x)}{h}$$

$$=\underset{h\to 0}{\text{Lt}}\frac{\sqrt{[(x+h)^2+a^2]}-\sqrt{(x^2+a^2)}}{h}$$

$$=\underset{h\to 0}{\text{Lt}}\frac{[(x+h)^2+a^2]-(x^2+a^2)}{\{\sqrt{[(x+h)^2+a^2]}+\sqrt{(x^2+a^2)}\}h}$$

$$=\underset{h\to 0}{\text{Lt}}\frac{(2x+h)}{\sqrt{[(x+h)^2+a^2]}+\sqrt{(x^2+a^2)}}$$

$$=\frac{2x+0}{\sqrt{(x^2+a^2)}+\sqrt{(x^2+a^2)}}$$

$$=\frac{x}{\sqrt{(x^2+a^2)}}.$$

The process may **alternatively** be exhibited as under :

Let $y=\sqrt{(x^2+a^2)}$ Let δx be the change in x and δy the corresponding change in y.

$\therefore \quad y+\delta y=\sqrt{[(x+\delta x)^2+a^2]}.$

By subtraction, $\delta y=\sqrt{[(x+\delta x)^2+a^2]}-\sqrt{(x^2+a^2)}$

$$=\frac{[(x+\delta x)^2+a^2]-(x^2+a^2)}{\sqrt{[(x+\delta x)^2+a^2]}+\sqrt{(x^2+a^2)}}$$

$$=\frac{2x\,\delta x+(\delta x)^2}{\sqrt{[(x+\delta x)^2+a^2]}+\sqrt{(x^2+a^2)}}$$

$$\therefore \quad \frac{\delta y}{\delta x}=\frac{2x+\delta x}{\sqrt{[(x+\delta x)^2+a^2]}+\sqrt{(x^2+a^2)}}$$

Proceeding to the limit as $\delta x \to 0$, we get

$$\frac{dy}{dx}=\frac{2x+0}{\sqrt{(x^2+a^2)}+\sqrt{(x^2+a^2)}}$$

$$=\frac{x}{\sqrt{(x^2+a^2)}}$$

The process is essentially the same as given earlier, only it spreads out the various steps and is likely to be a little lengthier. A serious defect about it is that the notation is quite cumbersome if adopted for the calculation of the derivative at a particular point of the domain of $f(x)$.

Ex. 6. *Show that the function $f(x)$ where*

$$f(x)=x \sin 1/x, \qquad x\neq 0.$$
$$=0, \qquad x=0.$$

is continuous at $x=0$ but has no derivative for $x=0$. (*Panjab, 1942*)

Since $\sin (1/x)\leqslant 1$ for all x, therefore

$$|x \sin (1/x)| \leqslant |x| \text{ for all } x.$$

Hence $\underset{x\to 0}{\text{Lt}}\, f(x)=\underset{x\to 0}{\text{Lt}}\, x \sin (1/x)=0.$

Also $f(0)=0$. Hence $f(x)$ is continuous at $x=0$. Next.

$$\underset{h\to 0}{\text{Lt}}\, \frac{f(0+h)-f(0)}{h}=\underset{h\to 0}{\text{Lt}}\, \frac{h \sin (1/h)-0}{h}=\underset{h\to 0}{\text{Lt}}\, \sin (1/h)$$

which does not exist for $\sin (1/h)$ oscillates between -1 and 1 infinitely many times when $h\to 0$. Hence $f'(0)$ does not exist.

Ex. 7. *If $f(x)=|x|$, show that f is continuous at 0 and that $f'(0)$ does not exist.*

We have proved in Ex. 6, § 3 6, that f is continuous at $x=0$. Now

$$f'(0)=\underset{h\to 0}{\text{Lt}}\, \frac{f(0+h)-f(0)}{h}=\underset{h\to 0}{\text{Lt}}\, \frac{f(h)-f(0)}{h}$$

provided this limit exists.

But $\underset{h\to 0+}{\text{Lt}} \dfrac{f(h)-f(0)}{h} = \underset{h\to 0+}{\text{Lt}} \dfrac{h-0}{h}=1$

and $\underset{h\to 0-}{\text{Lt}} \dfrac{f(h)-f(0)}{h} = \underset{h\to 0-}{\text{Lt}} \dfrac{(-h)-0}{h}=-1.$

Hence $\underset{h\to 0}{\text{Lt}} \dfrac{f(h)-f(0)}{h}$ does not exist, *i.e.*, $f'(0)$ does not exist.

4·11. Examples 6 and 7 of the previous section bring out an **important fact,** *viz.*,

A function which is continuous at a point is not necessarily derivable at the point. In other words, continuity does not imply derivability.

We, however, assert that derivability at a point does imply continuity at the point.

Theorem. *If $f(x)$ is derivable at a, then it is continuous at a.*

Since $f(x)$ is derivable at a, $f'(a)$ exists as a finite quantity, *i.e.*,

$$\frac{f(a+h)-f(a)}{h} \to \text{a finite limit as } h\to 0.$$

If $f(a+h)-f(a)$ were not to tend to zero with h, $\dfrac{f(a+h)-f(a)}{h}$ would not tend to a finite limit as $h\to 0$. Hence in order that a finite derivative should exist, it is necessary that

$$\underset{h\to 0}{\text{Lt}} \{f(a+h)-f(a)\}=0 \quad i.e., \quad \underset{h\to 0}{\text{Lt}} f(a+h)=f(a).$$

Hence $f(x)$ is continuous at $x=a$.

Aliter. We have $f(a+h)-f(a)=\dfrac{f(a+h)-f(a)}{h}.h$

$$\therefore \quad \underset{h\to 0}{\text{Lt}} \{f(a+h)-f(a)\}= \underset{h\to 0}{\text{Lt}} \left\{\frac{f(a+h)-f(a)}{h}.h\right\}$$

$$i.e., \quad \underset{h\to 0}{\text{Lt}} \{f(a+h)-f(a)\}= \underset{h\to 0}{\text{Lt}} \frac{f(a+h)-f(a)}{h}. \underset{h\to 0}{\text{Lt}}\ h$$

$$=f'(a).0=0.$$

$$\therefore \quad \underset{h\to 0}{\text{Lt}} f'(a+h)=f(a).$$

Hence $f(x)$ is continuous at $x=a$.

Cor. *A function cannot possess a finite derivative at a point at which it is discontinuous.*

4·12. **Derivative as rate of change**. The difference $f(x+h)-f(x)$ is the total change in f over the interval $[x, x+h]$ whose length is h. Hence the difference quotient $\{f(x+h)-f(x)\}/h$ is equal to the average rate of change of f over $[x, x+h]$. As h decreases, this gives us the average rate of change of $f(x)$ in smaller and smaller

intervals measured from the point x. If $f'(x)$ exists, then the average rate of change of $f(x)$ in $[x, x+h]$ tends to $f'(x)$ as $h \to 0$, and therefore, we may define **the rate of change of $f(x)$ at x as $f'(x)$.**

The most important application of derivatives as rates of change is in Mechanics. If a particle moves in a straight line and $s = f(t)$ gives the distance of the particle at time t from a fixed origin in the line, then $\frac{ds}{dt}$ or $f'(t)$ gives the rate of change of the distance, *i.e.*, the velocity of the particle at time t. Further, if $v = g(t)$ gives the velocity at time t, then $g'(t)$ gives the rate of change of velocity, *i.e.*, the acceleration at time t.

We will consider the geometrical interpretation of the derivative and the geometrical and other applications of the derivative in Part II.

EXAMPLES V

1. Define (*i*) difference quotient (*ii*) derivative of a function f.
2. Calculate the difference quotient when $f(x)$ equals

   (*i*) x^3 (*ii*) $(2x+3)^2$ (*iii*) $1/(x+1)$
   (*iv*) x^2+1/x (*v*) $\sqrt{x}$ (*vi*) $1/\sqrt{x}$.
3. Find from definition the derivative of :

   (*i*) x^3 (*ii*) x^4 (*iii*) x^2+3x+2.
   (*iv*) ax^2+bx+c (*v*) $x^2(1+x)$ (*vi*) x^5+1.
4. Differentiate *ab initio*.

   (*i*) $\dfrac{1}{x}$ (*ii*) $\dfrac{1}{x^2}$ (*iii*) $\dfrac{1}{x^2+a^2}$.

   (*iv*) $\dfrac{x+1}{x^2+2}$ (*v*) $\dfrac{x^2+1}{x-1}$ (*vi*) $\dfrac{x}{x+a}$.
5. Differentiate from first principles :

   (*i*) $\sqrt{x}$ (*ii*) $\dfrac{1}{\sqrt{x}}$ (*iii*) $\sqrt{(ax+b)}$

   (*iv*) $\dfrac{1}{\sqrt{(x^2+1)}}$ (*v*) $\sqrt{(1-x^2)}$ (*vi*) $\dfrac{1}{\sqrt{(px+q)}}$.

4·2. Derivative of x^n. Let $f(x) = x^n$.

Then
$$f'(x) = \operatorname*{Lt}_{h\to 0} \frac{f(x+h)-f(x)}{h}$$
$$= \operatorname*{Lt}_{h\to 0} \frac{(x+h)^n - x^n}{h}$$
$$= \operatorname*{Lt}_{h\to 0} \frac{(x+h)^n - x^n}{(x+h)-x}.$$
$$= nx^{n-1} \qquad \text{[by II, Art. 3·5]}$$

i.e.,
$$\frac{\mathbf{d}}{\mathbf{dx}}(\mathbf{x}^n) = \mathbf{n}\mathbf{x}^{n-1}.$$

4·21. Derivative of $(\mathbf{ax+b})^n$. Let $f(x)=(ax+b)^n$.

Then
$$f'(x)=\underset{h\to 0}{\text{Lt}}\ \frac{f(x+h)-f(x)}{h}$$
$$=\underset{h\to 0}{\text{Lt}}\ \frac{\{a(x+h)+b\}^n-(ax+b)^n}{h}$$
$$=\underset{h\to 0}{\text{Lt}}\ \frac{\{(ax+b)+ah\}^n-(ax+b)^n}{\{(ax+b)+ah\}-(ax+b)}\cdot a$$
$$=n(ax+b)^{n-1}.a$$
$$=na(ax+b)^{n-1}.$$

$$\therefore\qquad \frac{\mathbf{d}}{\mathbf{dx}}(\mathbf{ax+b})^n=\mathbf{na}(\mathbf{ax+b})^{n-1}.$$

Ex. 1. (*i*) $\frac{d}{dx}(x^7)=7x^{7-1}=7x^6$

(*ii*) $\frac{d}{dx}(x^{\frac{3}{2}})=\frac{3}{2}x^{\frac{3}{2}-1}=\frac{3}{2}x^{\frac{1}{2}}$

(*iii*) $\frac{d}{dx}\left(\frac{1}{x^5}\right)=\frac{d}{dx}(x^{-5})=-5x^{-5-1}=-5x^{-6}.$

Ex. 2. (*i*) $\frac{d}{dx}(2x+3)^{11}=11.2(2x+3)^{10}=22(2x+3)^{10}.$

(*ii*) $\frac{d}{dx}(5-x)^{\frac{7}{5}}=\frac{7}{5}(-1)(5-x)^{\frac{2}{5}}=-\frac{7}{5}(5-x)^{\frac{2}{5}}$.

(*iii*) $\frac{d}{dx}\left[\frac{1}{\sqrt{(2x+3)}}\right]=\frac{d}{dx}(2x+3)^{-\frac{1}{2}}$
$$=-\tfrac{1}{2}\cdot 2\cdot(2x+3)^{-\frac{3}{2}}.$$
$$=-(2x+3)^{-\frac{3}{2}}.$$

EXAMPLES VI

Write down the derivatives of :

1. (*i*) x^2 (*ii*) x^{-3} (*iii*) $\frac{1}{\sqrt{x}}$ (*iv*) $^4/x^3$.

2. (*i*) $(3x+4)^7$ (*ii*) $\sqrt{(7-2x)}$ (*iii*) $(3x+1)^{-8}$ (*iv*) $\frac{1}{5-x}$.

4·3. General rules of differentiation. The process of finding the derivative of a function by calculating the limit defining the derivative becomes very difficult and tedious in all but the simplest cases. To avoid this difficulty, we establish some general rules for the differentiation of a sum, product, etc. of two or more functions. We also obtain the derivatives of the elementary functions. Since most other functions with which we shall be con-

cerned are formed from the elementary functions, the application of the rules established below for the differentiation of a sum, product, etc. will reduce the differentiation of the given functions to those of the elementary functions.

In what follows, we shall assume $u=f(x)$, $v=g(x)$, $w=h(x)$,...... where f, g, h,.........are functions of x which possess finite derivatives for all values of x under consideration.

4·31. **Rule I.** *If* $y=cu$, *then* $\dfrac{dy}{dx}=c\dfrac{du}{dx}$ (*c constant*).

$\because$ $$y=cu,$$

we have $$y+\delta y=c(u+\delta u)$$

δu, δy being changes in u and v corresponding to a change δx in x.

By subtraction, $$\delta y=c\delta u$$

$$\therefore \quad \frac{\delta y}{\delta x}=c\frac{\delta u}{\delta x}.$$

Taking limits as $\delta x\to 0$, we get

$$\frac{dy}{dx}=c\frac{du}{dx}.$$

Ex. 1. $\dfrac{d}{dx}(5x^4)=5\dfrac{d}{dx}(x^4)=5\cdot 4x^3=20x^3.$

Ex. 2. $\dfrac{d}{dx}\left[\frac{1}{3}(2x-1)^3\right]=\frac{1}{3}\dfrac{d}{dx}(2x-1)^3=\frac{1}{3}\,.\,3\,.\,2\,.\,(2x-1)^2$

$$=2(2x-1)^2.$$

4·32. **Rule II. Derivative of a sum.** *If* $y=u+v+w+$......, *the sum of a finite number of functions, then*

$$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}+\frac{dw}{dx}+\ldots$$

Let δx be a change in x and let $\delta u, \delta v, \delta w$,......, δy be the corresponding changes in u, v, w,......, y respectively. We then have

$$y+\delta y=(u+\delta u)+(v+\delta v)+(w+\delta w)+\ldots\ldots\ldots$$

or $$\delta y=\delta u+\delta v+\delta w+\ldots\ldots$$

$$\therefore \quad \frac{\delta y}{\delta x}=\frac{\delta u}{\delta x}+\frac{\delta v}{\delta x}+\frac{\delta w}{\delta x}+\ldots$$

Proceeding to the limits when $\delta x\to 0$, we get

$$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}+\frac{dw}{dx}+\ldots \qquad \ldots(1)$$

Hence *the derivative of a sum of a finite number of functions is equal to the sum of the derivatives of the various functions.*

Ex. 1. If $y=x^2+x^5$, then

$$\frac{dy}{dx}=\frac{d}{dx}(x^2+x^5)=\frac{d}{dx}(x^2)+\frac{d}{dx}(x^5)=2x+5x^4.$$

Ex. 2. If $y=7x^3-6x+5$, then

$$\frac{dy}{dx}=\frac{d}{dx}(7x^3-6x+5)$$

$$=\frac{d}{dx}(7x^3)-\frac{d}{dx}(6x)+\frac{d}{dx}(5)$$

$$=21x^2-6.1+0=21x^2-6.$$

EXAMPLES VII

Differentiate :

1. $x^4+x^2+1.$
2. $ax^3+bx^2+cx+d.$
3. $x^{1/2}-3x+5x^{3/2}.$
4. $x^2+x+\frac{1}{x}+\frac{1}{x^2}.$
5. $\frac{2x^3+3x^2+4x+5}{x}.$
6. $\frac{1}{x^3}+\frac{1}{x^2}+\frac{2}{x}-3.$
7. $\sqrt{x}+\frac{1}{\sqrt{x}}.$
8. $\left(x+\frac{1}{x}\right)\left(x^2+\frac{1}{x^2}\right).$

4·38. Rule III. Derivative of a product. Let at first $y=uv$ be a product of two functions only, then

$$\delta y=(u+\delta u)(v+\delta v)-uv=(u+\delta u)\delta v+v\delta u,$$

and therefore $\frac{\delta y}{\delta x}=(u+\delta u)\frac{\delta v}{\delta x}+v\frac{\delta u}{\delta x}.$

Proceeding to the limits when $\delta x\to 0$, we get

$$\frac{dy}{dx}=\text{Lt}\,(u+\delta u)\times\text{Lt}\,\frac{\delta v}{\delta x}+v\,\text{Lt}\,\frac{\delta u}{\delta x}=u\frac{dv}{dx}+v\frac{du}{dx}, \qquad \dots(2)$$

i.e., the derivative of a product of two factors is equal to the first factor into the derivative of the second plus the second factor into the derivative of the first.

By repeated application of this rule, we can find the derivative of the product of any finite number of factors, thus if $y=uvw$, we get

$$\frac{dy}{dx}=vw\frac{du}{dx}+u\frac{d}{dx}(vw)$$

$$=vw\frac{du}{dx}+wu\frac{dv}{dx}+uv\frac{dw}{dx}$$

$$=uvw\left(\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}+\frac{1}{w}\frac{dw}{dx}\right).$$

and generally, if $y=uvw\ldots\ldots$ to n factors, then

$$\frac{dy}{dx}=uvw\ldots\ldots\left(\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}+\ldots\ldots\right) \qquad \dots(3)$$

Cor. If we put $u=v=w=\ldots$ in (3), we get

$$y=u^n \text{ and } \frac{dy}{dx}=u^n\left(\frac{n}{u}\frac{du}{dx}\right)=nu^{n-1}\frac{du}{dx}.$$

Ex. 1. $\dfrac{d}{dx}\left\{ (x+1)^3(2x-1)^2 \right\}$

$$=(x+1)^3 \frac{d}{dx}(2x-1)^2+(2x-1)^2 \frac{d}{dx}(x+1)^3$$

$$=(x+1)^3.2.2(2x-1)+(2x-1)^2.3(x+1)^2$$

$$=(x+1)^2(2x-1)(10x+1).$$

Ex. 2. $\dfrac{d}{dx}\left[x^2(x+1)^{\frac{1}{2}} \right]$

$$=x^2\frac{d}{dx} (x+1)^{\frac{1}{2}} +(x+1)^{\frac{1}{2}} \frac{d}{dx}(x^2)$$

$$=x^2.\tfrac{1}{2}(x+1)^{-\frac{1}{2}} + (x+1)^{\frac{1}{2}} .2x$$

$$=\frac{x^2}{2\sqrt{(x+1)}} +2x\sqrt{(x+1)}$$

$$=\frac{5x^2+4x}{2\sqrt{(x+1)}}.$$

4·84. Rule IV. Derivative of a quotient.

Let $\quad y=\dfrac{u}{v}$, then $\;y+\delta y=\dfrac{u+\delta u}{v+\delta v}$, and

$$\delta y=\frac{u+\delta u}{v+\delta v}-\frac{u}{v}=\frac{v\delta u-u\delta v}{v(v+\delta v)}.$$

$$\therefore \qquad \frac{\delta y}{\delta x}=\frac{v\dfrac{\delta u}{\delta x}-u\dfrac{\delta v}{\delta x}}{v(v+\delta v)}.$$

Proceeding to the limits when $\delta x\to 0$, we get

$$\frac{dy}{dx}=\frac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}. \qquad \ldots(4)$$

As a particular case, when $u=1$, we get

$$y=\frac{1}{v} \text{ and } \frac{dy}{dx}=-\frac{1}{v^2}\frac{dv}{dx}.$$

It is assumed here that $v\neq 0$ for the value of x under consideration.

Ex. 1. *If* $y=\dfrac{x^2+1}{x+1}$, *find* $\dfrac{dy}{dx}$.

$$\frac{dy}{dx}=\frac{(x+1)\dfrac{d}{dx}(x^2+1)-(x^2+1)\dfrac{d}{dx}(x+1)}{(x+1)^2}$$

$$=\frac{(x+1)(2x+0)-(x^2+1)(1+0)}{(x+1)^2}$$

$$=\frac{x^2+2x-1}{(x+1)^2}.$$

Ex. 2. *If* $y=\dfrac{\sqrt{(a+bx)}}{\sqrt{(a-bx)}}$, *find* $\dfrac{dy}{dx}$.

$$\frac{dy}{dx}=\frac{\sqrt{(a-bx)}.\frac{d}{dx}\sqrt{(a+bx)}-\sqrt{(a+bx)}\frac{d}{dx}\sqrt{(a-bx)}}{[\sqrt{(a-bx)}]^2}$$

$$=\frac{\sqrt{(a-bx)}.\frac{1}{2}.\dfrac{b}{\sqrt{(a+bx)}}-\sqrt{(a+bx)}.\frac{1}{2}.\dfrac{-b}{\sqrt{(a-bx)}}}{(a-bx)}$$

$$=\frac{b.(a-bx)+b(a+bx)}{2(a-bx)\sqrt{(a^2-b^2x^2)}}$$

$$=\frac{ab}{(a-bx)\sqrt{(a^2-b^2x^2)}}.$$

EXAMPLES VIII

Differentiate :

1. $(x+1)(x+2)$.
2. $(x+3)^2(x^2+3x+4)$.
3. $(x+1)(x^2+1)(x^3+1)$.
4. $(x^2+2x+3)(x^2+3x+4)$.
5. $\dfrac{px+q}{rx+s}$.
6. $\dfrac{2x}{(x+1)^7}$.
7. $\dfrac{x^2-x+1}{x^2+x+1}$.
8. $\dfrac{(x+2)(2x+1)}{x^3-1}$.
9. $\dfrac{2(x+1)}{x^2+2x-3}$.
10. $\dfrac{\sqrt{(1+x)}}{\sqrt{(1-x)}}$.

4·35. Rule V. Derivative of an inverse function. Let $y=f(x)$, where f is the inverse function of φ so that $x=\varphi(y)$, and let $\varphi(y)$ have a derivative $\varphi'(y)$ which is finite and not equal to zero ; then $f(x)$ has a derivative and $f'(x)=1/\varphi'(y)$. For

$$\frac{\delta y}{\delta x}=\frac{1}{\dfrac{\delta x}{\delta y}} \text{ and Lt } \frac{\delta x}{\delta y}=\varphi'(y)$$

which is not zero. Hence proceeding to the limits,

$$\frac{dy}{dx}=\frac{1}{\dfrac{dx}{dy}}, \quad i.e., \quad f'(x)=\frac{1}{\varphi'(y)}. \qquad ...(5)$$

Hence the derivative of y considered as a function of x is the reciprocal of the derivative of x considered as a function of y.

4·36. Function of a function. Let $y=\varphi(u)$ and $u=f(x)$ so that $y=\varphi\{f(x)\}$. We say that y is a function of a function of x. For example, let $y=u^3$ and $u=\sin x$. Then $y=\sin^3 x$. Thus y is ultimately a function of x.

4·37. **Rule VI. Derivation of a function of a function** Let $y=\varphi(u)$ and $u=f(x)$ be both derivable and have finite derivatives $\varphi'(u)$ and $f'(x)$ respectively ; then y is also derivable and we show how to find the derivative of y. Let δx be the change of x and let δu and δy be the corresponding changes in u and y Let δx tend to zero without actually taking the value zero ; then δu and δy also tend to zero and might even actually be zero.

Since $\varphi'(u)$ and $f'(x)$ are both finite, we have

$$\delta y=\{\varphi'(u)+\epsilon\}\delta u \text{ and } \delta u=\{f'(x)+\epsilon'\}\delta x^*,$$

where ϵ and ϵ' tend to zero with δx. From these equations we get

$$\delta y=\{\varphi'(u)+\epsilon\}\{f'(x)+\epsilon'\}\delta x$$

or $$\frac{\delta y}{\delta x}=\{\varphi'(u)+\epsilon\}\{f'(x)+\epsilon'\}.$$

Hence proceeding to the limits, we get

$$\frac{dy}{dx}=\varphi'(u)f'(x), \text{ i.e., } \frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}. \qquad ...(6)$$

The rule can be extended to any number of functions. Thus, if $y=f(u)$, $u=\varphi(v)$, $v=\psi(w)$ and $w=F(x)$, then we get

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dw}\cdot\frac{dw}{dx}.$$

This is called **the chain rule** for the differentiation of functions of functions.

An 'easy' proof of the chain rule is generally given on these lines :

$\because$ $\dfrac{\delta y}{\delta x}=\dfrac{\delta y}{\delta u}\cdot\dfrac{\delta u}{\delta x}$, we have, on taking limits as $\delta x\to 0$,

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}.$$

The proof is defective in that δu may be actually zero and division by zero is not permissible.

Ex. 1. *Differentiate* $\sqrt{(x^2+a^2)}$.

Let $\qquad y=\sqrt{(x^2+a^2)}$.

Set $\qquad u=x^2+a^2$; then $y=u^{\frac{1}{2}}$.

$\therefore \qquad \dfrac{du}{dx}=2x$; $\qquad \dfrac{dy}{du}=\frac{1}{2}u^{-\frac{1}{2}}$.

---

* Since $\phi'(u)=\underset{\delta u\to 0}{\text{Lt}}\dfrac{\delta y}{\delta u}$, $\dfrac{\delta y}{\delta u}$ must differ from $\phi'(u)$ by a small quantity (ϵ, say) which ultimately $\to 0$.

Hence $\dfrac{\delta y}{\delta u}=\phi'(u)+\epsilon$ or $\delta y=\{\phi'(u)+\epsilon\}\delta u$.

Similarly, $\delta u=\{f'(x)+\epsilon'\}\delta x$.

Hence $\quad \dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dx}=\frac{1}{2}u^{-\frac{1}{2}}.2x=(x^2+a^2)^{-\frac{1}{2}}.x$

$$=\frac{x}{\sqrt{(x^2+a^2)}}.$$

Ex. 2. *Differentiate* (*i*) $\sin x^3$. (*ii*) $\sin^3 x$.

(*i*) Let $\quad y=\sin x^3$.

Set $\quad u=x^3$; then $y=\sin u$.

$\therefore \quad \dfrac{du}{dx}=3x^2$; $\quad \dfrac{dy}{du}=\cos u$. $\qquad$ [Ex. 4, Art. 4·1.]

Hence $\quad \dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dx}$

$$=\cos u\ .\ 3x^2=3x^2\cos x^3.$$

(*ii*) Let $\quad y=\sin^3 x$.

Set $\quad u=\sin x$; $\quad$ then $y=u^3$.

$\therefore \quad \dfrac{du}{dx}=\cos x$; $\quad \dfrac{dy}{du}=3u^2$.

Hence $\quad \dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dx}$

$$=3u^2\ .\ \cos x=3\sin^2 x\cos x.$$

Ex. 3. *Differentiate* $\sin^n(ax^2+bx+c)$

Let $\quad y=\sin^n(ax^2+bx+c)$

Set $\quad u=ax^2+bx+c$; $v=\sin u$; then $y=v^n$.

$\therefore \quad \dfrac{du}{dx}=2ax+b$; $\quad \dfrac{dv}{du}=\cos u$; $\quad \dfrac{dy}{dv}=nv^{n-1}$

Hence $\dfrac{dy}{dx}=\dfrac{dy}{dv}\cdot\dfrac{dv}{du}\cdot\dfrac{du}{dx}$

$$=n\,v^{n-1}\ .\ \cos u\ .\ (2ax+b)$$

$$=n\sin^{n-1}(ax^2+bx+c)\cos(ax^2+bx+c)\,(2ax+b).$$

EXAMPLES IX

Find the derivatives of :

1. $(x^2+x+1)^8$. $\quad$ 2. $\left(\dfrac{x^2-1}{x^2+1}\right)^{\frac{1}{2}}$. $\quad$ 3. $[x+\sqrt{(x^2+a^2)}]^n$.

4. $x\sqrt{(x^2+a^2)}$. $\quad$ 5. $\dfrac{x}{\sqrt{(ax^2+bx+c)}}$. $\quad$ 6. $\sqrt[3]{(3x^2-x^3)}$.

7. $\sin\sqrt{x}$. $\quad$ 8. $\sqrt{(\sin x)}$. $\quad$ 9. $\sqrt{(\sin\sqrt{x})}$.

4·38. Parametric equations. Let $x=f(t)$, $y=g(t)$ be the parametric representation of a function, t being the parameter. Then

$$\frac{dx}{dt}=f'(t) \text{ and } \frac{dy}{dt}=g'(t),$$

$$\therefore \quad \frac{dy}{dx}=\underset{\delta x\to 0}{\mathrm{Lt}}\frac{\delta y}{\delta x}=\underset{\delta t\to 0}{\mathrm{Lt}}\frac{\frac{\delta y}{\delta t}}{\frac{\delta x}{\delta t}}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{g'(t)}{f'(t)},$$

provided $f'(t)$ is not equal to zero for the value under consideration.

Ex. 1. *If* $x=a\dfrac{(1-t^2)}{1+t^2}$, $y=\dfrac{2bt}{1+t^2}$, *find* $\dfrac{dy}{dx}$.

We have $\dfrac{dx}{dt}=a\,\dfrac{(1+t^2)(-2t)-(1-t^2)2t}{(1+t^2)^2}=\dfrac{-4\,at}{(1+t^2)^2}$

$$\frac{dy}{dt}=2b.\frac{(1+t^2)\,.\,1-t\,.\,2t}{(1+t^2)^2}=\frac{2b(1-t^2)}{(1+t^2)^2}$$

$$\therefore \quad \frac{dy}{dx}=\frac{dy}{dt}\bigg/\frac{dx}{dt}=\frac{2b(1-t^2)}{(1+t^2)^2}\bigg/\frac{-4at}{(1+t^2)^2}=-\frac{b(1-t^2)}{2at}.$$

Ex. 2. *Find the derivative of* $\dfrac{x^2}{1+x^2}$ *w. r. to* x^4.

Let $y=\dfrac{x^2}{1+x^2}$ and $z=x^4$, then we have to find $\dfrac{dy}{dz}$.

Now $\dfrac{dy}{dx}=\dfrac{(1+x^2)\,.\,2x-x^2\,.\,2x}{(1+x^2)^2}=\dfrac{2x}{(1+x^2)^2}$

and $\dfrac{dz}{dx}=4x^3.$

$$\therefore \quad \frac{dy}{dz}=\frac{dy}{dx}\bigg/\frac{dz}{dx}=\frac{2x}{(1+x^2)^2}\cdot\frac{1}{4x^3}$$

$$=\frac{1}{2x^2(1+x^2)^2}$$

4·89. Differentiation of implicit functions. Let $\varphi(x, y)=0$ be the equation defining y implicitly as a function of x. To find the derivative of y *w. r.* to x, we differentiate each term in $\varphi(x, y)$ *w. r.* to x using the rule for the differentiation of function of a function and other rules for differentiation and equate the sum of the derivatives of all the terms to zero. The resulting equation is then solved for $\dfrac{dy}{dx}$. For example, if y^n is a term in $\varphi(x, y)$, then

$$\frac{d}{dx}(y^n)=\frac{d}{dy}(y^n)\,.\,\frac{dy}{dx}=ny^{n-1}\frac{dy}{dx},$$

and if $\sin(xy)$ is another term, then

$$\frac{d}{dx}\{\sin(xy)\}=\cos(xy)\cdot\frac{d}{dx}(xy)$$

$$=\cos(xy)\cdot\left\{x\frac{dy}{dx}+y\cdot\frac{dx}{dx}\right\}$$

$$=\left(y+x\frac{dy}{dx}\right)\cos(xy).$$

Ex. 1. *If* $x^2+y^2=a^2$, *find* $\frac{dy}{dx}$.

Differentiating both sides *w.r.* to x, we get

$$\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}(a^2)$$

or $$2x+2y\frac{dy}{dx}=0. \qquad \therefore \quad \frac{dy}{dx}=-\frac{x}{y}.$$

Ex. 2. *If* $x^3+y^3=3axy$, *find* $\frac{dy}{dx}$.

Differentiating both sides *w.r.* to x, we get

$$\frac{d}{dx}(x^3+y^3)=\frac{d}{dx}(3\,axy)$$

or $$3x^2+3y^2\frac{dy}{dx}=3a\left(x\frac{dy}{dx}+y\cdot 1\right)$$

or $$(y^2-ax)\frac{dy}{dx}=ay-x^2.$$

$$\therefore \quad \frac{dy}{dx}=\frac{ay-x^2}{y^2-ax}.$$

EXAMPLES X

Find $\frac{dy}{dx}$ when

1. $x=at^2$, $y=2at$. 2. $x=ct$, $y=c/t$.
3. $x=a\sqrt{(1-t^2)}$, $y=b\sqrt{(1+t^2)}$.
4. $x=ap^2+2bp+c$, $y=mp+n$, p being the parameter.
5. Differentiate (*i*) x^9 *w.r.* to x^3.

   (*ii*) $\frac{x^2}{1-x^2}$ *w.r.* to x^2.

Find $\frac{dy}{dx}$ in the following cases:

6. $x^3-y^3=a^3$. 7 $x^2/a^2+y^2/b^2=1$.
8. $ax^2+2hxy+by^2=1$. 9. $x^5+y^5=5ax^2y^2$.
10. $ax^2+2hxy+by^2+2gx+2fy+c=0$.

4·4. The Circular Functions. We now find the derivatives of the circular or trigonometric functions $\sin x$, $\cos x$, etc. It may be observed that in all these the angle x is measured in radians.

I. **sin x.** Let $y=\sin x$, then

$$\frac{dy}{dx}=\underset{h\to 0}{\text{Lt}}\ \frac{\sin(x+h)-\sin x}{h}=\underset{h\to 0}{\text{Lt}}\ \frac{2\cos(x+\frac{1}{2}h)\sin\frac{1}{2}h}{h}$$

$$=\underset{h\to 0}{\text{Lt}}\ \cos(x+\tfrac{1}{2}h)\cdot\frac{\sin\frac{1}{2}h}{\frac{1}{2}h}=\cos x\times 1=\cos x$$

since by Art. 3·5, I, $\underset{x\to 0}{\text{Lt}}\ \dfrac{\sin x}{x}=1.$

Hence $\dfrac{\mathbf{d}}{\mathbf{dx}}(\mathbf{sin\ x})=\mathbf{cos\ x}.$

II. **cos x.** Let $y=\cos x$, then

$$\frac{dy}{dx}=\underset{h\to 0}{\text{Lt}}\ \frac{\cos(x+h)-\cos x}{h}=\underset{h\to 0}{\text{Lt}}\ \frac{-2\sin(x+\frac{1}{2}h)\sin\frac{1}{2}h}{h}$$

$$=-\underset{h\to 0}{\text{Lt}}\ \sin(x+\tfrac{1}{2}h)\frac{\sin\frac{1}{2}h}{\frac{1}{2}h}=-\sin x\times 1=-\sin x.$$

Hence $\dfrac{\mathbf{d}}{\mathbf{dx}}(\mathbf{cos\ x})=-\mathbf{sin\ x}.$

III. **tan x.** Let $y=\tan x$, then

$$\frac{dy}{dx}=\underset{h\to 0}{\text{Lt}}\ \frac{\tan(x+h)-\tan x}{h}=\underset{h\to 0}{\text{Lt}}\ \frac{\sin(x+h-x)}{h\cos(x+h)\cos x}$$

$$=\underset{h\to 0}{\text{Lt}}\ \frac{1}{\cos(x+h)\cos x}\cdot\frac{\sin h}{h}=\frac{1}{\cos^2 x}.$$

Hence $\dfrac{\mathbf{d}}{\mathbf{dx}}(\mathbf{tan\ x})=\dfrac{\mathbf{1}}{\mathbf{cos^2 x}}=\mathbf{sec^2 x}.$

IV. **cot x.** Let $y=\cot x$, then

$$\frac{dy}{dx}=\underset{h\to 0}{\text{Lt}}\ \frac{\cot(x+h)-\cot x}{h}=\underset{h\to 0}{\text{Lt}}\ \frac{\sin(x-x-h)}{h\sin(x+h)\sin x}$$

$$=-\underset{h\to 0}{\text{Lt}}\ \frac{1}{\sin(x+h)\sin x}\cdot\frac{\sin h}{h}=-\frac{1}{\sin^2 x}.$$

Hence $\dfrac{\mathbf{d}}{\mathbf{dx}}(\mathbf{cot\ x})=-\dfrac{\mathbf{1}}{\mathbf{sin^2 x}}=-\mathbf{cosec^2 x}.$

V. **sec x.** Let $y=\sec x$, then

$$\frac{dy}{dx}=\underset{h\to 0}{\text{Lt}}\ \frac{\sec(x+h)-\sec x}{h}=\underset{h\to 0}{\text{Lt}}\ \frac{\cos x-\cos(x+h)}{h\cos(x+h)\cos x}$$

$$=\underset{h\to 0}{\text{Lt}}\ \frac{2\sin(x+\frac{1}{2}h)\sin\frac{1}{2}h}{h\cos(x+h)\cos x}$$

$$= \underset{h \to 0}{\mathrm{Lt}} \frac{\sin (x+\frac{1}{2}h)}{\cos (x+h) \cos x} \cdot \frac{\sin \frac{1}{2}h}{\frac{1}{2}h}.$$

$$= \frac{\sin x}{\cos^2 x} = \tan x \sec x.$$

Hence $\frac{\mathbf{d}}{\mathbf{dx}}$ **(sec x)=tan x sec x.**

VI. cosec x. Let $y = \operatorname{cosec} x$, then

$$\frac{dy}{dx} = \underset{h \to 0}{\mathrm{Lt}} \frac{\operatorname{cosec} (x+h) - \operatorname{cosec} x}{h}$$

$$= \underset{h \to 0}{\mathrm{Lt}} \frac{\sin x - \sin (x+h)}{h \sin (x+h) \sin x}$$

$$= \underset{h \to 0}{\mathrm{Lt}} \frac{-\cos (x+\frac{1}{2})}{\sin (x+h) \sin x} \cdot \frac{\sin \frac{1}{2}h}{\frac{1}{2}h}$$

$$= -\frac{\cos x}{\sin^2 x} = -\cot x \operatorname{cosec} x.$$

Hence $\frac{\mathbf{d}}{\mathbf{dx}}$ **(cosec x)= −cot x cosec x.**

Since $\tan x = \frac{\sin x}{\cos x}$, $\cot x = \frac{\cos x}{\sin x}$, we can also find the derivatives of $\tan x$, $\cot x$, $\sec x$, and $\operatorname{cosec} x$ by making use of the rule for the derivation of a quotient and the results of I and II above.

Ex. 1. *Differentiate*

(*i*) $\sin 3x$, (*ii*) $\sin^2 x$, (*iii*) $\sec^2 x^3$,

(*i*) Let $y = \sin 3x$.

Set $u = 3x$ so that $y = \sin u$.

$\therefore$ $\frac{du}{dy} = 3$ and $\frac{dy}{du} = \cos u$.

$\therefore$ $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos u.3 = 3 \cos 3x$.

(*ii*) Let $y = \tan^2 x$.

Set $u = \tan x$ so that $y = u^2$.

$\therefore$ $\frac{du}{dx} = \sec^2 x$ and $\frac{dy}{dx} = 2u$.

$\therefore$ $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2u.\sec^2 x = 2 \tan x.\sec^2 x$.

(*iii*) Let $y = \sec^2 x^3$.

Set $u = x^3$, $v = \sec u$, then $y = v^2$.

$\therefore$ $\frac{du}{dx} = 3x^2$, $\frac{dv}{du} = \sec u \tan u$, $\frac{dy}{dv} = 2v$.

$$\therefore \qquad \frac{dy}{dx}=\frac{dy}{dv}\cdot\frac{dv}{du}\cdot\frac{du}{dx}.$$
$$=2v.\sec u \tan u.3x^2.$$
$$=6x^2 \sec^2 x^3 \tan x^3.$$

The process may also be exhibited as under :

Let $\qquad y=\sec^2 x^3=(\sec x^3)^2.$

$$\therefore \qquad \frac{dy}{dx}=2 \sec x^3 . \frac{d}{dx}(\sec x^3)$$
$$=2 \sec x^3 \sec x^3 \tan x^3 . \frac{d}{dx}(x^3)$$
$$=2 \sec^2 x^3.\tan x^3.3x^2.$$
$$=6x^2 \sec^2 x^3 \tan x^3.$$

Ex. 2. *If* $x=3 \cos t-2 \cos^3 t$, $\quad y=3 \sin t-2 \sin^3 t$, *find* $\dfrac{dy}{dx}$ *at* $t=\pi/4$.

We have $\quad \dfrac{dx}{dt}=-3 \sin t-2.3 \cos^2 t.(-\sin t)$
$$=3 \sin t(1-2 \cos^2 t)$$
$$=-3 \sin t \cos 2t.$$
$$\frac{dy}{dt}=3 \cos t-2.3 \sin^2 t.\cos t$$
$$=3 \cos t\,(1-2 \sin^2 t)$$
$$=3 \cos t \cos 2t.$$
$$\therefore \qquad \frac{dy}{dx}=\frac{dy}{dt} \Big/ \frac{dx}{dt}=\frac{3 \cos t \cos 2t}{-3 \sin t \cos 2t}=-\cot t$$

At $t=\pi/4$, $\dfrac{dy}{dx}=-\cot \pi/4=-1.$

EXAMPLES XI

Differentiate

1. $\sin^2 x.$ **2.** $\cos mx.$ **3.** $\sin x^0.$

4. $\cos^m x.$ **5.** $\sin x^n.$ **6.** $a \cos^2 x+b \sin^2 x.$

7. $\cos(\sin x).$ **8.** $\tan^2 x^2.$ **9.** $\dfrac{1}{\sec x-\tan x}.$

10. $\sqrt{\left(\dfrac{1-\cos x}{1+\cos x}\right)}.$ **11.** $\sqrt{\left(\dfrac{1+\sin x}{1-\sin x}\right)}.$

12. $\dfrac{1+\tan x}{1-\tan x}.$ **13.** $x^2 \sin x.$

14. $\dfrac{a \cos x+b}{a+b \cos x}.$ **15.** $\tan\left(\dfrac{2+3x}{3+2x}\right).$

16. $\sqrt{[\tan \sqrt{(1+x^2)}]}$.

Find $\dfrac{dy}{dx}$ in each of the following cases :—

17. (i) $x=a \cos \theta$, $\quad y=b \sin \theta$.
(ii) $x=a \sec t$, $\quad y=b \tan t$.

18. $x=a \cos^3 t$, $\quad y=a \sin^3 t$.

19. $x=a(\theta-\sin \theta)$, $\quad y=a(1-\cos \theta)$.

20. $x=2 \cos t-\cos 2t$, $y=2 \sin t-\sin 2t$ at $t=\pi/2$.

4·5. **Inverse trigonometric functions.** Let $y=\sin x$, $-\frac{1}{2}\pi \leqslant x \leqslant \frac{1}{2}\pi$. As x changes from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$, y changes from -1 to $+1$, each value in this interval being taken by y just once. Conversely, given any y in the range $-1 \leqslant y \leqslant 1$, we can find just one x in $-\frac{1}{2}\pi \leqslant x \leqslant \frac{1}{2}\pi$ such that $\sin x=y$. The angle x is called the **inverse sine** of y or **arc sine** of y and we write $x=\sin^{-1} y$ Thus the two statements

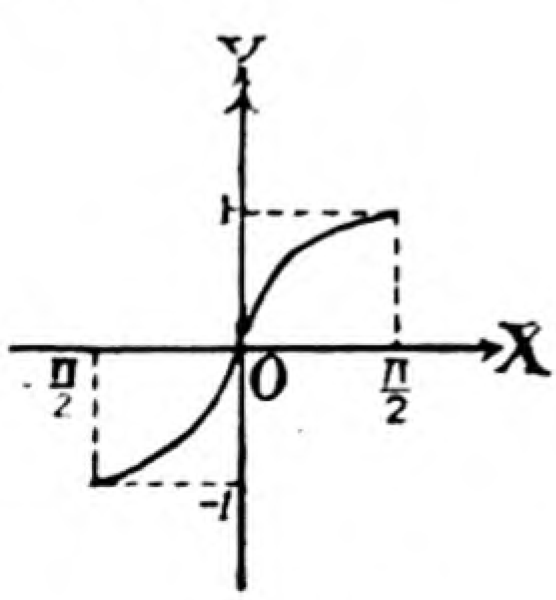
$$x=\sin^{-1} y \quad \text{and} \quad y=\sin x$$

are equivalent if x is restricted to the interval $-\frac{1}{2}\pi \leqslant x \leqslant \frac{1}{2}\pi$.

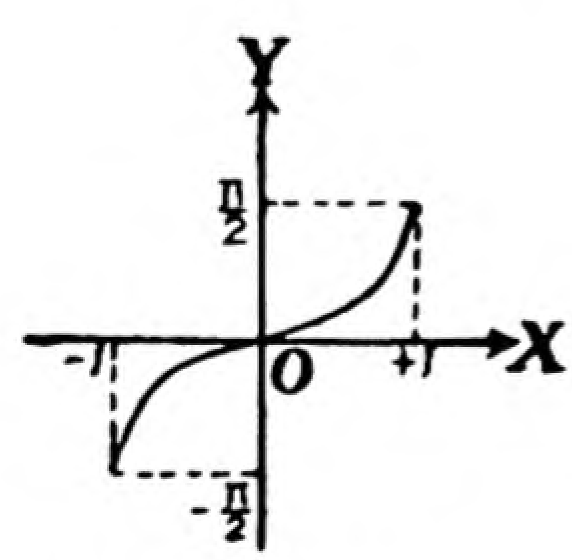
Instead of the interval $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$, we might have taken any other suitable interval like $[\frac{1}{2}\pi, \frac{3}{2}\pi]$ or $[-\frac{1}{2}\pi, -\frac{3}{2}\pi]$, etc for the variation of x. Then for any y in $-1 \leqslant y \leqslant 1$, the corresponding value of x is taken to lie in that particular interval. The value of x lying in $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$ is called the **principal value** of $\sin^{-1} y$. By $\sin^{-1} y$ we shall always mean its principal value. Similar remarks apply to $\cos^{-1} y$, $\tan^{-1} y$, etc. We observe, however, that the **principal value of $\cos^{-1}$ y** lies between 0 and π and the **principal value of $\tan^{-1}$y** lies between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$.

The ranges of principal values of $\operatorname{cosec}^{-1} x$, $\sec^{-1} x$ and $\cot^{-1} x$ are respectively the same as those of $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$.

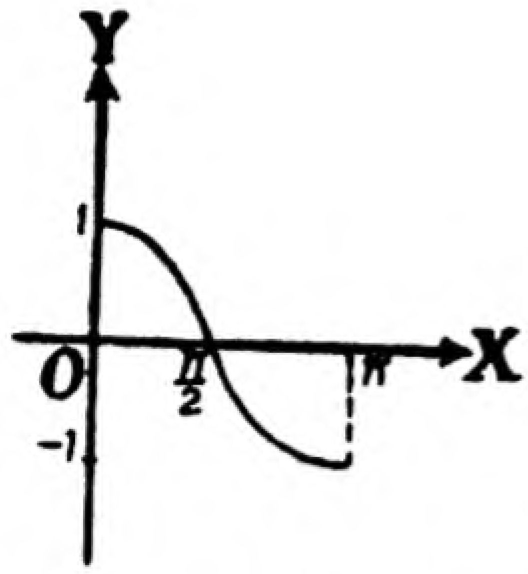
We draw the graphs of the trigonometric functions and their inverses side by side. In each case, the domain corresponds to the principal value.



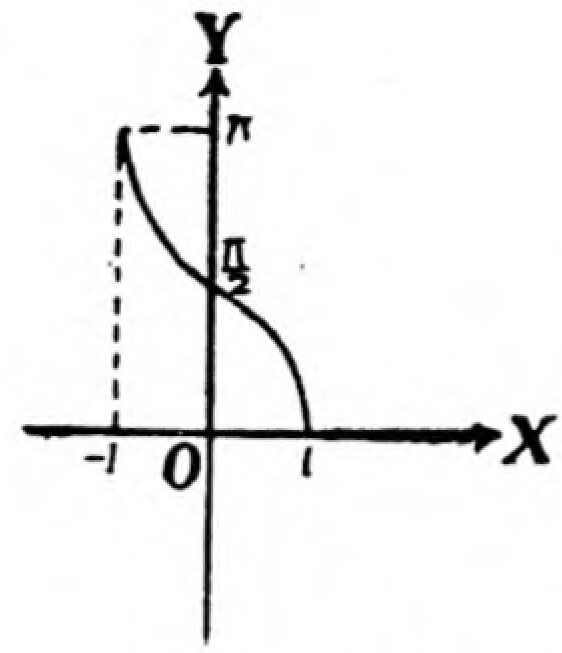
$y=\sin x$



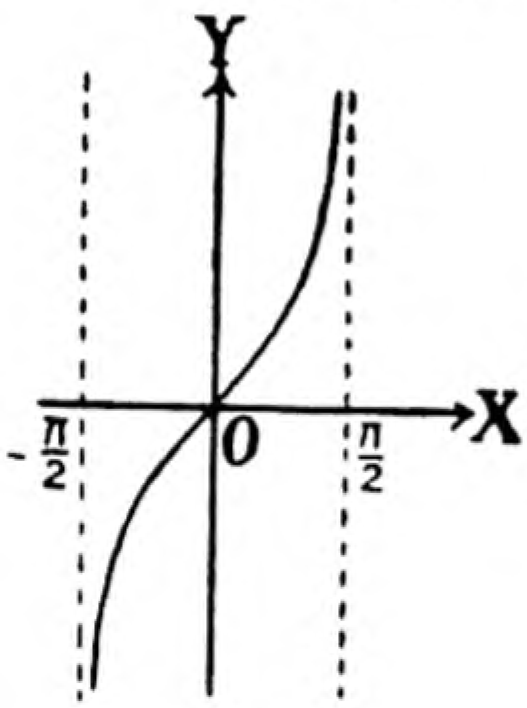
$y=\sin^{-1} x$.



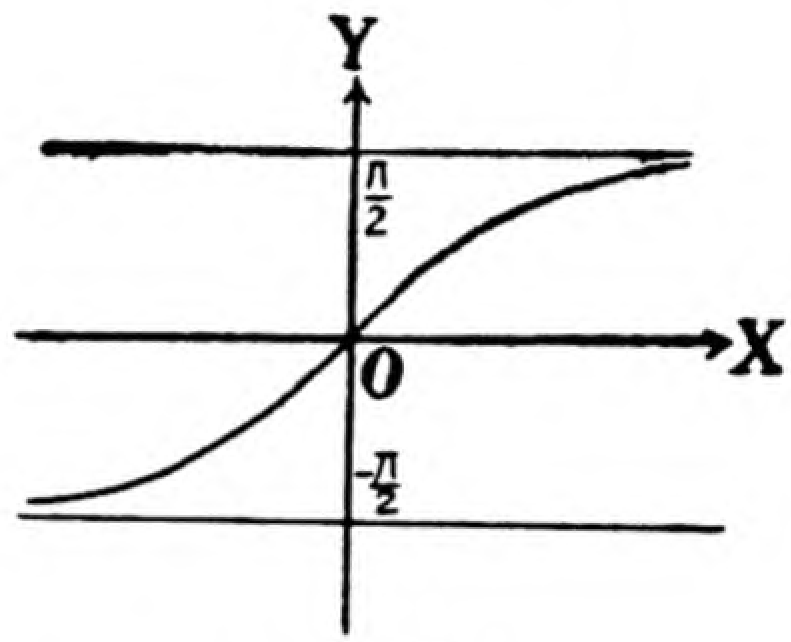
$y = \cos x$



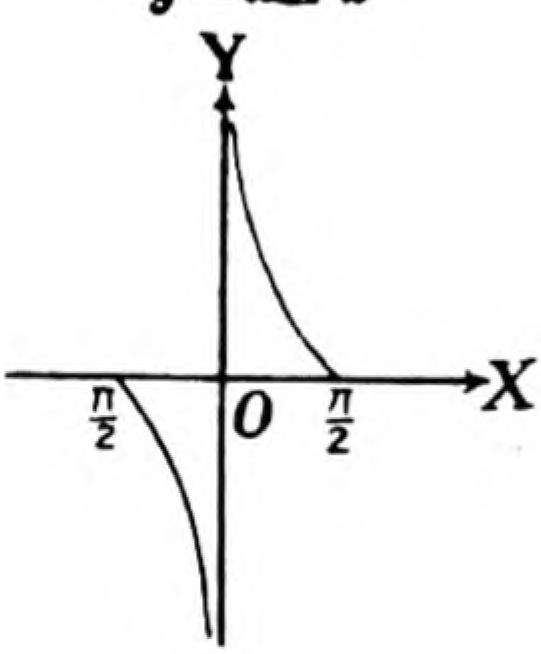
$y = \cos^{-1} x$



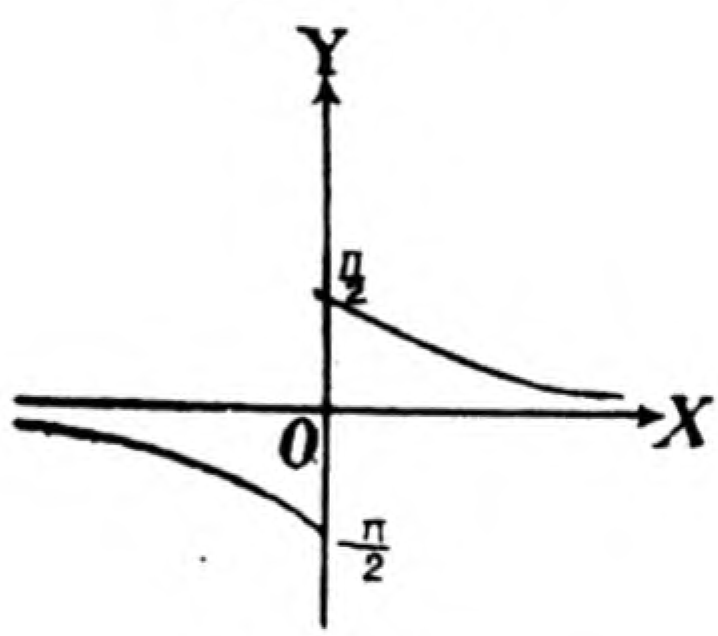
$y = \tan x$



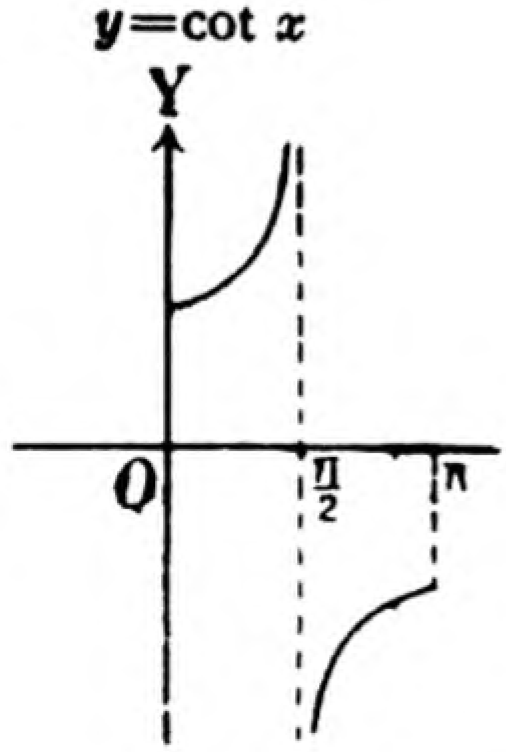
$y = \tan^{-1} x$



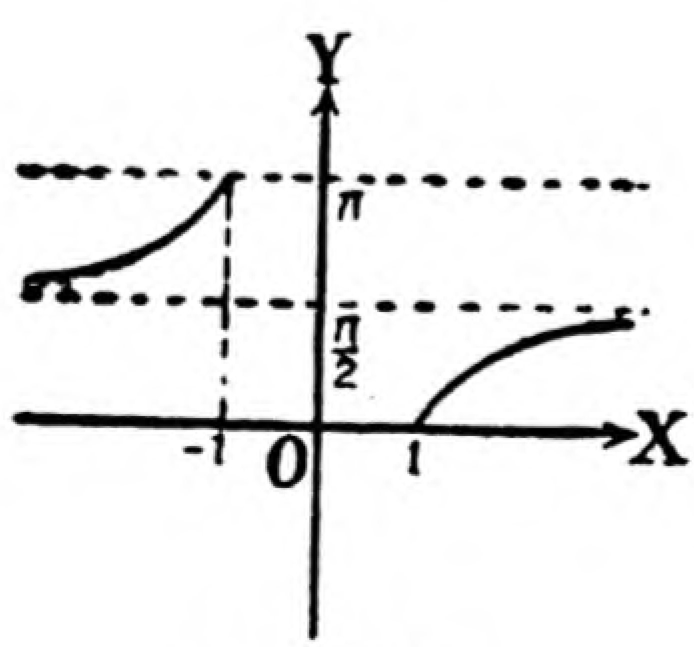
$y = \cot x$



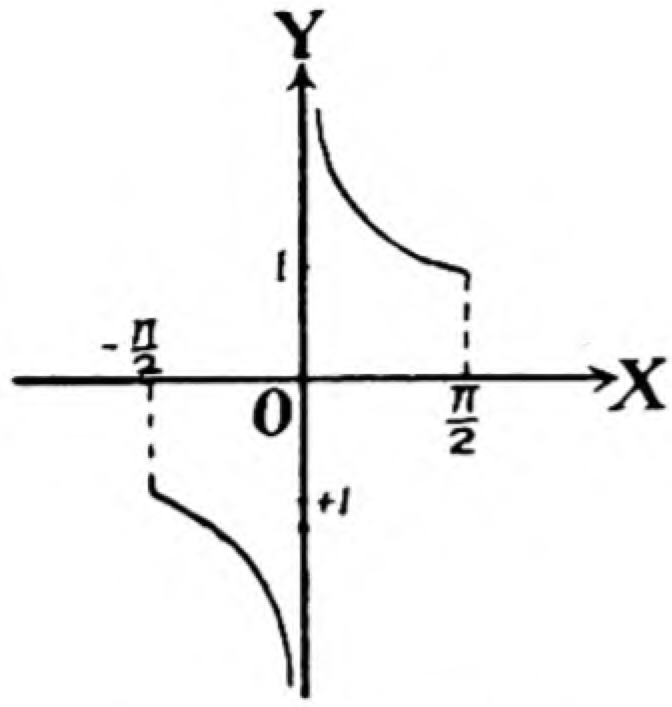
$y = \cot^{-1} x$



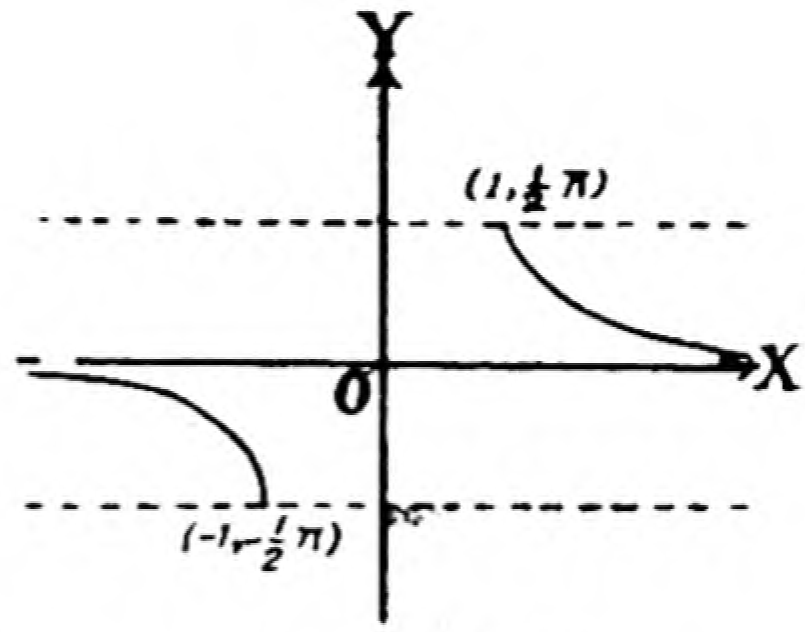
$y = \sec x$



$y = \sec^{-1} x$



$y = \text{cosec } x$



$y = \text{cosec}^{-1} x$

4·51. Differentiation of inverse circular functions. We now proceed to find the derivatives of the inverse circular functions $\sin^{-1}x$, $\cos^{-1}x$, etc.

I. $\mathbf{sin^{-1}x}$. Let $y=\sin^{-1}x$, then $x=\sin y$, and

$$\therefore \frac{dx}{dy}=\cos y, \text{ i.e., } \frac{dy}{dx}=\frac{1}{\cos y}=\pm\frac{1}{\sqrt{(1-x^2)}}.$$

The ambiguous sign is the same as that of $\cos y$, *i.e*, of $\cos(\sin^{-1}x)$. For the principal branch, *viz.*, for the interval $-\frac{1}{2}\pi<y\leqslant\frac{1}{2}\pi$, $\cos y$ is positive, therefore the positive sign of the radical must be taken. Hence

$$\frac{\mathbf{d}}{\mathbf{dx}}(\mathbf{sin^{-1}x})=\frac{1}{\sqrt{(1-\mathbf{x}^2)}} \text{ for } -\frac{\pi}{2}<\sin^{-1}x\leqslant\frac{\pi}{2}.$$

The other branches of $\sin^{-1}x$ may be divided into two classes, namely, $y=\sin^{-1}x+2k\pi$ and $(\pi-\sin^{-1}x)+2k\pi$. The derivative of the first class is the same as that of the principal branch and the derivative of the second class is to be taken with the negative sign. We shall usually take the derivative of the principal branch.

II. $\mathbf{cos^{-1}x}$. Let $y=\cos^{-1}x$, then $x=\cos y$, and

$$\therefore \frac{dx}{dy}=-\sin y, \text{ i.e., } \frac{dy}{dx}=-\frac{1}{\sin y}=\frac{-1}{\pm\sqrt{(1-x^2)}}.$$

The ambiguous sign is the same as that of $\sin y$, *i.e.*, of $\sin(\cos^{-1}x)$. For the principal branch, *i.e.*, for the range $0\leqslant y<\pi$, $\sin y$ is positive. Hence

$$\frac{\mathbf{d}}{\mathbf{dx}}(\mathbf{cos^{-1}x})=-\frac{1}{\sqrt{(1-\mathbf{x}^2)}} \text{ for } 0\leqslant\cos^{-1}x<\pi.$$

III. $\mathbf{tan^{-1}x}$. Let $y=\tan^{-1}x$, then $x=\tan y$, and

$$\therefore \quad \frac{dx}{dy}=\sec^2 y, \text{ i.e., } \frac{dy}{dx}=\frac{1}{\sec^2 y}=\frac{1}{1+x^2}.$$

$\tan y$ assumes all positive and negative values as y ranges over

any interval of π units, but $\sec^2 y$ is positive always. Hence there is no ambiguity and for every range

$$\frac{\mathbf{d}}{\mathbf{dx}}\ (\mathbf{tan^{-1}\ x}) = \frac{\mathbf{1}}{\mathbf{1+x^2}}.$$

IV. $\mathbf{cot^{-1}\ x}$. Let $y=\cot^{-1} x$, then $x=\cot y$ and

$$\therefore \quad \frac{dx}{dy}=-\operatorname{cosec}^2 y, \ \textit{i.e.,}\ \frac{dy}{dx}=-\frac{1}{\operatorname{cosec}^2 y}=-\frac{1}{1+x^2}.$$

As in the case of $\tan^{-1} x$, there is no ambiguity in sign and we have for every range of values of x

$$\frac{\mathbf{d}}{\mathbf{dx}}\ (\mathbf{cot^{-1}\ x}) = -\frac{\mathbf{1}}{\mathbf{1+x^2}}.$$

V. $\mathbf{sec^{-1}\ x}$. Let $y=\sec^{-1} x$, then $x=\sec y$, and

$$\therefore \quad \frac{dx}{dy}=\tan y \sec y=\pm x\sqrt{(x^2-1)}.$$

The ambiguous sign is the same as that of $\tan y$, *i.e.,* of $\tan(\sec^{-1} x)$.

For $0\leqslant y<\frac{1}{2}\pi$, $\tan y$ is positive and $=\sqrt{(x^2-1)}$

$$\therefore \quad \frac{dy}{dx}=\frac{1}{x\sqrt{(x^2-1)}}.$$

For $\frac{1}{2}\pi<y\leqslant\pi$, $\tan y$ is negative and $=-\sqrt{(x^2-1)}$

$$\therefore \quad \frac{dy}{dx}=\frac{-1}{x\sqrt{(x^2-1)}}.$$

$$\therefore \quad \frac{\mathbf{d}}{\mathbf{dx}}\ (\mathbf{sec^{-1}\ x}) = \frac{\mathbf{1}}{|\mathbf{x}|\ \sqrt{(\mathbf{x^2-1})}}.$$

VI. $\mathbf{cosec^{-1}\ x}$. Let $y=\operatorname{cosec}^{-1} x$, then $x=\operatorname{cosec} y$, and

$$\therefore \quad \frac{dx}{dy}=-\operatorname{cosec} y \cot y=-x\{\pm\sqrt{(x^2-1)}\}.$$

The ambiguous sign is the same as that of $\cot y$.

For $0<y\leqslant\frac{1}{2}\pi$, $\cot y$ is positive and $=\sqrt{(x^2-1)}$.

$$\therefore \quad \frac{dy}{dx}=\frac{-1}{x\sqrt{(x^2-1)}}.$$

For $-\frac{1}{2}\pi\leqslant y<0$, $\cot y$ is negative and $=-\sqrt{(x^2-1)}$

$$\therefore \quad \frac{dy}{dx}=\frac{-1}{-x\sqrt{(x^2-1)}}.$$

$$\therefore \quad \frac{\mathbf{d}}{\mathbf{dx}}\ (\mathbf{cosec^{-1}\ x}) = -\frac{\mathbf{1}}{|\mathbf{x}|\ \sqrt{(\mathbf{x^2-1})}}.$$

It should be observed that $\sec^{-1} x$ and $\operatorname{cosec}^{-1} x$ are not defined for values of x lying between -1 and $+1$.

It may also be remarked that the mere difference in signs of the derivatives of $\sin^{-1} x$ and $\cos^{-1} x$; $\tan^{-1} x$ and $\cot^{-1} x$; $\sec^{-1} x$ and

$\operatorname{cosec}^{-1} x$ is explained by the fact that for the principal ranges of the functions involved, we have the relations

$$\sin^{-1} x+\cos^{-1} x=\tfrac{1}{2}\pi,$$
$$\tan^{-1} x+\cot^{-1} x=\tfrac{1}{2}\pi,$$
$$\sec^{-1} x+\operatorname{cosec}^{-1} x=\tfrac{1}{2}\pi.$$

Ex. 1. *Differentiate* $\sec^{-1} x^2$.

Let $\qquad y=\sec^{-1} u$ where $u=x^2$.

$\therefore \qquad \dfrac{dy}{du}=\dfrac{1}{u\sqrt{(u^2-1)}}$ and $\dfrac{du}{dx}=2x.$

$\therefore \qquad \dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dx}=\dfrac{1}{u\sqrt{(u^2-1)}}\cdot 2x=\dfrac{2}{x\sqrt{(x^4-1)}}.$

Ex. 2. *Show that* $\dfrac{d}{dx}\{\tan^{-1}(\tfrac{1}{2}\tan x)\}=\dfrac{2}{4-3\sin^2 x}.$

Let $\qquad y=\tan^{-1} u$, where $u=\tfrac{1}{2}\tan x$, then

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\frac{1}{1+u^2}\cdot\tfrac{1}{2}\sec^2 x=\frac{\sec^2 x}{2(1+\frac{1}{4}\tan^2 x)}$$
$$=\frac{2}{4\cos^2 x+\sin^2 x}=\frac{2}{4-3\sin^2 x}.$$

Ex. 3. *Differentiate* $\sin^{-1}\dfrac{a+b\cos x}{b+a\cos x}$. *(Panjab, 1956)*

Let $\qquad y=\sin^{-1} u$, where $u=\dfrac{a+b\cos x}{b+a\cos x}$. Then

$$\frac{dy}{du}=\frac{1}{\sqrt{(1-u^2)}}=\frac{1}{\sqrt{\left\{1-\left(\dfrac{a+b\cos x}{b+a\cos x}\right)^2\right\}}}$$
$$=\frac{b+a\cos x}{\sqrt{\{(b^2-a^2)+(a^2-b^2)\cos^2 x\}}}$$
$$=\frac{b+a\cos x}{\sqrt{(b^2-a^2)}\sin x}, \text{ assuming } b^2>a^2.$$

Also $\qquad \dfrac{du}{dx}=\dfrac{(b+a\cos x)(-b\sin x)-(a+b\cos x)(-a\sin x)}{(b+a\cos x)^2}$

$$=\frac{(a^2-b^2)\sin x}{(b+a\cos x)^2}.$$

Hence $\dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dx}=\dfrac{b+a\cos x}{\sqrt{(b^2-a^2)}\sin x}\cdot\dfrac{(a^2-b^2)\sin x}{(b+a\cos x)^2}$

$$=-\frac{\sqrt{(b^2-a^2)}}{b+a\cos x}, \text{ assuming } b^2>a^2.$$

Ex. 4. *Differentiate* $\cos^{-1}\dfrac{1-x^2}{1+x^2}$.

Let $y=\cos^{-1} u$, where $u=\dfrac{1-x^2}{1+x^2}$. Then

$$\frac{dy}{du}=-\frac{1}{\sqrt{(1-u^2)}}=-\frac{1}{\sqrt{\left\{1-\left(\frac{1-x^2}{1+x^2}\right)^2\right\}}}$$

$$=-\frac{1+x^2}{\sqrt{\{(1+x^2)^2-(1-x^2)^2\}}}=-\frac{1+x^2}{2x}.$$

and $$\frac{du}{dx}=\frac{(1+x^2)(-2x)-(1-x^2)(2x)}{(1+x^2)^2}=\frac{-4x}{(1+x^2)^2}.$$

$$\therefore \quad \frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\frac{-(1+x^2)}{2x}\cdot\frac{-4x}{(1+x^2)^2}=\frac{2}{1+x^2}.$$

Second method. Put $x=\tan\theta$.

Then $$y=\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$
$$=\cos^{-1}(\cos 2\theta)=2\theta$$
$$=2\tan^{-1} x$$

$$\therefore \quad \frac{dy}{dx}=\frac{2}{1+x^2}.$$

Ex. 5. *Differentiate* $\tan^{-1}\dfrac{2x}{1-x^2}$ *w.r.t.* $\tan^{-1}\dfrac{\sqrt{(1+x^2)}-1}{x}$.

Let $y=\tan^{-1}\dfrac{2x}{1-x^2}$ and $z=\tan^{-1}\dfrac{\sqrt{(1+x^2)}-1}{x}$. Then we have to find $\dfrac{dy}{dz}$.

Let $x=\tan\theta$, then

$$y=\tan^{-1}\frac{2\tan\theta}{1-\tan^2\theta}=\tan^{-1}(\tan 2\theta)=2\theta$$

and $$z=\tan^{-1}\frac{\sqrt{(1+\tan^2\theta)}-1}{\tan\theta}=\tan^{-1}\frac{\sec\theta-1}{\tan\theta}=\tan^{-1}\frac{1-\cos\theta}{\sin\theta}$$
$$=\tan^{-1}(\tan\tfrac{1}{2}\theta)=\tfrac{1}{2}\theta.$$

$$\therefore \frac{dy}{dz}=\frac{dy}{d\theta}\Big/\frac{dz}{d\theta}=2/\tfrac{1}{2}=4.$$

EXAMPLES XII

Find the derivatives of :

1. $\sin^{-1} x^2$.
2. $\cos^{-1}\sqrt{x}$.
3. $\tan^{-1}\dfrac{x}{a}$.
4. $\operatorname{cosec}^{-1}\dfrac{ax}{b}$.
5. $\cot^{-1} x+\cot^{-1}\dfrac{1}{x}$.
6. $\tan^{-1}\dfrac{x+3}{x+2}$.
7. $\tan^{-1}(\sec x+\tan x)$.
8. $\sin^{-1}(\cos x)$.

9. $\dfrac{\sin^{-1} x}{x}$.

10. $x-\sqrt{(1-x^2)}\sin^{-1} x$.

11. $\cot^{-1}\sqrt{\left(\dfrac{1+\cos x}{1-\cos x}\right)}$.

12. $\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)$.

13. $\sin^{-1}[2x\sqrt{(1-x^2)}]$

14. $\sin^{-1}(1-2x^2)$.

15. $\tan^{-1}\dfrac{1+x^2}{1-x^2}$.

16. $\tan^{-1}\dfrac{a+x}{1-ax}$.

17. $\sin^{-1}\dfrac{2x}{1+x^2}$.

18. $\cos^{-1}\left(\dfrac{a\cos x+b}{a+b\cos x}\right)$.

19. Differentiate (*i*) $\tan^{-1}\dfrac{\sqrt{(1+x^2)}-1}{x}$ *w.r.* to $\tan^{-1}\dfrac{2x}{1-x^2}$.

(*ii*) $\sin^{-1}\dfrac{2x}{1+x^2}$ *w.r.* to $\cos^{-1}\dfrac{1-x^2}{1+x^2}$.

(*iii*) $\sec^{-1}\dfrac{1}{2x^2-1}$ *w.r.* to $\sqrt{(1-x^2)}$.

20. If $y=\dfrac{\sin^{-1} x}{\sqrt{(1-x^2)}}$, show that $(1-x^2)\dfrac{dy}{dx}-xy=1$.

4·6. The exponential function e^x. The graph of this function is drawn in Art. 2·4, Ex. 6. In Art. 3·5, we found that

$$\underset{n\to\infty}{\text{Lt}}\left(1+\frac{x}{n}\right)^n=e^x$$

for all x. This limit is very often taken to define e^x. The exponential function satisfies the laws of indices, *viz.*,

$$e^u\ e^v=e^{u+v},\ e^u\div e^v=e^{u-v},\ (e^u)^v=e^{uv}.$$

Since $e=2.71......>1$, $\therefore$ e^x increases as x increases and $e^x\to\infty$ as $x\to+\infty$. On the other hand $1/e^x$, *i.e.*, e^{-x}, decreases as x increases and $e^{-x}\to0$ as $x\to+\infty$ or, in other words, $e^x\to0$ as $x\to-\infty$.

4·61. Derivative of e^x.

Let $y=e^x$, then by definition,

$$\frac{dy}{dx}=\underset{h\to0}{\text{Lt}}\frac{e^{x+h}-e^x}{h}=e^x\cdot\underset{h\to0}{\text{Lt}}\frac{e^h-1}{h}=e^x,$$

since by Art. 3 5, V, Cor. 1, $\underset{h\to0}{\text{Lt}}\dfrac{e^h-1}{h}=1$.

Hence $\dfrac{\mathbf{d}}{\mathbf{dx}}(\mathbf{e^x})=\mathbf{e^x}$.

4·62. Derivative of a^x.

Let $y=a^x$,

$$\therefore \frac{dy}{dx}=\underset{h\to 0}{\text{Lt}}\left(\frac{a^{x+h}-a^x}{h}\right)=a^x \underset{h\to 0}{\text{Lt}}\left(\frac{a^h-1}{h}\right)$$

$$=a^x \log a \qquad \left[\because \underset{h\to 0}{\text{Lt}}\left(\frac{a^h-1}{h}\right)=\log a.\right.$$

$$\therefore \qquad \frac{\mathbf{d}}{\mathbf{dx}}(a^x)=\mathbf{a^x \log a}.$$

Alternatively. Let $y=a^x=e^{x \log a}$.

Let $\quad z=x \log a$ so that $y=e^z$.

$$\therefore \qquad \frac{dz}{dx}=\log a,\ \frac{dy}{dz}=e^z$$

$$\therefore \qquad \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=e^z.\log a=a^x \log a.$$

Ex. 1. *Differentiate* $e^{\sin x}$.

Let $\quad y=e^{\sin x}$.

Set $\quad u=\sin x$ so that $y=e^u$.

$$\therefore \quad \frac{du}{dx}=\cos x, \qquad \frac{dy}{du}=e^u.$$

$$\therefore \quad \frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=e^u\ .\cos x=e^{\sin x}\ .\cos x.$$

Ex. 2. *If* $y=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$, *prove that* $\dfrac{dy}{dx}=1-y^2$.

We have $$\frac{dy}{dx}=\frac{\left(x^x+e^{-x}\right)\frac{d}{dx}\left(e^x-e^{-x}\right)-\left(e^x-e^{-x}\right)\frac{d}{dx}\left(e^x+e^{-x}\right)}{(e^x+e^{-x})^2}$$

$$=\frac{(e^x+e^{-x})(e^x+e^{-x})-(e^x-e^{-x})(e^x-e^{-x})}{(e^x-e^{-x})^2}$$

$$=1-\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right)^2$$

$$=1-y^2.$$

Ex. 3. *If* $u=e^{ax}\sin bx$, $v=e^{ax}\cos bx$, *prove that*

$$\left(\frac{du}{dx}\right)^2+\left(\frac{dv}{dx}\right)^2=(a^2+b^2)e^{2ax}.$$

We have $\dfrac{du}{dx}=e^{ax}\cos bx\ .\ b+\sin bx\ .\ e^{ax}\ .\ a$

$$=e^{ax}(a \sin bx+b \cos bx) \qquad \ldots(1)$$

and $$\frac{dv}{dx}=e^{ax}(-\sin bx)\, b+\cos bx\ .\ e^{ax}\ .\ a$$

$$=e^{ax}(a \cos bx-b \sin bx) \qquad \ldots(2)$$

Square (1) and (2) and add.

$$\therefore \left(\frac{du}{dx}\right)^2+\left(\frac{dv}{dx}\right)^2=e^{2ax}[(a\sin bx+b\cos bx)^2+(a\cos bx-b\sin bx)^2]$$
$$=e^{2ax}[a^2(\sin^2 bx+\cos^2 bx)+b^2(\cos^2 bx+\sin^2 bx)]$$
$$=e^{2ax}(a^2+b^2).$$

4·63. The logarithmic function $\log_e x$. The graph of this function was drawn in Art. 2·4, Ex. 6. This is the inverse of the exponential function. We recall below some properties of the logarithmic function. The base e is usually omitted.

(*i*) $\log(xy)=\log x+\log y$,
$\log(x/y)=\log x-\log y$,
$\log x^m=m\log x$.

(*ii*) $\log e=1$, $\log 1=0$.

(*iii*) $\log x$ is defined only for $x>0$.

(*iv*) $\log x$ increases as x increases and $\log x\to\infty$ as $x\to+\infty$.

(*v*) When $0<x<1$, $\log x$ is negative and increases negatively as x decreases. We have $\log x\to-\infty$ for $x\to 0$.

4·64. Derivative of log x.

Let $y=\log_e x$, then by definition,

$$\frac{dy}{dx}=\operatorname*{Lt}_{h\to 0}\frac{\log(x+h)-\log x}{h}$$
$$=\operatorname*{Lt}_{h\to 0}\frac{1}{h}\cdot\log\left(1+\frac{h}{x}\right)$$
$$=\operatorname*{Lt}_{h\to 0}\frac{1}{x}\cdot\frac{x}{h}\log\left(1+\frac{h}{x}\right)$$
$$=\operatorname*{Lt}_{h\to 0}\frac{1}{x}\log(1+u)^{1/u},\text{ where } u=\frac{h}{x}$$
$$=\frac{1}{x}\log\operatorname*{Lt}_{u\to 0}(1+u)^{1/u}\quad[\text{Since } u\to 0 \text{ as } h\to 0$$
$$=\frac{1}{x}\log e=\frac{1}{x}\cdot 1=\frac{1}{x}.$$

$$\therefore\quad \mathbf{\frac{d}{dx}(\log x)=\frac{1}{x}}.$$

Ex. 1. *Differentiate* (*i*) $\log\log x$ (*ii*) $\log_{10}(1+\sqrt{x})$

(*i*) Let $y=\log u$, where $u=\log x$, then

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\frac{1}{u}\times\frac{1}{x}=\frac{1}{x\log x}.$$

(*ii*) $$\frac{d}{dx}\{\log_{10}(1+\sqrt{x})\}=\frac{d}{dx}\left[\frac{\log_e(1+\sqrt{x})}{\log_e 10}\right]$$
$$=\frac{1}{\log_e 10}\cdot\frac{1}{1+\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}=\frac{1}{2(x+\sqrt{x})\log_e 10}$$

Ex. 2. *If* $\log(xy)=x^2+y^2$, *find* $\frac{dy}{dx}$.

We have $$\log(xy)=x^2+y^2$$

$$\therefore \quad \log x+\log y=x^2+y^2$$

Differentiating both sides *w.r.* to x, we get

$$\frac{1}{x}+\frac{1}{y}\ \frac{dy}{dx}=2x+2y\frac{dy}{dx}$$

or $$\frac{dy}{dx}\ (\frac{1}{y}-2y)=2x-\frac{1}{x}.$$

$$\therefore \quad \frac{dy}{dx}=\frac{(2x^2-1)y}{(1-2y^2)x}.$$

Ex. 3. *Find* $\frac{dy}{dx}$ *when* $y=\log_x \tan x$.

Here $y=\log_x \tan x=\frac{\log_e \tan x}{\log_e x}$; therefore differentiating by the quotient rule,

$$\frac{dy}{dx}=\frac{\log\ x\times\frac{\sec^2 x}{\tan\ x}-\frac{1}{x}\log\tan x}{(\log x)^2}$$

$$=\frac{x\log x-\sin x\cos x\log\tan x}{x\sin x\cos x\ (\log x)^2}.$$

Ex. 4. *If* $\frac{x}{x-y}=\log\frac{a}{x-y}$ *prove that* $\frac{dy}{dx}=2-\frac{x}{y}$.

From the given relation, $\frac{x}{x-y}=\log a—\log(x-y)$.

Differentiating both sides *w.r.* to x, we get

$$\frac{(x-y)\ .\ 1-x(1-\frac{dy}{dx})}{(x-y)^2}=0-\frac{1}{x-y}\ (1-\frac{dy}{dx})$$

or $$(x-y)-x(1-\frac{dy}{dx})=-(x-y)(1-\frac{dy}{dx})$$

or $$x-y=y\ (1-\frac{dy}{dx})$$

or $$y\frac{dy}{dx}=2y-x$$

$$\frac{dy}{dx}=2-\frac{x}{y}.$$

EXAMPLES XIII

Find the derivatives of :

1. $e^{2x}-e^{-2x}$.
2. e^{x^2+2x}.
3. e^{x^n}.
4. 5^{2x-5}.
5. $a^{\sqrt{x}}$.
6. xe^x.
7. $a^x.x^a$.
8. $\log(2x^3+3)$.
9. $\log\{x+\sqrt{(x^2+a^2)}\}$.
10. $\log\{\sqrt{(x+a)}+\sqrt{(x+b)}\}$.
11. $\log_{10}\sqrt{(1+x^2)}$.
12. $\log_x 10$.
13. $\log\dfrac{a+bx}{a-bx}$.
14. $x^2\log x$.
15. $\dfrac{\log x}{x}$.
16. $\log\log\log x$.
17. $e^x(1+\log x)$.
18. $x^n e^x\log x$.
19. $\dfrac{1+e^x}{1-e^x}$.
20. $\dfrac{1+\log x}{1-\log x}$.

4·65. Logarithmic differentiation. If $y=f(x)$ be a function of x, or more generally, there may be any relation between x and y, then the process of first taking the lagarithms of both sides of the given equation and then differentiating the two sides is called logarithmic differentiation. The method is employed either when

$$y=\frac{u_1.u_2\ldots\ldots u_n}{v_1.v_2\ldots\ldots v_m}\quad\text{or}\quad y=u^v$$

where $u_1, u_2,\ldots\ldots\ldots, u_n, v_1, v_2,\ldots\ldots\ldots, v_m, u, v$ are functions of x. It is illustrated by the following solved examples.

Ex. 1. *Differentiate* $y=\dfrac{(x+1)^{\frac{3}{4}}(x^2+3)^{\frac{5}{3}}(x^3+7)^7}{\sqrt{x}(\sqrt{x}+2)^{\frac{4}{3}}}$.

Taking logarithms of both sides, we get

$$\log y=\tfrac{3}{4}\log(x+1)+\tfrac{5}{3}\log(x^2+3)+7\log(x^3+7)-\tfrac{1}{2}\log x-\tfrac{4}{3}\log(\sqrt{x}+2).$$

The derivative of $\log y$ with respect to x is, by the rule for the derivation of a function of a function, equal to $\dfrac{1}{y}\dfrac{dy}{dx}$. Hence, taking derivatives of both sides with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx}=\frac{3}{4(x+1)}+\frac{10x}{3(x^2+3)}+\frac{21x^2}{x^3+7}-\frac{1}{2x}-\frac{2}{3\sqrt{x}(\sqrt{x}+2)}.$$

This gives $\dfrac{dy}{dx}$.

Ex. 2. *Find the derivative of* $\sin x\times\log x\times e^x\times\sqrt{x}$.

Let
$$y=\sin x\times\log x\times e^x\times\sqrt{x},$$
then
$$\log y=\log\sin x+\log\log x+x+\tfrac{1}{2}\log x.$$

$$\therefore \quad \frac{1}{y}\frac{dy}{dx}=\frac{\cos x}{\sin x}+\frac{1}{x\log x}+1+\frac{1}{2x}.$$

or $$\frac{dy}{dx}=y\left(\cot x+\frac{1}{x\log x}+\frac{2x+1}{2x}\right).$$

Ex. 3. *Differentiate* : (*i*) $(\sin x)^{\cos x}$.

(*ii*) $x^x+(\tan x)^{\log x}$. *(Panjab, 1951)*

(*i*) Let $y=(\sin x)^{\cos x}$, then $\log y=\cos x \log \sin x$.

Taking derivatives of both sides

$$\frac{1}{y}\frac{dy}{dx}=\cos x.\frac{\cos x}{\sin x}-\sin x\log\sin x$$

or $$\frac{dy}{dx}=(\sin x)^{\cos x}(\cos x\cot x-\sin x\log\sin x).$$

(*ii*) Here we cannot take logarithms directly.

Let $u=x^x$ and $v=(\tan x)^{\log x}$, then

$$y=u+v \text{ and } \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}.$$

Now $u=x^x$; therefore $\log u=x\log x$, and

$$\frac{1}{u}\frac{du}{dx}=x\,.\,\frac{1}{x}+\log x=1+\log x=\log ex$$

$$\therefore \quad \frac{du}{dx}=x^x\log ex. \qquad \ldots(1)$$

Also $v=(\tan x)^{\log x}$. $\therefore$ $\log v=\log x\log\tan x$

and $$\frac{1}{v}\frac{dv}{dx}=\frac{1}{x}\log\tan x+\frac{\sec^2 x}{\tan x}\log x.$$

$$\therefore \frac{dv}{dx}=(\tan x)^{\log x}\left\{\frac{\log\tan x}{x}+\frac{\log x}{\sin x\cos x}\right\}. \qquad \ldots(2)$$

$\frac{dy}{dx}$ is obtained by adding (1) and (2).

Ex. 4. *If* $x^y=e^{x-y}$, *prove that* $\frac{dy}{dx}=\frac{\log x}{(1+\log x)^2}$.

(Agra, 1945 ; Delhi, 1950 ; Panjab, 1956)

Taking logarithms of both sides, we get

$$y\log x=x-y. \qquad (1)$$

Differentiating both sides *w.r.* to x, we get

$$\frac{dy}{dx}\log x+\frac{y}{x}=1-\frac{dy}{dx} \text{ or } \frac{dy}{dx}=\frac{x-y}{x(1+\log x)} \qquad (2)$$

But from (1), $y=\frac{x}{1+\log x}$. Substituting this value of y in (2), we get

$$\frac{dy}{dx}=\frac{\log x}{(1+\log x)^2}.$$

Or we could have solved (1) for y and differentiated.

EXAMPLES XIV

Differentiate with respect to x :

1. $\dfrac{(x+1)(x+2)(x+3)(x+4)}{(x-1)(x-2)(x-3)(x-4)}$.   2. $\sqrt{x}.\sqrt{(\sin x)}.\sqrt{(\log x)}$.

3. x^{1+x}.   4. $(\tan x)^{\cot x}$.   5. $(\sin^{-1}x)^x$.   6. $x^{\sin x}$.

7. $(\sin x)^{\log x}$. *(Calcutta, 1943)*   8. $\cos x^x$.   9. a^{b^x}.

10. x^{x^x} . *(Nagpur, 1950)*   11. $x^x+x^{1/x}$. *(Patna, 1945)*

12. $(\sin x)^{\cos x}+(\cos x)^{\sin x}$.   13. $(\sec x)^{\operatorname{cosec} x}+(\operatorname{cosec} x)^{\sec x}$.

14. $(\tan x)^x+x^{\tan x}$.

4·7. Hyperbolic functions. The hyperbolic sine, cosine tangent, etc. are defined by the following equations :

$$\sinh x=\tfrac{1}{2}(e^x-e^{-x}),\ \cosh x=\tfrac{1}{2}(e^x+e^{-x}),$$

$$\tanh x=\frac{\sinh x}{\cosh x},\qquad \coth x=\frac{\cosh x}{\sinh x},$$

$$\operatorname{sech} x=\frac{1}{\cosh x},\ \operatorname{cosech} x=\frac{1}{\sinh x}.$$

The student can easily verify the relations

$$\coth^2 x-\sinh^2 x=1,$$
$$1-\tanh^2 x=\operatorname{sech}^2 x,$$
and
$$\coth^2 x-1=\operatorname{cosech}^2 x$$

from the definitions of the hyperbolic functions.

We now proceed to obtain their derivatives.

I. Let $y=\sinh x=\frac{1}{2}(e^x-e^{-x})$, then

$$\frac{dy}{dx}=\frac{1}{2}(e^x+e^{-x})=\cosh x.$$

Hence $\dfrac{\mathbf{d}}{\mathbf{dx}}$ **(sinh x) = cosh x.**

II. Let $y=\cosh x=\frac{1}{2}(e^x-e^{-x})$, then

$$\frac{dy}{dx}=\frac{1}{2}(e^x+e^{-x})=\sinh x.$$

Hence $\dfrac{\mathbf{d}}{\mathbf{dx}}$ **(cosh x) = sinh x.**

III. Let $y=\tanh x=\dfrac{\sinh x}{\cosh x}$, then, applying the rule for the differentiation of a quotient,

$$\frac{dy}{dx}=\frac{\cosh x.\cosh x-\sinh x.\sinh x}{\cosh^2 x}=\frac{\cosh^2 x-\sinh^2 x}{\cosh^2 x}$$

$$=\frac{1}{\cosh^2 x}=\operatorname{sech}^2 x.$$

Hence $\dfrac{d}{dx}(\tanh x)=\text{sech}^2 x.$

Proceeding similarly, we can show that:

IV. $\dfrac{d}{dx}(\coth x)=-\text{cosech}^2 x$

V. $\dfrac{d}{dx}(\text{sech } x)=-\text{sech } x \tanh x.$

VI. $\dfrac{d}{dx}(\text{cosech } x)=-\text{cosech } x \coth x.$

It should be observed that the derivatives follow the same rule as for the circular functions, but here the first three are positive and the remaining three are negative.

4·71. Inverse hyperbolic functions.

I. Let $y=\sinh^{-1} x$, then $x=\sinh y$, and

$$\therefore \quad \frac{dx}{dy}=\cosh y, \text{ i.e., } \frac{dy}{dx}=\frac{1}{\cosh y}=\frac{1}{\sqrt{(1+x^2)}}$$

There is no ambiguity of sign as $\cosh y$ is always positive.

Hence $$\frac{d}{dx}(\sinh^{-1}x)=\frac{1}{\sqrt{(1+x^2)}}.$$

II. Let $y=\cosh^{-1}x$, then $x=\cosh y$, and

$$\therefore \quad \frac{dx}{dy}=\sinh y, \text{ i.e., } \frac{dy}{dx}=\frac{1}{\sinh y}=\frac{1}{\pm\sqrt{(x^2-1)}}.$$

For any value of $x>1$, there are two values of y and for these values $\dfrac{dy}{dx}$ has opposite signs We usually take the derivative with the plus sign.

Hence $$\frac{d}{dx}(\cosh^{-1} x)=\frac{1}{\sqrt{(x^2-1)}}, |x|>1$$

III. Let $y=\tanh^{-1} x$, then $x=\tanh y$, and

$$\therefore \quad \frac{dx}{dy}=\text{sech}^2 y, \text{ i.e., } \frac{dy}{dx}=\frac{1}{\text{sech}^2 y}=\frac{1}{1-x^2}.$$

Since $\tanh y$ is less than 1 numerically, therefore $|x|<1$. Also, there is no ambiguity of sign. Hence

$$\frac{d}{dx}(\tanh^{-1} x)=\frac{1}{1-x^2}, |x|<1.$$

Proceeding similarly, we obtain :

IV. $\dfrac{d}{dx}(\coth^{-1}x)=-\dfrac{1}{x^2-1}, |x|>1.$

V. $\dfrac{d}{dx}(\text{sech}^{-1}x)=-\dfrac{1}{x\sqrt{(1-x^2)}}, |x|<1.$

VI. $\dfrac{\mathbf{d}}{\mathbf{dx}}(\mathbf{cosech}^{-1}\mathbf{x}) = -\dfrac{\mathbf{1}}{\mathbf{x}\sqrt{(\mathbf{x}^2+\mathbf{1})}}$.

Ex. *Find the derivatives of* (*i*) $\tan^{-1}(\tanh x)$, (*ii*) $\tanh^{-1}\dfrac{a+x}{1+ax}$.

(*i*) Let $y = \tan^{-1} u$, where $u = \tanh x$, then

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{1+u^2}\cdot \operatorname{sech}^2 x$$

$$= \frac{1}{1+\tanh^2 x}\cdot \operatorname{sech}^2 x = \frac{1-\tanh^2 x}{1+\tanh^2 x}.$$

(*ii*) Let $y = \tanh^{-1} u$, where $u = \dfrac{a+x}{1+ax}$, then

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{1-u^2}\cdot\frac{(1+ax)\cdot 1-(a+x)\cdot a}{(1+ax)^2}$$

$$= \frac{(1+ax)^2}{(1+ax)^2-(a+x)^2}\cdot\frac{1-a^2}{(1+ax)^2}$$

$$= \frac{1-a^2}{(1-a^2)+(a^2-1)x^2} = \frac{1}{1-x^2}.$$

EXAMPLES XV

Find the derivatives of the following functions :

1. $\sinh x + \frac{1}{3}\sinh^3 x$, $\dfrac{\cosh x+\cos x}{\sinh x+\sin x}$, $\dfrac{\cosh x+\sin x}{\sinh x+\cos x}$.

2. $\log\cosh x$, $\log\tanh x$, $\log \operatorname{cosech} x$, $a^{\sinh x}$

$\log\dfrac{1+\sqrt{(1-x^2)}}{x}$, $\log\dfrac{1+\sqrt{(1+x^2)}}{x}$

$\sinh^{-1}\sqrt{(x^2-1)}$, $\tanh^{-1}\dfrac{2x}{1+x^2}$, $\tanh^{-1}\dfrac{x^2-1}{x^2+1}$, $\sinh^{-1}(\tan x)$.

5. Prove that each of the functions $\sec^{-1}(\cosh x)$, $\tan^{-1}(\sinh x)$ $\cos^{-1}(\operatorname{sech} x)$ has the derivative $\operatorname{sech} x$.

6. Prove that each of the functions $2\tanh^{-1}(\tan \frac{1}{2}x)$ $\cosh^{-1}(\sec x)$ has the derivative $\sec x$.

4·8. Differentiation from definition. We now add a few examples of differentiation from definition or from first principles *i.e.*, we obtain the derivatives by the limiting process given in the definition in Art. 4·1.

Ex. *Find from first principles the derivatives of*

(*i*) $\sqrt{(\sin x)}$, (*ii*) $\tan^{-1}x$, (*iii*) $x/(x^2+a^2)$.

(*i*) $$\frac{dy}{dx} = \operatorname*{Lt}_{h\to 0}\frac{\sqrt{\{\sin(x+h)\}}-\sqrt{(\sin x)}}{h}$$

$$= \operatorname*{Lt}_{h\to 0}\frac{\sin(x+h)-\sin x}{h[\sqrt{\{\sin(x+h)\}}+\sqrt{(\sin x)}]}$$

$$=\underset{h\to 0}{\mathrm{Lt}}\ \frac{\cos(x+\frac{1}{2}h)}{\sqrt{\{\sin(x+h)\}}+\sqrt{(\sin x)}}\cdot\frac{\sin\frac{1}{2}h}{\frac{1}{2}h}$$

$$=\frac{\cos x}{2\sqrt{(\sin x)}}.\qquad\left[\because\ \underset{x\to 0}{\mathrm{Lt}}\ \frac{\sin x}{x}=1.\right.$$

(*ii*) Let $y=\tan^{-1}x$, then $y+\delta y=\tan^{-1}(x+\delta x)$, so that $x=\tan y$, $x+\delta x=\tan(y+\delta y)$ and $\delta x=\tan(y+\delta y)-\tan y$.

$$\therefore\quad \frac{dy}{dx}=\underset{\delta x\to 0}{\mathrm{Lt}}\ \frac{\delta y}{\delta x}=\underset{\delta y\to 0}{\mathrm{Lt}}\ \frac{\delta y}{\tan(y+\delta y)-\tan y}$$

$$=\underset{\delta y\to 0}{\mathrm{Lt}}\ \cos y\cos(y+\delta y)\frac{\delta y}{\sin\delta y}$$

$$=\cos^2 y=\frac{1}{\sec^2 y}=\frac{1}{1+x^2}.$$

(*iii*) Here $y=\dfrac{x}{x^2+a^2}$, $\therefore\ y+\delta y=\dfrac{x+\delta x}{(x+\delta x)^2+a^2}$, and

$$\therefore\quad \frac{dy}{dx}=\underset{\delta x\to 0}{\mathrm{Lt}}\ \frac{\delta y}{\delta x}=\underset{\delta x\to 0}{\mathrm{Lt}}\ \frac{1}{\delta x}\left\{\frac{x+\delta x}{(x+\delta x)^2+a^2}-\frac{x}{x^2+a^2}\right\}$$

$$=\underset{\delta x\to 0}{\mathrm{Lt}}\ \frac{x\{x^2+a^2-(x+\delta x)^2-a^2\}+\delta x(x^2+a^2)}{\delta x(x^2+a^2)\{(x+\delta x)^2+a^2\}}$$

$$=\underset{\delta x\to 0}{\mathrm{Lt}}\ \frac{-2x^2-\delta x\,x+x^2+a^2}{(x^2+a^2)\{(x+\delta x)^2+a^2\}}=\frac{a^2-x^2}{(x^2+a^2)^2}.$$

EXAMPLES XVI

Find from definition the derivatives of :

1. All the inverse trigonometric functions.

2. $\sin x^2$. (*Agra, 1948*)

3. $e^{\sqrt{x}}$. (*Nagpur, 1934*)

4. $\log\sin x$.

5. $\dfrac{x^2+2}{x^2+3}$.

6. $x\sin x$.

7. $e^{\cos x}$.

8. $\tan^2 x$.

CHAPTER V

SUCCESSIVE DIFFERENTIATION

5·1. Successive derivatives. Let $y=f(x)$ be a function of x which has a derivative for each x in some interval, then the derivative $f'(x)$ is itself a function of x in the same interval. If $f'(x)$ has a derivative at a point x, then this derivative is called the *second* derivative of y or $f(x)$. If this new derivative has a derivative, then this is called the *third* derivative of y or $f(x)$, and so on.

Various notations are used for denoting the successive derivatives of a function. Thus the first, second, third,........., nth,......... derivatives of y are represented by

$$y_1,\ y_2,\ y_3,\ldots\ldots,y_n,\ldots\ldots$$

or by
$$y',\ y'',\ y''',\ldots\ldots,y^{(n)},\ldots\ldots,$$

or by
$$\frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \frac{d^3y}{dx^3},\ldots\ldots,\frac{d^ny}{dx^n},\ldots\ldots,$$

or by
$$Dy,\ D^2y,\ D^3y,\ldots\ldots,D^ny,\ldots\ldots,$$

and those of $f(x)$ by

$$f_1(x), f_2(x), f_3(x),\ldots\ldots, f_n(x),\ldots\ldots$$

or by
$$f'(x), f''(x), f'''(x)\ldots\ldots, f^{(n)}(x),\ldots\ldots$$

Ex. 1. *If $y=x^5$, find the successive derivatives of y.*

Here $y_1=5x^4$, $\quad y_2=20x^3$, $\quad y_3=60x^2$,

$y_4=120x$, $\quad y_5=120$, $\quad y_6=0$,

and all subsequent derivatives are zero.

It is an obvious generalization that, if $y=x^n$, where n is a positive integer, then all the derivatives of y beginning with the $(n+1)$th are zero.

Ex. 2. *If $y=\log\sin x$, find y_1, y_2, y_3.*

Here $y_1=\cot x$, $y_2=-\operatorname{cosec}^2 x$ and $y_3=2\operatorname{cosec}^2 x\cot x$.

Ex. 3. *Find the fourth derivative of* $\dfrac{x-3}{2-x}$.

Here $\quad y=\dfrac{x+3}{2-x}=-1+\dfrac{5}{2-x}=-1+5(2-x)^{-1}$,

Hence $y_1=5(-1)(-1)(2-x)^{-2}=5(2-x)^{-2}$,

$y_2=5(-2)(-1)(2-x)^{-3}=10(2-x)^{-3}$,

$y_3=30(2-x)^{-4}$,

$y_4=120(2-x)^{-5}=\dfrac{120}{(2-x)^5}$.

Ex. 4. *If $y=\sin x$, find the successive derivatives of y.*

Here $y_1=\cos x$, $y_2=-\sin x$, $y_3=-\cos x$, $y_4=\sin x$.
$y_5=\cos x$, $y_6=-\sin x$, $y_7=-\cos x$, $y_8=\sin x$,
....

We see that $y_1, y_5, y_9, \ldots$ are all equal to $\cos x$; $y_2, y_6, y_{10}, \ldots\ldots$ to $-\sin x$; $y_3, y_7, y_{11}, \ldots\ldots$ to $-\cos x$, and $y_4, y_8, y_{12}, \ldots\ldots$ to $\sin x$. These values can all be derived from one single formula,

$$y_n=\sin(x+\tfrac{1}{2}n\pi),$$

where n is any positive integer. This will be proved in Art. 5·2. The student may verify it by giving different values to n.

Ex. 5. *If $y=A\{x+\sqrt{(x^2-1)}\}^n$, prove that*

$$(x^2-1)y_2+xy_1-n^2y=0.$$

We can calculate y_1 and y_2 by the rules of differentiation, substitute in the L.H.S of the equation to be proved and verify that the equation is satisfied However, in many such problems, the following or a similar method may be used. Differentiating y, we get

$$y_1=An\,\{x+\sqrt{(x^2-1)}\}^{n-1}\left\{1+\frac{x}{\sqrt{(x^2-1)}}\right\}$$

or
$$y_1=\frac{An}{\sqrt{(x^2-1)}}\{x+\sqrt{(x^2-1)}\}^n=\frac{ny}{\sqrt{(x^2-1)}}.$$

Cross-multiplying and squaring, we get

$$(x^2-1)y_1^2=n^2y^2.$$

Differentiating this equation *w.r.* to x, we get

$$(x^2-1)\,2y_1y_2+2x.y_1^2=2n^2yy_1$$

Assuming that $y_1\neq 0$ we can cancel $2y_1$ on both sides and get

$$(x^2-1)y_2+xy_1=n^2y$$

or
$$(x^2-1)y_2+xy_1-n^2y=0.$$

5·11. When x and y are given as functions of a parameter t, that is,

$$x=f(t),\ y=g(t), \qquad \ldots(1)$$

the second derivative of y with respect to x is found as follows. Regarding y as a function of t and t as a function of x, and applying the rule for the differentiation of a function of a function, we get

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}=\frac{f'(t)}{g'(t)}. \qquad \ldots(2)$$

Differentiating both sides of (2) *w.r.* to x, we get

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx} \qquad \ldots(3)$$

$$=\frac{d}{dt}\left\{\frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}\right\}\cdot\frac{1}{\dfrac{dx}{dt}}=\frac{\dfrac{dx}{dt}\cdot\dfrac{d^2y}{dt^2}-\dfrac{dy}{dt}\cdot\dfrac{d^2x}{dt^2}}{\left(\dfrac{dx}{dt}\right)^3} \qquad \ldots(4)$$

or $$\frac{d^2y}{dx^2}=\frac{f'(t)\,g''(t)-g'(t)\,f''(t)}{\{f'(t)\}^3}. \qquad \ldots(4')$$

The step (3) should be noted. Both sides of (2) have to be differentiated *w.r.* to the same variable x. Since the *r.h.s.* of (2) is a function of t while the *l.h.s.* is apparently that of x, the mistake is very often made of differentiating the *r.h.s.* with respect to t and the *l.h.s.* with respect to x. The formula (4) or (4′) is of frequent use.

The third order derivative of y *w.r.* to x may be found similarly by differentiating both sides of (4) *w.r.* to x. Higher order derivatives can be calculated in a similar manner.

Ex. *If* $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$, *find* $\dfrac{d^2y}{dx^2}$.

Here

$$\frac{dx}{d\theta}=a(1-\cos\theta),\ \frac{dy}{d\theta}=a\sin\theta, \text{ and so } \frac{d\theta}{dx}=\frac{1}{a(1-\cos\theta)}.$$

$$\therefore\ \frac{dy}{dx}=\frac{dy}{d\theta}\cdot\frac{d\theta}{dx}=\frac{a\sin\theta}{a(1-\cos\theta)}=\frac{\sin\theta}{1-\cos\theta}.$$

Differentiating both sides *w.r.* to x,

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{\sin\theta}{1-\cos\theta}\right)=\frac{d}{d\theta}\left(\frac{\sin\theta}{1-\cos\theta}\right)\cdot\frac{d\theta}{dx}$$

$$=\frac{(1-\cos\theta)\cos\theta-\sin\theta.\sin\theta}{(1-\cos\theta)^2}\cdot\frac{1}{1-\cos\theta}$$

$$=\frac{\cos\theta-1}{(1-\cos\theta)^3}=-\frac{1}{(1-\cos\theta)^2}.$$

5·12. When y is given as an implicit function of x, that is,

$$f(x,y)=0,$$

the method of procedure is indicated by the solved example below.

Ex. *If* $ax^2+2hxy+by^2=1$, *show that* $\dfrac{d^2y}{dx^2}=\dfrac{h^2-ab}{(hx+by)^3}$.

Differentiating the given equation *w.r.* to x, we get

$$2ax+2h\left(y+x\frac{dy}{dx}\right)+2by\frac{dy}{dx}=0, \text{ whence } \frac{dy}{dx}=-\frac{ax+hy}{hx+by} \qquad \ldots(1)$$

Differentiating both sides of (1) *w.r.* to x, we get

$$\frac{d^2y}{dx^2}=-\frac{(hx+by)\left(a+h\dfrac{dy}{dx}\right)-(ax+hy)\left(h+b\dfrac{dy}{dx}\right)}{(hx+by)^2}$$

$$=\frac{(h^2-ab)(ax^2+2hxy+by^2)}{(hx+by)^3}$$

[substituting the value of $\dfrac{dy}{dx}$ from (1)]

$$=\frac{h^2-ab}{(hx+by)^3}, \text{ since } ax^2+2hxy+by^2=1.$$

EXAMPLES XVII

1. Find the (*i*) fourth derivative of x^9,
 (*ii*) third derivative of $\tan x+\cot x$,
 (*iii*) second derivative of $\{x+\sqrt{(x^2-1)}\}^n$,
 (*iv*) fourth derivative of $x^3 \log x$.

2. Find the first six derivatives of the following functions and guess a formula for the nth derivative where n is a positive integer.

(*i*) e^x, (*ii*) e^{ax}, a constant, (*iii*) $1/x$,
(*iv*) $\log x$ (*v*) $1/(ax+b)$, (*vi*) $\log (3-x)$,
(*vii*) $\cos x$, (*viii*) $\cos^5 x$.

3. If $y=(ax+b)/(cx+d)$, prove that $2y'y'''=3y''^2$.

4. If $y=Ae^{mx}+Be^{nx}$, prove that

$$\frac{d^2y}{dx^2}-(m+n)\frac{dy}{dx}+mny=0.$$

5. If $y=A\cos nx+B\sin nx$, then $\frac{d^2y}{dx^2}+n^2y=0$.

6. If $y=\tan^{-1}\left(\frac{\sqrt{(1+x^2)}-1}{x}\right)+\tan^{-1}\frac{2x}{1-x^2}$, show that

$$y_2=-5x/(1+x^2)^2.$$

7. If $y=\sin(\sin x)$, prove that

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}\tan x+y\cos^2 x=0.$$ (*Delhi, 1953*)

8. If $x=(a+bt)e^{-nt}$, show that

$$\frac{d^2x}{dt^2}+2n\frac{dx}{dt}+n^2x=0.$$

9. If $y=e^{-kx/2}(A\cos nx+B\sin nx)$, then

$$\frac{d^2y}{dx^2}+k\frac{dy}{dx}+\left(n^2+\frac{1}{4}k^2\right)y=0.$$

10. If $y=(A+Bx)\cos kx+(C+Dx)\sin kx$, prove that

$$\frac{d^4y}{dx^4}+2k^2\frac{d^2y}{dx^2}+k^4y=0.$$ (*Panjab, 1931*)

11. For $y=x^ne^{ax}$, $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=0$, find n and a.

12. If $2y=x(1+y_1)$, prove that y_2 is a constant.
[*Hint.* Show that $y_3=0$.]

13. If $p^2=a^2\cos^2\theta+b^2\sin^2\theta$, prove that $p+\frac{d^2p}{d\theta^2}=\frac{a^2b^2}{p^3}$.
(*Rajputana, 1950*)

14. If $x=a\cos\theta$, $y=b\sin\theta$, find $\frac{d^2y}{dx^2}$.

15. If $x=2\cos t-\cos 2t$, $y=2\sin t-\sin 2t$, find the value of $\frac{d^2y}{dx^2}$ when $t=\frac{1}{2}\pi$. (*Panjab, 1937*)

16. If $x=\sin t$, $y=\sin pt$, prove that

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+p^2y=0. \qquad (Panjab,\ 1938)$$

17. If $ax^2+2hxy+by^2+2gx+2fy+c=0$, prove that

$$\frac{d^2y}{dx^2}=\frac{abc+2fgh-af^2-bg^2-ch^2}{(hx+by+f)^3}. \qquad (Panjab,\ 1930)$$

18. Prove that if $y^3-3ax^2+x^3=0$, then $\dfrac{d^2y}{dx^2}+\dfrac{2a^2x^2}{y^5}=0.$

(Delhi, 1954)

19. If $x^3+y^3-3axy=0$, show that $\dfrac{d^2y}{dx^2}=-\dfrac{2a^3xy}{(y^2-ax)^3}.$

20. If $ky=\sin(x+y)$, where k is a constant, prove that

$$y_2=-y(1+y_1)^3.$$

5·2. **Standard results.** The calculation of a derivative of any given order of a function is a straightforward, though usually a tedious, process. But to determine a formula for obtaining the nth derivative, where n is an arbitrary positive integer, of a given function is much more difficult. In some simple cases, we can find such formulae quite easily and we proceed to do so.

I. *To find the nth derivative of* $(\mathbf{ax+b})^{\mathbf{m}}$.

Here
$$y=(ax+b)^m$$
$$y_1=ma(ax+b)^{m-1}$$
$$y_2=m(m-1)a^2(ax+b)^{m-2}$$
$$\dots\dots\dots\dots\dots\dots$$
$$\mathbf{y_n=m(m-1)\dots(m-n+1)a^n(ax+b)^{m-n}}.$$

If m is not a positive integer, then this formula holds for all positive integral values of n.

If, however, m is a positive integer, then

$$y_m=m(m-1)\dots\dots2.1.a^m(ax+b)^0=m!\,a^m$$

is a constant. Hence

$$y_{m+1}=0,\ y_{m+2}=0, \text{ and so on, } y_n=0 \text{ for all } n>m.$$

As a corollary it follows that the nth derivative of a polynomial of degree less than n is zero.

Two particular cases of this formula should be noted :

(*i*) If $\mathbf{y=x^m}$, then $a=1$, $b=0$ and

$$\therefore\quad \mathbf{y_n=m(m-1)\dots(m-n+1)x^{m-n}}.$$

(*ii*) If $y=\dfrac{1}{ax+b}$, then $m=-1$ and

$$\therefore\quad y_n=\frac{(-1)^n(n!)a^n}{(ax+b)^{n+1}}.$$

II. *To find the nth derivative of* **log (ax+b).**

Here
$$y=\log (ax+b)$$
$$y_1=\frac{a}{ax+b}=a(ax+b)^{-1}$$
$$y_2=(-1)a^2(ax+b)^{-2}$$
$$y_3=(-1)^2\, 2a^3(ax+b)^{-3}$$
$$\dots\dots\dots\dots$$
$$y_n=(-1)^{n-1}\frac{(n-1)!\,a^n}{(ax+b)^n}.$$

Cor. If $y=\log x$, then $a=1$, $b=0$ and
$$y_n=(-1)^{n-1}\frac{(n-1)!}{x^n}.$$

III. *To find the nth derivative of* a^x.

Here
$$y=a^x$$
$$y_1=a^x \log a$$
$$y_2=a^x (\log a)^2$$
$$\dots\dots\dots\dots$$
$$y_n=a^x(\log a)^n.$$

Cor. 1. If $y=e^x$, then $a=e$, $\log a=\log e=1$, and
$$\therefore \quad y_n=e^x.$$

Cor. 2. If $y=e^{ax}$, then $y_n=a^n e^{ax}$

IV. *To find the nth derivatives of* **sin (ax+b)** *and* **cos (ax+b).**

Let $y=\sin (ax+b)$ then
$$y_1=a \cos (ax+b)=a \sin (ax+b+\tfrac{1}{2}\pi),$$
$$y_2=a^2 \cos (ax+b+\tfrac{1}{2}\pi)=a^2 \sin (ax+b+\tfrac{1}{2}\pi+\tfrac{1}{2}\pi)$$
$$=a^2 \sin (ax+b+2.\tfrac{1}{2}\pi),$$
$$y_3=a^3 \sin (ax+b+3.\tfrac{1}{2}\pi),$$
$$\dots\dots\dots\dots$$
$$y_n=a^n \sin (ax+b+\tfrac{1}{2}n\pi).$$

Similarly, if $y=\cos (ax+b)$ then
$$y_n=a^n \cos (ax+b+\tfrac{1}{2}n\pi).$$

When $a=1$, $b=0$, we get the particular cases
$$D^n (\sin x)=\sin (x+\tfrac{1}{2}n\pi),\quad D^n (\cos x)=\cos (x+\tfrac{1}{2}n\pi).$$

V. *To find the nth derivatives of* $e^{ax} \sin (bx+c)$ *and* $e^{ax} \cos (bx+c)$.

Let $y=e^{ax} \sin (bx+c)$, then
$$y_1=ae^{ax} \sin (bx+c)+be^{ax} \cos (bx+c).$$

Put $a=r \cos \theta$ and $b=r \sin \theta$, then
$$r^2=a^2+b^2 \text{ and } \tan \theta=b/a.$$

and
$$y_1=re^{ax} [\sin(bx+c) \cos \theta+\cos(bx+c) \sin \theta]$$
$$=re^{ax} \sin(bx+c+\theta).$$

Similarly, $y_2=r^2e^{ax}\sin(bx+c+2\theta)$,

$y_3=r^3e^{ax}\sin(bx+c+3\theta)$,

....

$\mathbf{y_n=r^ne^{ax}\sin(bx+c+n\theta)}$,

where $r=\sqrt{(a^2+b^2)}$ and $\theta=\tan^{-1}(b/a)$.

Similarly, if $y=e^{ax}\cos(bx+c)$, then

$$\mathbf{y_n=r^ne^{ax}\cos(bx+c+n\theta)},$$

where r and θ have the same values.

VI. *To find the nth derivatives of* $\cot^{-1}x$ *and* $\tan^{-1}x$.

Let $y=\cot^{-1}x$, so that $x=\cot y$, then

$$y_1=-\frac{1}{1+x^2}=-\frac{1}{1+\cot^2 y}=-\sin^2 y,$$

$$y_2=-2\sin y\cos y\frac{dy}{dx}=(-1)^2\sin 2y\sin^2 y,$$

$$y_3=(\sin 2y.2\sin y\cos y+\sin^2 y.2\cos 2y)\frac{dy}{dx}$$

$$=2\sin y.\sin 3y.(-\sin^2 y)$$

$$=(-1)^3 .\ 2\,!\sin 3y\sin^3 y.$$

Similarly, $y_4=(-1)^4.3\,!\sin 4y\sin^4 y$, and so on. Hence

$$\mathbf{y_n=(-1)^n(n-1)\,!\sin ny\sin^n y}.$$

Similarly, if $y=\tan^{-1}x$, then

$$\mathbf{y_n=(-1)^{n-1}(n-1)\,!\sin n(\tfrac{1}{2}\pi-y)\sin^n(\tfrac{1}{2}\pi-y)}.$$

5·21. Preliminary simplifications or trigonometric transformations will sometimes reduce a given function to one of the standard forms above and then the nth derivative can be found easily. This is illustrated in the following solved examples.

Ex. 1. *If* $y=\sin^2 x\cos^3 x$, *find* y_n.

Here $y=\sin^2 x\cos^2 x\cos x=\frac{1}{4}\sin^2 2x\cos x$

$$=\tfrac{1}{8}(1-\cos 4x)\cos x=\tfrac{1}{8}\cos x-\tfrac{1}{8}\cos x\cos 4x$$

$$=\tfrac{1}{16}(2\cos x-\cos 3x-\cos 5x)$$

Each of the terms is now in the standard form IV. Hence, applying the result of that form to each of the terms,

$$y_n=\tfrac{1}{16}\{2\cos(x+\tfrac{1}{2}n\pi)-3^n\cos(3x+\tfrac{1}{2}n\pi)-5^n\cos(5x+\tfrac{1}{2}n\pi)\}.$$

Ex. 2. *Find the nth derivative of* $e^x\sin x\sin 2x$.

Here $\sin x\sin 2x=\frac{1}{2}(\cos x-\cos 3x)$ and so

$$y=\tfrac{1}{2}e^x\cos x-\tfrac{1}{2}e^x\cos 3x.$$

Each of the two terms is now in the standard form V. Hence, applying the result of that form,

$$y_n=\tfrac{1}{2}(\sqrt{2})^ne^x\cos(x+n\tan^{-1}1)-\tfrac{1}{2}(\sqrt{10})^ne^x\cos(3x+n\tan^{-1}3)$$

$$=\tfrac{1}{2}e^x\{(\sqrt{2})^n\cos(x+\tfrac{1}{4}n\pi)-(\sqrt{10})^n\cos(3x+n\tan^{-1}3)\}.$$

EXAMPLES XVIII

Find the nth derivatives of :

1. $(2x+3)^{2m}$. **2.** $\dfrac{1}{1+x}$. **3.** $\dfrac{1}{a-x}$. **4.** $\dfrac{1}{a-bx}$.

5. $\dfrac{1}{(2x+3)^2}$. **6.** $\dfrac{x+2}{3x+7}$. **7.** $\sqrt{(x+a)}$. **8.** $\log(x^2)$.

9. e^{2x}. **10.** $\sin x \sin 2x$. **11.** $a \sin^2 x + b \cos^2 x$.

12. $\cos x \cos 2x \cos 3x$. (*Banaras, 1945*) **13.** $\sin^2 x$.

14. $\cos^4 x$. (*Panjab, 1936*) **15.** $e^x \cos^3 x$. **16.** $e^{3x} \sin^2 x \cos^3 x$.

17. $e^x \cos x \cos 2x$. **18.** $e^x \cos^3 x$.

19. $e^x (\cos x + \sin x)$. **20.** $e^{x \cos \alpha} \cos (x \sin \alpha)$

21. Show that the nth derivative of $y = \tan^{-1} x$ is

$$(-1)^n (n-1)!\, \sin n(\tfrac{1}{2}\pi - y) \sin^n (\tfrac{1}{2}\pi + y). \qquad (\textit{Panjab, 1935})$$

22. If $y = \tan^{-1} x$, show that

$$y_n = (n-1)!\, \cos \{ny + \tfrac{1}{2}\pi(n-1)\} \cos^n y.$$

5·3. **Use of partial fractions.** If the denominator of a rational fraction consists entirely of repeated or unrepeated linear factors then its nth derivative can be found by putting it into partial fractions and applying the result of standard form I of Art 5·2.

Ex. 1. *Find the nth derivative of* $\dfrac{2x-1}{(x-2)(x+1)}$.

Let $y = \dfrac{2x-1}{(x-2)(x+1)} \equiv \dfrac{A}{x-2} + \dfrac{B}{x+1}$, then

$$2x-1 \equiv A(x+1) + B(x-2).$$

Putting $x=2$ and $x=-1$ in succession, we get

$$A = B = 1.$$

Hence
$$y = \frac{1}{x-2} + \frac{1}{x+1},$$

and
$$\therefore \quad y_n = \frac{(-1)^n (n!)}{(x-2)^{n+1}} + \frac{(-1)^n (n!)}{(x+1)^{n+1}}.$$

Ex. 2. *If* $y = \dfrac{x^2}{(x-1)^3 (x-2)}$, *find* y_n.

Let
$$\frac{x^2}{(x-1)^3 (x-2)} \equiv \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{x-2}.$$

Multiplying both sides by $(x-1)^3 (x-2)$, we get

$$x^2 = A(x-1)^2 (x-2) + B(x-1)(x-2) + C(x-2) + D(x-1)^3. \quad \dots(1)$$

Putting $x=1$ and 2 in succession in (1), we obtain

$$C = -1 \text{ and } D = 4.$$

Equating the coefficients of x^3 and the constant term on both sides of (1), we get

$$A+D=0, \text{ whence } A=-D=-4\ ;$$

and $$-2A+2B-2C-D=0,$$

whence $$B=A+C+\tfrac{1}{2}D=-4-1+2=-3.$$

Hence $$y=-\frac{4}{x-1}-\frac{3}{(x-1)^2}-\frac{1}{(x-1)^3}+\frac{4}{x-2},$$

and $$\therefore\ y_n=(-1)^{n+1}\left[\frac{4(n\,!)}{(x-1)^{n+1}}+\frac{3(n+1)\,!}{(x-1)^{n+2}}+\frac{(n+2)\,!}{2(x-1)^{n+3}}-\frac{4(n\,!)}{(x-2)^{n+1}}\right]$$

5·31. Use of auxiliary functions. This is illustrated in the solved examples below.

Ex 1 *Find y_n if* (*i*) $y=\dfrac{1}{x^2+a^2}$, (*ii*) $y=\dfrac{x}{x^2+a^2}$

(*i*) Put $x=a\cot\theta$, so that

$$\frac{dx}{d\theta}=-a\operatorname{cosec}^2\theta \text{ and } \frac{d\theta}{dx}=-\frac{1}{a}\sin^2\theta.$$

With this substitution, we get

$$y=\frac{1}{a^2}\sin^2\theta.$$

Differentiating this *w.r.* to x successively and replacing $d\theta/dx$ at each differentiation by its value found above, we have

$$y_1=\frac{dy}{dx}=\frac{2}{a^2}\sin\theta\cos\theta\frac{d\theta}{dx}$$

$$=-\frac{2}{a^3}\sin^3\theta\cos\theta=-\frac{1}{a^3}\sin 2\theta\sin^2\theta,$$

$$y_2=-\frac{1}{a^3}(\sin 2\theta.2\sin\theta\cos\theta+2\cos 2\theta\sin^2\theta)\frac{d\theta}{dx}$$

$$=\frac{2}{a^4}(\sin 2\theta\cos\theta+\cos 2\theta\sin\theta)\sin^3\theta$$

$$=\frac{(-1)^2}{a^4}(2\,!)\sin 3\theta\sin^3\theta.$$

Similarly,

$$y_3=\frac{(-1)^3}{a^5}(3\,!)\sin 4\theta\sin^4\theta,$$

$$y_4=\frac{(-1)^4}{a^6}(4\,!)\sin 5\theta\sin^5\theta, \text{ and so on.}$$

Hence

$$y_n=\frac{(-1)^n(n\,!)}{a^{n+2}}\sin(n+1)\theta\sin^{n+1}\theta$$

(*ii*) The same substitution gives

$$y=\frac{1}{a}\sin\theta\cos\theta$$

$$y_1=-\frac{1}{a}(\cos^2\theta-\sin^2\theta)\frac{d\theta}{dx}=-\frac{1}{a^2}\cos 2\theta\sin^2\theta.$$

Proceeding as in (*i*) we can show that

$$y_n=\frac{(-1)^n(n\,!)}{a^{n+1}}\cos(n+1)\theta\sin^{n+1}\theta.$$

Cor If we replace x by $x+b$, where b is a constant, we obtain the following results :

(*i*) If $y=\dfrac{1}{(x+b)^2+a^2}$, then

$$_n=\frac{(-1)^n(n\,!)}{a^{n+1}}\sin(n+1)\theta\sin^{n+1}\theta,$$

and (*ii*) if $y=\dfrac{x+b}{(x+b)^2+a^2}$, then

$$y_n=\frac{(-1)^n(n\,!)}{a^{n+1}}\cos(n+1)\theta\sin^{n+1}\theta,$$

where $x+b=a\cot\theta$.

Ex. 2. *Find the nth derivative of* $1/(x^3+1)$.

Here $$y=\frac{1}{x^3+1}=\frac{1}{(x+1)(x^2-x+1)}$$

$$=\frac{1}{3}\frac{1}{x+1}-\frac{1}{3}\frac{x-2}{x^2-x+1}.$$

$$=\frac{1}{3}\frac{1}{x+1}-\frac{1}{3}\frac{x-\frac{1}{2}}{(x-\frac{1}{2})^2+\frac{3}{4}}+\frac{1}{2}\frac{1}{(x-\frac{1}{2})^2+\frac{3}{4}}.$$

Hence, by the corollary to Ex. 2, we have

$$y_n=\frac{(-1)^n(n\,!)}{3(x+1)^{n+1}}-\frac{(-1)^n(n\,!)}{3}\left(\frac{2}{\sqrt{3}}\right)^{n+1}\{\cos(n+1)\theta-\sqrt{3}\sin(n+1)\theta)\}$$
$$\sin^{n+1}\theta,$$

where $\tan\theta=\sqrt{3}/(2x-1)$.

EXAMPLES XIX

Find the nth derivatives of :

1. $\dfrac{x}{x^2-3x+2}$. 2. $\dfrac{1}{2-x-x^2}$. 3. $\dfrac{x^2+4x+1}{x^3+2x^2-x-2}$.

4. $\dfrac{2x+1}{(x+2)(x-3)^2}$. 5. $\dfrac{1}{x^4-a^4}$. 6. $\dfrac{1}{x^2+x+1}$.

7. $\dfrac{1}{(x+1)(x^2+1)}$. 8. $\tan^{-1}x$. (*Patna, 1945.*) 9. $\cot^{-1}\dfrac{x}{a}$.

10. $\dfrac{x}{x^4+x^2+1}$. 11. $\tan^{-1}\dfrac{2x}{1-x^2}$. (*Panjab, 1949 ; Agra, 1950*)

12. $\tan^{-1}\dfrac{x\sin\alpha}{1-x\cos\alpha}$.

5·4. Leibnitz's Theorem for the nth derivative of the product of two functions of x. *If $y=uv$, where u and v are functions of x having derivatives of any desired order, then*

$$y_n={}^nC_0u_nv+{}^nC_1u_{n-1}v_1+{}^nC_2v_{n-2}v_2+\dots\dots\dots+{}^nC_ru_{n-r}v_r+\dots\dots+{}^nC_nuv_n,$$

where suffixes of u and v denote differentiations with respect to x, and the symbol ${}^nC_r$ denotes the number of combinations of n different things taken r at a time.

The theorem is proved easily by induction. Suppose that it is true for any *particular* value of n, so that for this value the given equation is true, then differentiating this equation once with respect to x, we get

$$\begin{aligned}y_{n+1}&={}^nC_0(u_{n+1}v+u_nv_1)+{}^nC_1(u_nv_1+u_{n-1}v_2)+\dots\dots\\&\qquad+{}^nC_r(n_{n-r+1}v_r+u_{n-r}v_{r+1})+\dots+{}^nC_n(u_1v_n+uv_{n+1})\\&={}^nC_0u_{n+1}v+({}^nC_0+{}^nC_1)u_nv_1+({}^nC_1+{}^nC_2)u_{n-1}v_2+\dots\dots\\&\qquad+({}^nC_{r-1}+{}^nC_r)u_{n-r+1}v_r+\dots\dots+{}^nC_nuv_{n+1}\end{aligned}$$

Now ${}^nC_{r-1}+{}^nC_r={}^{n+1}C_r$, therefore putting $r=1$, 2, etc.,

$$\begin{gathered}{}^nC_0+{}^nC_1={}^{n+1}C_1,\\{}^nC_1+{}^nC_2={}^{n+1}C_2,\\\dots\dots\dots\dots\dots\dots\dots\end{gathered}$$

and so on. Also ${}^nC_0=1={}^{n+1}C_0$ and ${}^nC_n=1={}^{n+1}C_{n+1}$. Hence

$$\begin{aligned}y_{n+1}&={}^{n+1}C_0u_{n+1}v+{}^{n+1}C_1u_nv_1+{}^{n+1}C_2u_{n-1}v_2+\dots\dots\\&\qquad+{}^{n+1}C_ru_{n+1-r}v_r+\dots\dots+{}^{n+1}C_{n+1}uv_{n+1},\end{aligned}$$

which is of exactly the same form as the given formula with n replaced by $n+1$. Hence if the formula is true for any particular value of n, then it is also true for the next higher value $n+1$.

But, by actual differentiation,

$$y_1=u_1v+uv_1$$

$$y_2=u_2v+2u_1v_1+uv_2={}^2C_0u_2v+{}^2C_1u_1v_1+{}^2C_2uv_2$$

and so the theorem is true for $n=2$. Hence by the above reasoning it is true for $n=2+1$, *i.e.*, for $n=3$, and, therefore, for $n=3+1$, *i.e.*, for $n=4$ and so on. Hence the theorem is true for all positive integral values of n.

In practice, the theorem can be gainfully employed if we know the formula for the nth derivative of each factor of the product. If x^m, where m is a positive integer, is one of the two factors, then taking $v=x^m$ simplifies the writing of the nth derivative because $v_{m+1}=v_{m+2}=\dots\dots=0$.

Ex. 1. *Find the nth derivative of $x^2 \sin ax$.*

Let $u=\sin ax$, $v=x^2$, then v_3, v_4, etc., are all zero and therefore

$$\begin{aligned}D^n(x^2\sin ax)&=x^2D^n(\sin ax)+{}^nC_1.2xD^{n-1}(\sin ax)+{}^nC_2\,2.D^{n-2}(\sin ax)\\&=x^2a^n\sin(ax+\tfrac{1}{2}n\pi)+2nxa^{n-1}\sin\{ax+\tfrac{1}{2}(n-1)\pi\}\\&\qquad+n(n-1)a^{n-2}\sin\{ax+\tfrac{1}{2}(n-2)\pi\}.\end{aligned}$$

Ex. 2. *Prove that*

$$D^n\left(\frac{\log x}{x}\right)=\frac{(-1)^n(n!)}{x^{n+1}}\left(\log x-1-\tfrac{1}{2}-\tfrac{1}{3}-\ldots\ldots-\frac{1}{n}\right)$$

Let $u=\dfrac{1}{x}$ and $v=\log x$, then

$$u_1=\frac{-1}{x^2},\quad u_2=\frac{(-1)^2(2\,!)}{x^3},\ldots\ldots,\; u_n=\frac{(-1)^n(n\,!)}{x^{n+1}},$$

and $$v_1=\frac{-1}{x},\; v_2=\frac{1}{x^2},\ldots\ldots\quad v_n=\frac{(-1)^{n-1}(n-1)!}{x^n}.$$

Therefore $D^n\left(\dfrac{\log x}{x}\right)$

$$=\frac{(-1)^n(n\,!)}{x^{n+1}}\log x+n\cdot\frac{(-1)^{n-1}(n-1)!}{x^n}\cdot\frac{1}{x}$$
$$+\frac{n(n-1)}{2\,!}\;\frac{(-1)^{n-2}(n-2)\,!}{x^{n-1}}\;\frac{-1}{x^2}+\ldots\ldots$$
$$+\frac{n(n-1)\ldots(n-r+1)}{r\,!}\;\frac{(-1)^{n-r}(n-r)\,!}{x^{n-r+1}}\cdot\frac{(-1)^{r-1}(r-1)\,!}{x^r}+\ldots$$
$$+\frac{1}{x}\cdot\frac{(-1)^{n-1}(n-1)\,!}{x^n}$$
$$=\frac{(-1)^n(n\,!)}{x^{n+1}}\left(\log x-1-\tfrac{1}{2}-\tfrac{1}{3}-\ldots-\frac{1}{n}\right).$$

Ex. 3. *Differentiate n times the equation*

$$(1-x^2)y_2-xy_1+a^2y=0.$$ *(Allahabad, 1950)*

We differentiate each of the terms by Leibnitz's theorem and then add up the results and equate to zero. Here

$$D^n\{(1-x^2)y_2\}=(1-x^2)y_{n+2}+n(-2x)y_{n+1}+\frac{n(n-1)}{1\cdot 2}(-2)y_n$$
$$D^n(-xy_1)=-xy_{n+1}-n.\,1.y_n$$
$$D^n(a^2y)=a^2y_n.$$

Hence adding up and equating to zero, the required result is

$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}+(a^2-n^2)y_n=0.$$

5·5. **Calculation of the value of the nth derivative for $x=0$.** In many cases, though it may not be possible to find a formula for the nth derivative of a given function, yet we may be able, with the help of Leibnitz's theorem, to find the value of the nth derivative for $x=0$. This is illustrated by the following solved example.

Ex. *If $y=(\sin^{-1}x)^2$, prove that $(1-x^2)y_2-xy_1=2$ and hence find the value of y_n at $x=0$.*

Since $y=(\sin^{-1}x)^2$, $\quad\therefore\; y_1=\dfrac{2\sin^{-1}x}{\sqrt{(1-x^2)}},$

or $$(1-x^2)\,y_1^2=4(\sin^{-1}x)^2=4y.$$

Differentiating this equation with respect to x, we get

$$(1-x^2)2y_1y_2-2xy_1^2=4y_1$$

or cancelling out the common factor $2y_1$, we get

$$(1-x^2)y_2-xy_1=2. \qquad \dots (1)$$

Differentiating this equation n times by **Leibnitz's theorem**, we obtain

$$(1-x^2)y_{n+2}-2nxy_{n+1}-n(n-1)y_n-xy_{n+1}-ny_n=0,$$

i.e.,

$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-n^2y_n=0. \qquad \dots (2)$$

Put $x=0$ in this relation and let $(y_n)_0$ denote the value of y_n for $x=0$, then

$$(y_{n+2})_0-n^2(y_n)_0=0 \quad \text{or} \quad (y_{n+2})_0=n^2(y_n)_0. \qquad (3)$$

This gives us a recurrence formula connecting the values of y_n and y_{n+2} for $x=0$. From this we can obtain the values of the successive derivatives for $x=0$, if we know the values of y, y_1 and y_2 for $x=0$. Now putting $x=0$ in the original equation, in its derivative and in (1), we get

$$(y)_0=0, \quad (y_1)_0=0 \quad \text{and} \quad (y_2)_0=2.$$

Since $(y_1)_0=0$, by putting $n=1, 3, 5, \dots\dots$ in succession in (3), we find that

$$(y_1)_0=(y_3)_0=\dots\dots=(y_{2m+1})_0=\dots\dots=0.$$

Again by putting $n=2, 4, 6, \dots\dots$ in succession in (3), we get

$$(y_4)_0=2^2(y_2)_0=2.2^2$$
$$(y_6)_0=4^2(y_4)_0=2.2^2.4^2$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$(y_{2m})_0=(2m-2)^2(y_{2m-2})_0$$
$$=2.2^2.4^2\dots\dots(2m-2)^2.$$

Hence when $x=0$, $y_n=0$ when n is odd and $y_n=2.2^2.4^2 \dots\dots\dots\dots (2m-2)^2$ when n is even and equal to $2m$.

It may be remarked that it is necessary for the validity of the above process that all the derivatives be continuous for $x=0$.

EXAMPLES XX

Find the nth derivatives of :

1. xe^{ax}.   2. x^2a^x.   3. $x^2\cos x$.   4. $x^3\sin ax$.
5. $x^2\log x$.   6. $e^x\log x$. (*Panjab, 1956*)   7. $x\tan^{-1}x$.
8. $xe^{ax}\sin bx$.

9. If $y=x^2e^x$, show that

$$\frac{d^ny}{dx^n}=\frac{1}{2}n(n-1)\frac{d^2y}{dx^2}-n(n-2)\frac{dy}{dx}+\frac{1}{2}(n-1)(n-2)y.$$

(*Delhi, 1955, ; Agra, 1949*)

10. If $y=x^3e^x\cos x$, prove that

$$y_n=2^{(n-2)/2}e^x\{2x(n+x)\cos(x+\tfrac{1}{4}n\pi)+n(2x+n-1)\sin(x+\tfrac{1}{4}n\pi)\}.$$

11. If $u=\tan^{-1}x$, prove that

$$(1+x^2)\frac{d^2u}{dx^2}+2x\frac{du}{dx}=0,$$

and hence determine the values of all the derivatives of u when $x=0$. (*M.T.*)

12. If $y=x\log x$, prove that, for $n>1$,

$$y_n=(-1)^n(n-2)!/x^{n-1}.$$

13. If $y=x^{n-1}\log x$, prove that $xy_n=(n-1)!$

14. If $y=A\{x+\sqrt{(x^2-1)}\}^n+B\{x-\sqrt{(x^2-1)}\}^n$, prove that

$$(x^2-1)y_2+xy_1-n^2y=0.$$

Differentiate this equation n times by Leibnitz's theorem.

15. If $y=e^{x\cos\alpha}\sin(x\sin\alpha)$, prove that

$$y_{n+2}-2y_{n+1}\cos\alpha+y_n=0.$$

16. If $y=\sin(m\sin^{-1}x)$, prove that

$$(1-x^2)y_2-xy_1+m^2y=0, \qquad (\textit{Agra, 1948})$$

and $$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2-m^2)y_n=0. \qquad (\textit{Delhi, 1954})$$

17. If $y=e^{a\sin^{-1}x}$, show that

$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+a^2)y_n=0. \qquad (\textit{Panjab, 1951})$$

18. If $y=a\cos(\log x)+b\sin(\log x)$, show that

$$x^2y_2+xy_1+y=0$$

and $$x^2y_{n+2}+(2n+1)xy_{n+1}+(n^2+1)y_n=0.$$

(*Aligarh, 1948*; *Delhi, 1952*)

19. If $y^{1/m}+y^{-1/m}=2x$, prove that

$$(x^2-1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0,$$

where y_n denotes the nth derivative of y.

(*Panjab, 1959*; *Bombay, 1947*)

20. If $y=\dfrac{\sin^{-1}x}{\sqrt{(1-x^2)}}$, prove that, for $x=0$, $y_n=(n-1)^2y_{n-2}$.

21. If $y=[\log\{x+\sqrt{(1+x^2)}\}]^2$, prove that

$$(1+x^2)y_2+xy_1=2.$$

Hence find the value of y_n at $x=0$.

22. If $y=\{x+\sqrt{(1+x^2)}\}^m$, find $(y_n)_0$.

(*Lucknow, 1950*; *Agra, 1951*)

23. Show that if

$$u=\sin nx+\cos nx,$$

then $$u_r=n^r\{1+(-1)^r\sin 2nx\}^{1/2},$$

where u_r denotes the rth derivative of u with respect to x.

(*Agra, 1943*)

24. If $y=e^{-x}\cos x$, show that the value of y_n at $x=0$ is

$$2^{n/2}\cos\tfrac{3}{4}n\pi$$

25. Show that the nth derivative of $(e^x-e^{-x})\sin x$ vanishes for $x=0$ unless n is of the form $4m+2$ when its value is

$$(-1)^m2^{2m+2}.$$

MISCELLANEOUS EXAMPLES I

1. Explain, giving suitable examples, the distinction between the value of a function $f(x)$ for $x=a$, and the limit of $f(x)$ for $x=a$.

(*Allahabad, 1950*)

2. Define the limit of a function as x tends to a. Use your definition to find the limit of $(x^2-3x+2)/(x-2)$ as x tends to 2.

(*Delhi, 1956*)

3. Explain what you understand by the term limit of a function at a point. For the functions

(*i*) $f(x)=\sin(1/x)$ when $x\neq 0$, $f(0)=0$,
(*ii*) $f(x)=x\cos(1/x)$ when $x\neq 0$, $f(0)=0$,

ivestigate the existence of limit and continuity at $x=0$.

(*Panjab, 1946*)

4. The function y of x is defined as follows :

$$\begin{aligned} y &= 5x-4 \quad \text{when } 0<x\leqslant 1 \\ &= 4x^2-3x \text{ when } 1<x<2. \end{aligned}$$

Examine whether or not it is continuous at $x=1$. (*Panjab, 1959*)

5. Let $f(x)=[x]$, where $[x]$ represents the integral part of x, show that $f(x)$ is discontinuous for all integral values of x. Draw its graph.

[Let x be any real number. Write it as $x=n+\theta$, where n is an integer and θ is a positive proper fraction so that $0\leqslant\theta<1$. Then n is called the *integral part* of x and we write $n=[x]$.]

6. Prove that $f(x)=x-[x]$ is dicontinuous for all integral values of x. Draw its graph.

7. Examine whether or not the function

$$\begin{aligned} f(x) &= (\sin 2x)/x \qquad \text{when } x\neq 0, \\ &= 1 \qquad\qquad\quad \text{when } x=0 \end{aligned}$$

is continuous at $x=0$. (*Panjab, 1958*)

8. Find from definition the derivatives of the following :—

(*i*) $\tan(1/x)$. (*ii*) $\sqrt{(a^2x^2+b^2)}$.

Find dy/dx from *first principles* when

$$ax^2+2hxy+by^2=1.$$

(*Panjab, 1953*)

9. Find $\dfrac{dy}{dx}$ when $x=\dfrac{3at}{1+t^3}$, $y=\dfrac{3at^2}{1+t^3}$. (*Delhi, 1959*)

Find the deriavtives of :

10. $\cot^{-1}\dfrac{\sqrt{(1+\sin x)}+\sqrt{(1-\sin x)}}{\sqrt{(1+\sin x)}-\sqrt{(1-\sin x)}}$.

11. $\tan^{-1}\dfrac{\sqrt{(1+x^2)}-\sqrt{(1-x^2)}}{\sqrt{(1+x^2)}+\sqrt{(1-x^2)}}$.

12. $\sin^{-1}\dfrac{x^2}{\sqrt{(x^4+a^4)}}$.

13. $\tan^{-1}\dfrac{1}{\sqrt{(x^2-1)}}$.

14. $\sin\left\{2\tan^{-1}\sqrt{\left(\frac{1-x}{1+x}\right)}\right\}$. 15. $\log_{\cos x}\sin x$.

16. Differentiate (*i*) $\tanh^{-1}\frac{x^2-1}{x^2+1}$ *w.r.t.* $\log x$.

(*ii*) $\log_{10}x$ *w.r.t.* $\log_x 10$.

17. If $x=a\cos^{-1}\sqrt{\left(\frac{y}{a}\right)}+\sqrt{(ay-y^2)}$, find $\frac{dy}{dx}$.

18. Find $\frac{dy}{dx}$ if :

(*i*) $y=x+\frac{1}{x+}\,\frac{1}{x+}$.........*ad inf*

(*ii*) $y=\frac{\sqrt{x}}{1+}\,\frac{\sqrt{x}}{1+}\,\frac{\sqrt{x}}{1+}$.........*ad inf.*

(*iii*) $y=(\cos x)^{(\cos x)^{(\cos x)\ldots\ldots\ldots ad\ inf.}}$

(*iv*) $y=x^{x^{x\ldots\ldots\ldots ad\ inf}}$

19. If $\sinh x\tan y=1$, prove that $\frac{dy}{dx}=-\sin y$

20. If $x\sqrt{(1+y)}+y\sqrt{(1+x)}=0$, prove that

$$\frac{dy}{dx}=-\frac{1}{(1+x)^2}.$$ (*Agra, 1943*)

21. If $\sqrt{(1-x^2)}+\sqrt{(1-y^2)}=k\sqrt{(x-y)}$, prove that

$$\frac{dy}{dx}=\frac{\sqrt{(1-y^2)}}{\sqrt{(1-x^2)}}.$$ (*Panjab, 1952*)

22. Differentiate

(*i*) $x^{\sin x}$ *w.r.* to $(\sin x)^x$. (*Panjab, 1959*)

(*ii*) $(\log x)^{\tan x}$ *w.r.* to $\sin(m\cos^{-1}x)$. (*Panjab, 1958*)

(*iii*) $e^{\tan x}$ *w.r.* to $\sin x$. (*Delhi, 1954*)

(*iv*) $\sin^2 x$ *w r.* to $(\log x)^2$. (*Delhi, 1952*)

(*v*) $\tan^{-1}\sqrt{\left(\frac{1-x^2}{1+x^2}\right)}$ *w.r.* to $\cos^{-1}x^2$. (*Panjab, 1952 S*)

23. (*i*) Let $f(x)=\sin x$, find $f'(\frac{1}{2}\pi)$ from definition.

(*ii*) If $f(x)=(x^3-8x^2+13x-6)/(x^2-11x+10)$, find for which values of x, $f'(x)$ vanishes.

24. Given $(1+x)^n=1+{}^nC_1x+{}^nC_2x^2+\ldots\ldots+{}^nC_nx^n$, ($n$, a positive integer greater than 1), using rules of differentiation, show that

$${}^nC_1-2\,{}^nC_2+3\,{}^nC_3-\ldots+(-1)^n n\,{}^nC_n=0.$$ (*Calcutta, 1958*)

25. Prove that the value of the nth derivative of $x^2/(x^2-1)$ for $x=0$, is zero if n is even, and is $-(n\,!)$ if n is odd and greater than 1. (*Panjab, 1949*)

26. If $\cos^{-1}(y/b)=\log(x/a)^n$, prove that

$$x^2y_{n+2}+(2n+1)xy_{n+1}+2n^2y_n=0$$

27. Prove that the nth derivative of $\tan x$ is a ploynomial in $\tan x$ of degree $n+1$.

If $f(x)=\tan x$, prove that

$$f^n(0)-{}^nC_2f^{n-2}(0)+{}^nC_4f^{n-4}(0)-\ldots=\sin \tfrac{1}{2}n\pi. \qquad \textit{(Panjab, 1936 S)}$$

28. Find the nth derivative of $\dfrac{1}{1+x+x^2+x^3}$ and show that for $x=0$, it is zero when n is of the form $4p+2$ or $4p+3$ and it is $(n\,!)$ or $-(n\,!)$ according as n is of the form $4p$ or $4p+1$.

(Panjab, 1952)

29. By forming in two different ways the nth derivative of x^{2n}, prove that

$$1+\frac{n^2}{1^2}+\frac{n^2(n-1)^2}{1^2\,2^2}+\frac{n^2(n-1)^2(n-2)^2}{1^2\,2^2.3^2}+\ldots\ldots=\frac{(2n)\,!}{(n\,!)^2}.$$

(Lucknow, 1944)

30. If $I_n=D^n(x^n \log x)$, prove that

$$I_n=nI_{n-1}+(n-1)\,!$$

and hence show that

$$I_n=n\,!\,(\log x+1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots+\frac{1}{n}).$$

APPENDIX

A.1. Infinitesimals. *A* **variable** *which tends to zero is called an* **infinitesimal**. The infinitesimal of the calculus is different from the infinitesimal of ordinary speech. In ordinary speech, when we talk of an infinitesimal quantity, we usually mean a fixed quantity whose value is exceedingly small so as to be almost negligible. The infinitesimal of the calculus is not a fixed quantity ; it is a variable quantity which is approaching the limit zero. Thus $\sin x$ when $x\to 0$ is an infinitesimal. Also the increment δx of x in the definition of the derivative is an infinitesimal.

In order to compare the relative magnitudes of several related infinitesimals, we usually select one of them as the **principal infinitesimal** and measure the magnitudes of the other infinitesimals by considering the limits of the ratio of these infinitesimals to the principal infinitesimal. Thus, if α be the principal infinitesimal and β any other related infinitesimal, then *α and β are said to be infinitesimal of the same order if* Lt β/α *is a non-zero finite quantity. If, however,* Lt $\beta/\alpha=0$, *then β is said to be an infinitesimal of higher order as compared to α* and *α is said to be of lower order as compared to β.* If, further, Lt β/α^n is a non-zero finite quantity, then β is said to be an infinitesimal of the nth order as compared to α.

Ex. 1. *Show that as* $\theta \to 0$, θ, $\sin\theta$ *and* $\tan\theta$ *are all three infinitesimals of the same order.*

Since $\underset{\theta\to 0}{\text{Lt}} \frac{\sin\theta}{\theta} = 1$ and $\underset{\theta\to 0}{\text{Lt}} \frac{\tan\theta}{\theta} = 1$, it follows that $\sin\theta$ and $\tan\theta$ are infinitesimals of the same order as θ. Hence also $\sin\theta$ and $\tan\theta$ are infinitesimals of the same order. Hence all three are of the same order.

Ex. 2. *Show that as* $\theta \to 0$, $1-\cos\theta$ *is an infinitesimal of the second order with respect to* θ.

For
$$\underset{\theta\to 0}{\text{Lt}} \frac{1-\cos\theta}{\theta^2} = \underset{\theta\to 0}{\text{Lt}} \frac{2\sin^2\frac{1}{2}\theta}{\theta^2}$$
$$= \underset{\theta\to 0}{\text{Lt}} \frac{1}{2}\left(\frac{\sin\frac{1}{2}\theta}{\frac{1}{2}\theta}\right)^2 = \frac{1}{2}.$$

Hence the result.

A 11. Principal part of an infinitesimal. If an infinitesimal consists of a sum of several infinitesimals of different orders, then the terms consisting of the lowest order infinitesimals are called the **principal part** of the infinitesimal. Thus the principal part of the infinitesimal $5x+7x^3+3x^4$ as $x\to 0$ is $5x$.

Two infinitesimals α and β are said to be **equivalent if** $\text{Lt}\ \alpha/\beta = 1$. It is easy to see that two infinitesimals are equivalent if, and only if, they have the same principal parts. For, if δ be the common principal part of α and β, then $\alpha = \delta +$ infinitesimals of higher orders and similarly for β. Hence

$$\text{Lt}\ \frac{\alpha}{\beta} = \text{Lt}\ \frac{\delta + \text{higher order infinitesimals}}{\delta + \text{higher order infinitesimals}} = 1.$$

Conversely, if $\text{Lt}\ \alpha/\beta = 1$, then we can write $\alpha/\beta = 1+\varepsilon$, where $\varepsilon\to 0$ and $\beta\to 0$. Hence $\alpha = \beta + \beta\varepsilon$. Hence $\beta\varepsilon$ is obviously of higher other than β. Hence it follows that α and β have the same principal parts.

In particular, it follows that an infinitesimal and its principal parts are equivalent.

A.12. Theorems on infinitesimals. (*i*) *The limit of the ratio of two infinitesimals* α *and* β *is unaltered when these are replaced by two equivalent infinitesimals* α_1 *and* β_1.

For $\text{Lt}\ \alpha/\alpha_1 = 1$ and $\text{Lt}\ \beta/\beta_1 = 1$, therefore

$$\text{Lt}\ \frac{\alpha_1}{\beta_1} = \text{Lt}\ \frac{\alpha}{\beta}\cdot\frac{\alpha_1}{\alpha}\cdot\frac{\beta}{\beta_1}.$$
$$= \text{Lt}\ \frac{\alpha}{\beta}\cdot \text{Lt}\ \frac{\alpha_1}{\alpha}\cdot \text{Lt}\ \frac{\beta}{\beta_1} = \text{Lt}\ \frac{\alpha}{\beta}.$$

In particular, the theorem is true when α and β are replaced by their principal parts respectively.

(*ii*) *In any sum or quotient of infinitesimals, we can replace each term by its principal part without affecting the limits in any way.*

This follows easily from the previous result.

A.2. Differential of a function. Let $y=f(x)$ have a finite derivative $f'(x)$ at a point x, then

$$\underset{x\to 0}{\text{Lt}} \frac{\delta y}{\delta x}=f'(x) \qquad \text{or} \qquad \frac{\delta y}{\delta x}=f'(x)+\varepsilon,$$

where ε is a quantity which $\to 0$ as $\delta x\to 0$. Hence

$$\delta y=f'(x)\delta x+\varepsilon\delta x. \qquad \ldots(1)$$

Now δy is an infinitesimal whose principal part is $f'(x)\delta x$, for $f'(x)$ is a finite quantity while $\varepsilon\to 0$ as $\delta x\to 0$ and, therefore, $\varepsilon\delta x$ is an infinitesimal of higher order as compared to $f'(x)\,\delta x$. *The principal part of δy or the part that is linear in δx, viz., $f'(x)\,\delta x$, is called the* **differential** *of y or $f(x)$* and is denoted by the symbol dy. Thus

$$dy=f'(x)\delta x. \qquad \ldots(2)$$

In the particular case when $f(x)=x$, then $y=x$, $\delta y=\delta x$ and so $dy=\delta x$, *i.e.*, $dx=\delta x$ since $y=x$. Hence the differential of the independent variable x is equal to its arbitrary increment δx. The equation (2) can now be written as

$$dy=f'(x)\,dx, \qquad \ldots(3)$$

i.e., the differential of $y=\dfrac{dy}{dx}\times$the differential of x.

The derivative $f'(x)$ is the coefficient of dx in the differential of $f(x)$, and is for this reason also called the **differential coefficient** of $f(x)$. If we divide (3) by dx we get

$$\frac{dy}{dx}=f'(x)$$

Hence the derivative of a function is also equal to the ratio of the differential of the function to that of the independent variable. Hence, if the derivative of a function is finite, the symbol $\dfrac{dy}{dx}$ may be regarded either as one single indivisible entity standing for the derivative of y or it may be regarded as the ratio of the two differentials dy and dx.

Ex. *If $y=x^3+x$, find the differential dy.*

Since $y=x^3+x$, $y+\delta y=(x+\delta x)^3+x+\delta x$, and so

$\delta y=(x+\delta x)^3+x+\delta x-x^3-x=(3x^2+1)\delta x+3x(\delta x)^2+(\delta x)^3.$

Here $(\delta x)^2$ and $(\delta x)^3$ are infinitesimals of order higher than δx. Hence the principal part of δy is $(3x^2+1)\delta x$ and therefore

$$dy=(3x^2+1)\delta x=(3x^2+1)dx,$$

since $\delta x=dx$, the differential of x.

ANSWERS

Examples I, Page 20

1. 3, -6, -5, $(1+2x-5x^2)/x^2$. 2. 1, $\frac{1}{2}$, 0. 6. $2a+h$.

7. (*i*) $x=1$. (*ii*) $x=2, 3$. (*iii*) $x=n\pi$, n any integer.
   (*iv*) $x=0$. (*v*) $x=2$. (*vi*) $x=1$.

10. (*i*) $[-2, 2]$, $[0, 2]$. (*ii*) $x\leqslant -2$ and $x\geqslant 2$, $f(x)\geqslant 0$.
    (*iii*) $(-2, 2)$, $f(x)>2$.
    (*iv*) All real numbers except $x=1, 2$; The range is all real numbers excepting those lying in the interval $(-3-2\sqrt{2},\ -3+2\sqrt{2})$.

Examples II, Pages 25, 26

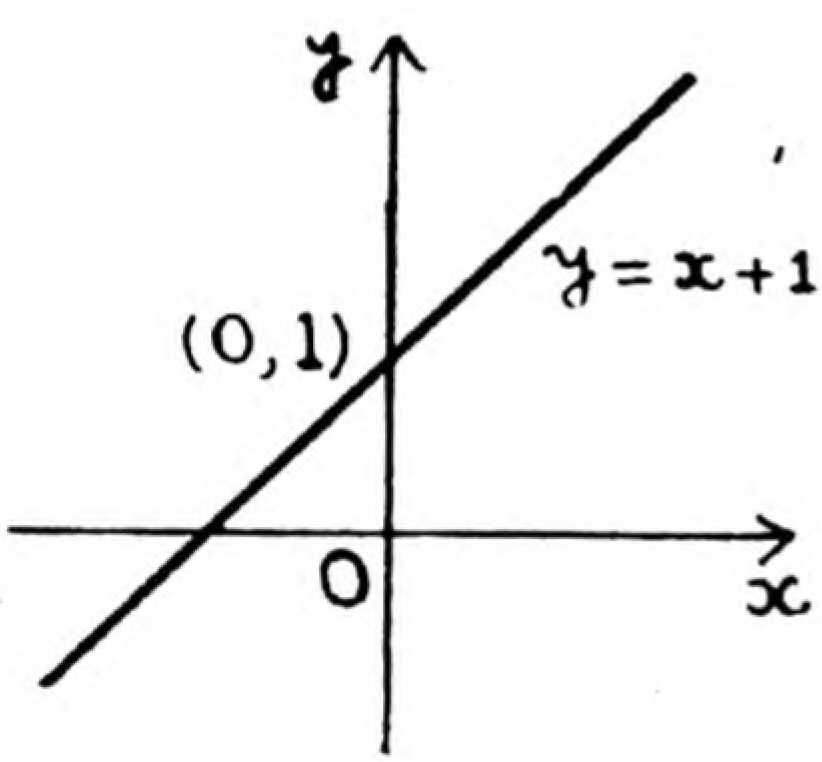


Fig. 1

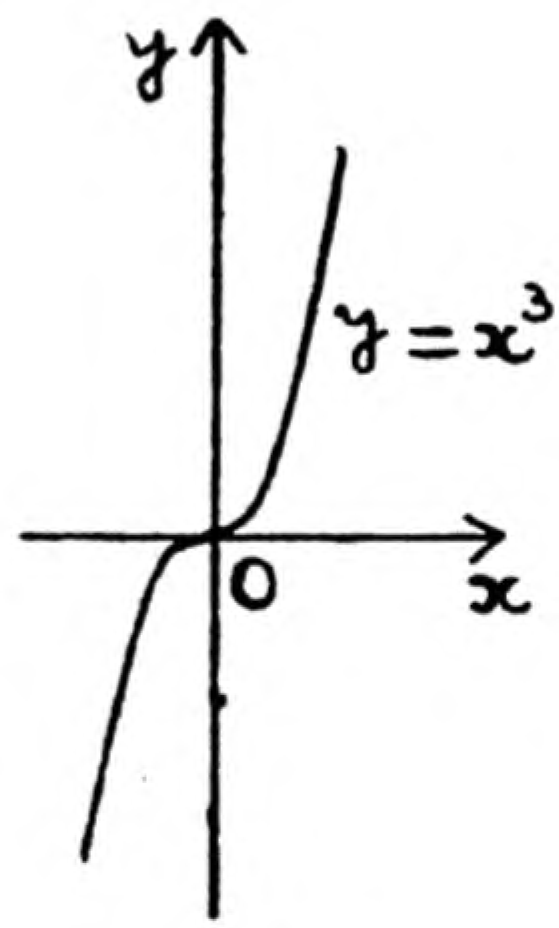


Fig. 2

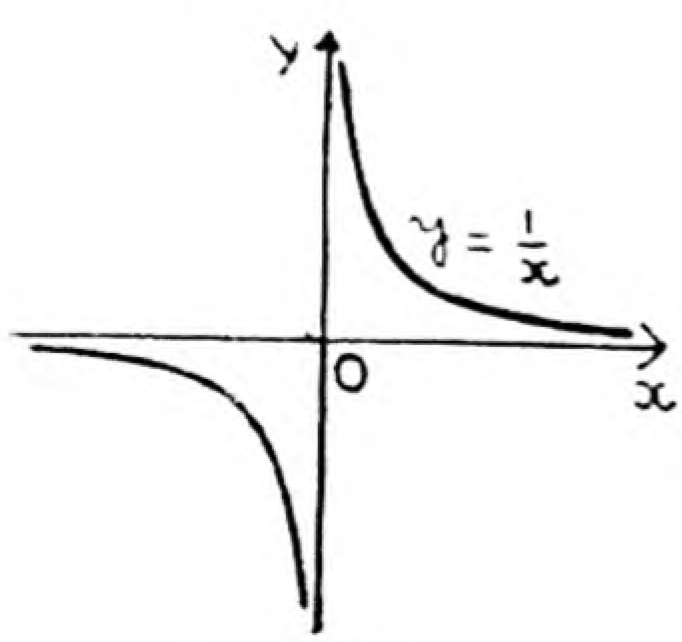


Fig. 3

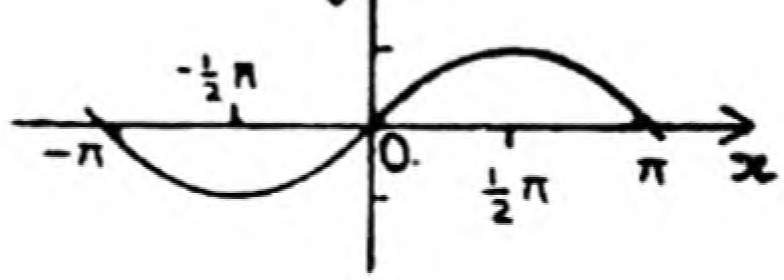


Fig. 4

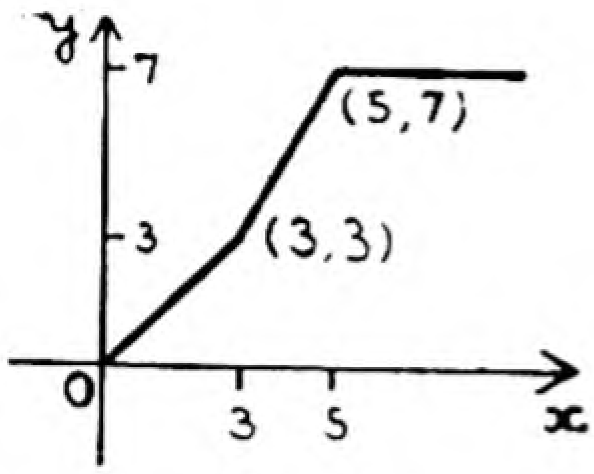


Fig. 5

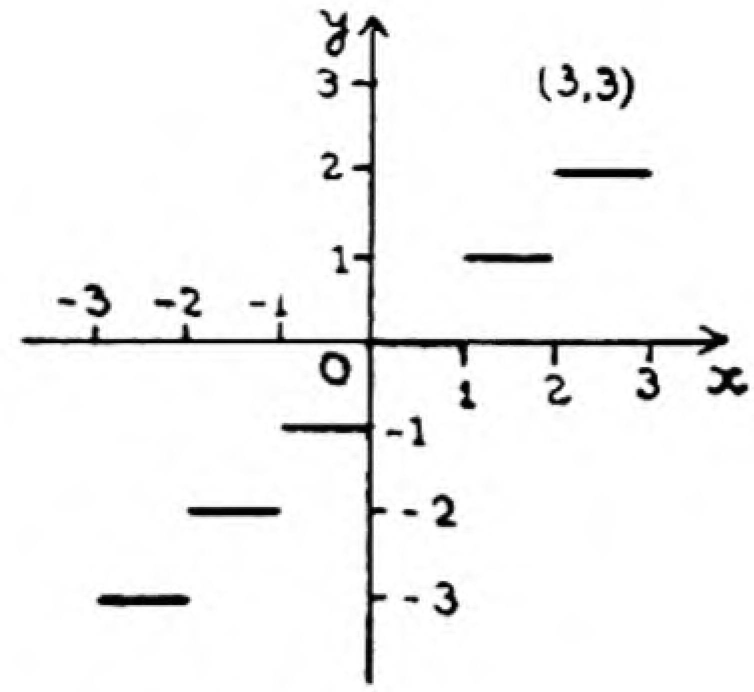


Fig. 6

1—5. Figures 1—5 respectively.

6. Fig. 6. The right-hand extremity of every segment is not a part of the graph.

7. (*i*) No ; g is not defined at $x=1$.
(*ii*) No ; $f=g$ only when $|x| \geqslant 1$.

8. $y=1+\sqrt{(1-x^2)}$ and $y=1-\sqrt{(1-x^2)}$; domain $-1 \leqslant x \leqslant 1$ for both.

9. (*i*) Even. (*ii*) Odd. (*iii*) Neither even nor odd.

10. Write $f(x)=\frac{1}{2}\{f(x)+f(-x)\}+\frac{1}{2}\{f(x)-f(-x)\}$.

Examples III, Pages 44, 45

1. (*i*) 10. (*ii*) -10.
2. (*i*) 6. (*ii*) 5/3. (*iii*) $(m/n)\, a^{m-n}$.
(*iv*) a/a_1 if $a_1 \neq 0$. If $a_1=0$ and $a \neq 0$, then no unique limit. If $a=a_1=0$, then the limit is b/b_1, provided $b_1 \neq 0$, and so on. (*v*) 1.
3. (*i*) 0. (*ii*) 0, a/a_1 or infinity according as $m >, =$ or $< n$.
4. $\frac{1}{2}$. 5. -1. 6. $\log_b a$.
8. (*i*) $\cos x$. (*ii*) $\sec x \tan x$.
9. (*i*) $\frac{3}{5}$. (*ii*) a^k/b^k. (*iii*) $\frac{3}{2}$.

Examples IV, Pages 52, 53

3. Discontinuous. 5. Discontinuous at $x=0$ and $x=1$.
9. (*i*) Discontinuous. (*ii*) Discontinuous. (*iii*) Continuous.
(*iv*) Discontinuous. (*v*) Continuous
10. (*i*) $x=-1, 2$. (*ii*) $x=\pm 3$. (*iii*) $x=0, \pm 1$.
(*iv*) $x=2n\pi \pm \frac{1}{2}\pi$.

Examples V, Page 58

2. (*i*) $3x^2+3hx+h^2$. (*ii*) $4(2x+3)+4h$.
(*iii*) $-1/(x+1)(x+1+h)$. (*iv*) $2x+h-1/\{x(x+h)\}$.
(*v*) $1/\{\sqrt{x}+\sqrt{(x+h)}\}$.
(*vi*) $-1/[\sqrt{x}\,\sqrt{(x+h)}\{\sqrt{x}+\sqrt{(x+h)}\}]$.
3. (*i*) $2x$. (*ii*) $4x^3$. (*iii*) $2x+3$.
(*iv*) $2ax+b$. (*v*) $3x^2+2x$. (*vi*) $5x^4$.

4. (*i*) $-1/x^2$. (*ii*) $-2/x^3$. (*iii*) $-2x/(x^2+a^2)^2$.
(*iv*) $(2-2x-x^2)/(x^2+2)^2$. (*v*) $(x^2-2x-1)/(x-1)^2$.
(*vi*) $a/(x+a)^2$.

5. (*i*) $1/2\sqrt{x}$. (*ii*) $-\frac{1}{2}x^{-3/2}$. (*iii*) $a/2\sqrt{(ax+b)}$.
(*iv*) $-x(x^2+1)^{-3/2}$. (*v* $-x(1-x^2)^{-1/2}$.
(*vi*) $-\frac{1}{2}p(px+q)^{-3/2}$.

Examples VI, Page 59

1. (*i*) $2x$. (*ii*) $-3x^{-4}$. (*iii*) $-\frac{1}{2}x^{-3/2}$. (*iv*) $\frac{3}{4}x^{-1/4}$.
2. (*i*) $21(3x+4)^6$. (*ii*) $-(7-2x)^{-1/2}$.
(*iii*) $-24(3x+1)^{-9}$. (*iv*) $(5-x)^{-2}$.

Examples VII, Page 61

1. $4x^3+2x$. 2. $3ax^2+2bx+c$. 3. $\frac{1}{2}x^{-1/2}-3+\frac{15}{2}x^{1/2}$.
4. $2x+1-x^{-2}-2x^{-3}$. 5. $4x+3-5x^{-2}$.
6. $-3x^{-4}-2x^{-3}-2x^{-2}$. 7. $\frac{1}{2}x^{-1/2}-\frac{1}{2}x^{-3/2}$.
8. $3x^2+1-x^{-2}-3x^{-4}$.

Examples VIII, Page 63

1. $2x+3$. 2. $(x+3)(4x^2+15x+17)$.
3. $6x^5+5x^4+4x^3+6x^2+2x+1$.
4. $4x^3+15x^2+26x+17$.
5. $(sp-qr)/(rx+s)^2$. 6. $2(1-6x)/(x+1)^8$.
7. $2(x^2-1)/(x^2+x+1)^2$
8. $-(2x^4+10x^3+6x^2+4x+5)/(x^3-1)^2$.
9. $-2(x^2+2x+5)/(x^2+2x-3)^2$.
10. $(1+x)^{-1/2}(1-x)^{-3/2}$

Examples IX, Page 65

1. $3(2x+1)(x^2+x+1)^2$. 2. $2x/\{(x^2-1)^{1/2}(x^2+1)^{3/2}\}$.
3. $n\{x+\sqrt{(x^2+a^2)}\}^n/\sqrt{(x^2+a^2)}$. 4. $(2x^2+a^2)/\sqrt{(x^2+a^2)}$.
5. $(\frac{1}{2}bx+c)/(ax^2+bx+c)^{3/2}$. 6. $(1-x^2)(3x-x^3)^{-2/3}$.
7. $\frac{1}{2}x^{-1/2}\cos\sqrt{x}$. 8. $\cos x/2\sqrt{(\sin x)}$.
9. $\frac{1}{4}\cos\sqrt{x}/\{\sqrt{x}\sqrt{(\sin\sqrt{x})}\}$

Examples X, Page 67

1. $1/t$. 2. $-1/t^2$. 3. $-b\sqrt{(1-t^2)}/a\sqrt{(1+t^2)}$.
4. $\frac{1}{2}m/(ap+b)$. 5. (*i*) $3x^8$. (*ii*) $(1-x^2)^{-2}$.
6. x/y. 7. $-b^2x/a^2y$ 8. $-(ax+hy)/(hx+by)$.
9. $(x^3-2axy^2)/(2ax^2y-y^4)$.
10. $-(ax+hy+g)/(hx+by+f)$.

Examples XI, Pages 70, 71

1. $\sin 2x$. 2. $-m\sin mx$. 3 $(\pi/180)\cos x^\circ$
4. $-m\cos^{m-1}x\sin x$. 5. $nx^{n-1}\cos x^n$.
6. $(b-a)\sin 2\alpha$. 7. $-\cos x\sin(\sin x)$.
8. $4x\tan x^2\sec^2 x^2$. 9. $1/(1-\sin x)$.
10. $\frac{1}{2}\sec^2\frac{1}{2}x$. 11. $\frac{1}{2}\operatorname{cosec}^2(\frac{1}{4}\pi-\frac{1}{2}x)$.

12. $2 \sec^2 x/(1-\tan x)^2$. 13. $x^2 \cos x + 2x \sin x$.
14. $(b^2-a^2) \sin x/(a+b \cos x)^2$.
15. $5(3+2x)^{-2} \sec^2 \{(2+3x)/(3+2x)\}$.
16. $\frac{1}{2}x \sec^2 \sqrt{(1+x^2)}/[\sqrt{(1+x^2)}\sqrt{\{\tan\sqrt{(1+x^2)}\}}]$.
17. (*i*) $-(b/a \cot \theta$. (*ii*) $(b/a) \operatorname{cosec} t$.
18. $-\tan t$. 19. $\cot \frac{1}{2}\theta$. 20. -1.

Examples XII, Pages 76, 77

1. $2x/\sqrt{(1-x^4)}$. 2. $-1/2\sqrt{\{x(1-x)\}}$.
3. $a/(a^2+x^2)$. 4. $-b/x\sqrt{(a^2x^2-b^2)}$.
5. 0. 6. $-1/(2x^2+10x+13)$.
7. $\frac{1}{2}$. 8. -1. 9. $1/x\sqrt{(1-x^2)}-(\sin^{-1} x)/x^2$.
10. $(x \sin^{-1} x)/\sqrt{(1-x^2)}$. 11. $\frac{1}{2}$. 12. $3/(1+x^2)$
13. $2/\sqrt{(1-x^2)}$. 14. $-2/\sqrt{(1-x^2)}$
15. $2x/(1+x^4)$. 16. $1/(1+x^2)$.
17. $2/(1+x^2)$. 18. $\sqrt{(a^2-b^2)}/(a+b \cos x)$.
19. (*i*) $\frac{1}{4}$. (*ii*) 1. (*iii*) $2/x$.

Examples XIII, Page 81

1. $2(e^{2x}+e^{-2x})$. 2. $2(x+1)e^{x^2+2x}$. 3. $nx^{n-1}e^{x^n}$.
4. $(2 \log 5)5^{2x-5}$. 5. $(a^{\sqrt{x}} \log a)/2\sqrt{x}$. 6. $(x+1)e^x$.
7. $ax^{a-1}a^x + x^a a^x \log a$. 8. $6x^2/(2x^3+3)$.
9. $1/\sqrt{(x^2+a^2)}$. 10. $1/2\{\sqrt{(x+a)}\sqrt{(x+b)}\}$.
11. $x/\{(1+x^2) \log 10\}$. 12. $-(\log 10)/\{x(\log x)^2\}$.
13. $2ah/(a^2-b^2x^2)$. 14. $x(1+2 \log x)$.
15. $(1-\log x)/x^2$. 16. $1/(x \log x \log \log x)$.
17. $e^x(1+x^{-1}+\log x)$ 18. $x^{n-1} e^x\{1+(n+x) \log x\}$.
19. $2e^x/(1-e^x)^2$. 20. $2/\{x(1-\log x)^2\}$.

Examples XIV, Page 83

1. $$\frac{20}{(x-1)^2}-\frac{180}{(x-2)^2}+\frac{420}{(x-3)^2}-\frac{280}{(x-4)^2}.$$

2. $$\frac{1}{2}\sqrt{(x)}\sqrt{(\sin x)}\sqrt{(\log x)}\left\{\frac{1}{x}+\cot x+\frac{1}{x \log x}\right\}.$$

3. $x^x(1+x+x \log x)$. 4. $(\tan x)^{\cot x}\operatorname{cosec}^2 x(1-\log \tan x)$

5. $$(\sin^{-1} x)^x\left\{\log \sin^{-1} x+\frac{x}{\sin^{-1}x\sqrt{(1-x^2)}}\right\}.$$

6. $x^{\sin x}\{(1/x) \sin x + \cos x \log x\}$.

7. $(\sin x)^{\log x}\{(1/x) \log \sin x + \cot x \log x\}$.
8. $-x^x \log ex \sin x^x$.

9. $b^x a^{b^x} \log a \log b$. 10. $x^x x^{x^x}\{(1/x)+\log x+(\log x)^2\}$.
11. $x^x(1+\log x)+(1/x^2)x^{1/x}(1-\log x)$.

12. $(\sin x)^{\cos x}(\cot x \cos x - \sin x \log x) + (\cos x)^{\sin x}$
$\times(-\tan x \sin x + \cos x \log \cos x)$.

13. $(\sec x)^{\operatorname{cosec} x}(\sec x - \operatorname{cosec} x \cot x \log \sec x) + (\operatorname{cosec} x)^{\sec x}$
$\times(\sec x \tan x \log \operatorname{cosec} x - \operatorname{cosec} x)$.

14. $(\tan x)^x(\log \tan x + 2x \operatorname{cosec} 2x) + x^{\tan x}\{\sec^2 x \log x$
$+(1/x)\tan x\}$

Examples XV, Page 85

1. $\cosh^3 x$, $-2(1+\cos x \cosh x)/(\sinh x + \sin x)^2$,
$(2 \sinh x \cos x)/(\sinh x + \cos x)^2$

2. $\tanh x$, $2 \operatorname{cosech} 2x$, $-\coth x$, $a^{\sinh x}\cosh x \log a$.

3. $\dfrac{-1}{x\sqrt{(1-x^2)}}$, $\dfrac{-1}{x\sqrt{(1+x^2)}}$. 4. $\dfrac{1}{\sqrt{(x^2-1)}}$, $\dfrac{2}{1-x}$. $\dfrac{1}{x}$, $\sec x$.

Examples XVI, Page 86

1. See Art. 4·51.
2. $2x \cos x^2$.
3. $e^{\sqrt{x}}/2\sqrt{x}$.
4. $\cot x$.
5. $2x/(x^2+3)^2$.
6. $\sin x + x \cos x$.
7. $-\sin x \, . \, e^{\cos x}$.
8. $2 \tan x \sec^2 x$.

Examples XVII, Pages 90, 91

1. (*i*) $3024x^5$.
(*ii*) $4 \tan^2 x \sec^2 x + 2 \sec^4 x - 4 \cot^2 x \operatorname{cosec}^2 x - 2 \operatorname{cosec}^4 x$.
(*iii*) $\left\{\dfrac{n^2}{x^2-1} - \dfrac{nx}{(x^2-1)^{3/2}}\right\}\{x+\sqrt{(x^2-1)}\}^n$. (iv) $6/x$

2. The nth derivatives are :
(*i*) e^x, (*ii*) $a^n e^{ax}$, (*iii*) $(-1)^n (n\,!)x^{-n-1}$,
(*iv*) $(-1)^{n-1}(n-1)\,!\,x^{-n}$, (*v*) $(-1)^n n\,!\,a^n(ax+b)^{-n-1}$,
(*vi*) $-\{(n-1)\,!\}(3-x)^{-n}$, (*vii*) $\cos(x+\frac{1}{2}n\pi)$,
(*viii*) $5^n \cos(5x+\frac{1}{2}n\pi)$.

11. $a=2$, $n=0, 1$. 14. $-(b/a^2) \operatorname{cosec}^3 \theta$. 15. $-3/2$.

Examples XVIII, Page 94

1. $2^{n+1} n(2n-1)(2n-2)\ldots\ldots(n+1)(2x+3)^n$.
2. $(-1)^n(n\,!)/(1+x)^{n+1}$.
3. $n\,!/(a-x)^{n+1}$.
4. $b^n(n\,!)/(a-bx)^{n+1}$.
5. $(-1)^n(n+1)\,!\,2^n/(2x+3)^{n+2}$
6. $(-1)^{n-1}(n\,!)\,3^{n-1}/(3x+7)^{n+1}$.
7. $(-1)^{n-1}.3.5\ldots\ldots(2n-1)2^{-n}(x+a)^{(1-2n)/2}$.
8. $2(-1)^{n-1}(n-1)\,!/x^n$.
9. $2^n e^{2x}$.
10. $\frac{1}{2}\cos(x+\frac{1}{2}n\pi) - \frac{1}{2}\,3^n \cos(3x+\frac{1}{2}n\pi)$.
11. $(b-a)2^{n-1}\cos(2x+\frac{1}{2}n\pi)$.
12. $\frac{1}{4}\{2^n \cos(2x+\frac{1}{2}n\pi) + 4^n \cos(4x+\frac{1}{2}n\pi) + 6^n \cos(6x+\frac{1}{2}n\pi)\}$.
13. $\frac{1}{4}\{3 \sin(x+\frac{1}{2}n\pi) - 3^n \sin(3x+\frac{1}{2}n\pi)\}$
14. $\frac{1}{8}\{4^n \cos(4x+\frac{1}{2}n\pi) + 2^{n+2}\cos(2x+\frac{1}{2}n\pi)\}$.
15. $\frac{1}{2}e^x\{1+(\sqrt{5})^n \cos(2x+n \tan^{-1} 2)\}$.

16. $\frac{1}{16} e^{3x}\{2(10)^{n/2} \cos (x+n \tan^{-1} \frac{1}{3})-(18)^{n/2} \cos (3x+\frac{1}{4}n\pi)$
$-(34)^{n/2} \cos (5x+n \tan^{-1} \frac{3}{5})\}$.

17. $\frac{1}{4}e^{x}\{2^{n/2} \cos (x+\frac{1}{4}n\pi)+10^{n/2} \cos (3x+n \tan^{-1} 3)\}$.

18. $\frac{1}{4}e^{x}\{3.2^{n/2} \cos (x+\frac{1}{4}n\pi)+10^{n/2} \cos (3x+n \tan^{-1} 3)\}$.

19. $(\sqrt{2})^{n+1} e^{x} \cos \{x+\frac{1}{4}(n+1)\pi\}$.

20. $e^{x \cos \alpha} \cos (x \sin \alpha+n\pi)$.

Examples XIX, Page 96

1. $(-1)^n (n!) \left\{ \frac{2}{(x-2)^{n+1}} - \frac{1}{(x-1)^{n+1}} \right\}$.

2. $\frac{1}{3} (n!) \left\{ \frac{1}{(1-x)^{n+1}} + \frac{(-1)^n}{(2+x)^{n+1}} \right\}$.

3. $(-1)^n(n!) \left\{ \frac{1}{(x-1)^{n+1}} + \frac{1}{(x+1)^{n+1}} - \frac{1}{(x+2)^{n+1}} \right\}$.

4. $\frac{(-1)^n (n!)}{25} \left\{ \frac{-3}{(x+2)^{n+1}} + \frac{3}{(x-3)^{n+1}} + \frac{35(n+1)}{(x-3)^{n+2}} \right\}$.

5. $\frac{(-1)^n(n!)}{4a^3} \left\{ \frac{1}{(x-a)^{n+1}} - \frac{1}{(x+a)^{n+1}} - \frac{2}{a^{n+1}} \sin (n+1)\theta \sin^{n+1} \theta \right\}$
where $x=a \cot \theta$.

6. $(-1)^n (n!)(2/\sqrt{3})^{n+2} \sin (n+1) \theta \sin^{n+1} \theta$,
where $2x+1=\sqrt{3} \cot \theta$.

7. $\frac{1}{2}(-1)^n (n!) [(x+1)^{-n-1}+\{\sin(n+1)\theta -\cos (n+1)\theta\}\sin^{n+1} \theta]$
where $x=\cot \theta$.

8. $(-1)^{n-1}(n-1)! \sin n\theta \sin^n \theta$, where $x=\cot \theta$.

9. $(-1)^n (n-1)! a^{-n} \sin n\theta \sin^n \theta$, where $x=a \cot \theta$.

10. $\frac{1}{2}(-1)^n (n!)(2/\sqrt{3})^{n+2}\{\sin (n+1)\theta \sin^{n+1} \theta-\sin (n+1) \phi$
$\sin^{n+1} \phi\}$, where $2x-1=\cot \theta$ and $2x+1=\cot \phi$.

11. $2(-1)^{n-1} (n-1)! \sin n\theta \sin^n \theta$, where $x=\cot \theta$.

12. $\frac{(-1)^{n-1}(n-1)!}{\sin^n \alpha} \sin n\theta \sin^n \theta$, where $x-\cos \alpha=\sin \alpha \cot \theta$.

Examples XX, Pages 99,100

1. $e^{ax}(a^n x+na^{n-1})$.

2. $a^x\{x^2(\log a)^n+2nx(\log a)^{n-1}+n(n-1)(\log a)^{n-2}\}$.

3. $x^2 \cos (x+\frac{1}{2}n\pi)+2nx \cos \{x+\frac{1}{2}(n-1)\pi\}$
$+n(n-1) \cos \{x+\frac{1}{2}(n-2)\pi\}$.

4. $x^3a^n \sin (ax+\frac{1}{2}n\pi)+3nx^2a^{n-1} \sin \{ax+\frac{1}{2}(n-1)\pi\}$
$+3n(n-1)xa^{n-2} \sin \{ax+\frac{1}{2}(n-2)\pi\}$
$+n(n-1)(n-2)a^{n-3} \sin \{ax+\frac{1}{2}(n-3)\pi\}$.

5. $\frac{(-1)^{n-1} (n!)}{x^{n-2}} \left\{ \frac{1}{n} - \frac{2}{n-1} + \frac{1}{n-2} \right\}$, if $n>2$.

6. $e^x\left[\log x+ \frac{n}{x} - \frac{n(n-1)}{2x^2} + \frac{n(n-1)(n-2)}{3x^3} - \dots + \frac{(-1)^{n-1}(n!)}{nx^n} \right]$

7. $(-1)^{n-1}(n-2)!\sin^{n-1}\theta\cos\theta\cos n\theta\{n\tan\theta-\tan n\theta\}$, where $x=\cot\theta$.

8. $(a^2+b^2)^{n/2}xe^{ax}\sin\{bx+n\tan^{-1}(b/a)\}$
$+n(a^2+b^2)^{(n-1)/2}e^{ax}\sin\{bx+(n-1)\tan^{-1}(b/a)\}$.

11. $(u_{2n})_0=0$, $(u_{2n+1})_0=(-1)^n.(2n)!$

21. $y_{2n-1}=0$, $y_{2n}=(-1)^{n-1}2.2^2.4^2......(2n-2)^2$.

22. $(y_{2n})_0=\{m^2-(2n-2)^2\}\{m^2-(2n-4)^2\}......(m^2-2^2)m^2$;
$(y_{2n+1})_0=\{m^2-(2n-1)^2\}\{m^2-(2n-3)^2\}......(m^2-1^2)m$.

Miscellaneous Examples I, Pages 101—103

2. 3.

3. (*i*) Limit does not exist, discontinuous.
(*ii*) Limit $=0$, continuous.

4 Continuous. 7. Discontinuous.

8. (*i*) $-1/(1+x^2)$, (*ii*) $a^2x/\sqrt{(a^2x^2+b^2)}$. $-(ax+hy)/(hx+by)$.

9. $t(2-t^3)/(1-2t^3)$. 10. $\frac{1}{2}$. 11. $x/\sqrt{(1-x^4)}$.

12. $2a^2x/(a^4+x^4)$. 13. $-1/\{x\sqrt{(x^2-1)}\}$.

14. $-x/\sqrt{(1-x^2)}$.

15. $(\cot x\log\cos x+\tan x\log\sin x)/(\log\cos x)^2$.

16. (*i*) 1. (*ii*) $-(\log_{10}x)^2$.

17. $-\sqrt{(ay-y^2)}/y$.

18. (*i*) $y/(2y-x)$. (*ii*) $1/2\sqrt{\{x(1+4\sqrt{x})\}}$.
(*iii*) $(y^2\tan x)/(y\log\cos x-1)$. (*iv*) $y^2/x(1-y\log x)$.

22. (*i*) $x^{\sin x}\left[\dfrac{\sin x}{x}+\cos x\log x\right]\dfrac{1}{(\sin x)^x(x\cot x+\log\sin x)}$

(*ii*) $-\dfrac{(\log x)^{\tan x}\sqrt{(1-x^2)}}{m\cos(m\cos^{-1}x)}\left[\sec^2 x\log\log x+\dfrac{\tan x}{x\log x}\right]$.

(*iii*) $e^{\tan x}\sec^3 x$, (*iv*) $x\sin x\cos x/(\log x)$. (*v*) $\frac{1}{2}$.

23. (*i*) 0. (*ii*) 4, 16.

AN INTRODUCTION TO CALCULUS

PART II

DIFFERENTIAL CALCULUS

CHAPTER V

APPLICATIONS OF THE DERIVATIVE

5·1 In this and the following two chapters we consider simple applications of the derivative to inequalities, rates of growth of quantities, tangents and normals to plane curves, and maxima and minima of functions of a single variable.

5·2 Functions increasing or decreasing at a point. Let $y=f(x)$ be a function of x. Give an increment δx to x and let δy be the corresponding increment of y. If δy be of the same sign as δx, then y or $f(x)$ is said to be an **increasing function at the point x**. This means that when x increases, y increases, and when x decreases, y also decreases.

If, on the other hand, δy has the sign opposite to that of δx, then y or $f(x)$ is said to be a **decreasing function at the point x**. In this case, y decreases when x increases and increases when x decreases.

5·21 Interpretation of the sign of f′(x). Let $y=f(x)$ have a derivative $f'(x)$, finite or not, at the point x, then we have the following

Theorem. *If $f'(x)>0$, then $f(x)$ is increasing at the point x and if $f'(x)<0$, then $f(x)$ is decreasing at the point x.*

If the derivative $f'(x)$, finite or infinite, is positive at the point x, then the quotient $\delta y/\delta x$ has a positive limit when $\delta x \to 0$, therefore δy is different from zero and has the same sign as δx for all sufficiently small values of $|\delta x|$. Hence y is an increasing function at the point x.

If, on the other hand, $f'(x)$ is negative, then $\delta y/\delta x$ has a negative limit, therefore δy is different from zero and has a sign opposite to that of δx for all sufficiently small values of $|\delta x|$. Hence y is a decreasing function at the point x. This proves the theorem.

Cor. *If $f'(x)>0$ for all x in $[a, b]$, then $f(x)$ is an increasing function of x for each x in (a, b) and $f(b)>f(a)$.*

In particular, if $f(a)=0$, then it follows that $f(x)$ is positive for all $x>a$ in $[a, b]$ and so $f(b)>0$.

It should be observed that if $f'(x)$ is different from zero at a point $x=c$, then, in the immediate neighbourhood of this point, $f(x)$ assumes always values both greater than as well as smaller than $f(c)$. The greater values are, for example, assumed to the right of $x=c$ if $f(x)$ is increasing at $x=c$, that is, if $f'(c)$ is positive, and

to the left of $x=c$ if $f(x)$ is decreasing at $x=c$, that is, if $f'(c)$ is negative. It follows, therefore, that if $f'(c)\neq 0$, then $f'(c)$ can neither be the greatest nor the smallest value of $f(x)$ in the immediate neighbourhood of $x=c$.

5·22 Applications to inequalities. The results proved in Art. 5·21 can be immediately applied to prove some simple inequalities. This is illustrated in the following solved examples.

Ex. 1. *Show that $x^3-6x^2+15x+3$ is positive if x be positive.*

Let $\qquad f(x)=x^3-6x^2+15x+3,$

then $\qquad f'(x)=3x^2-12x+15=3\{(x-2)^2+1\},$

which shows that $f'(x)$ is always positive. Hence $f(x)$ is an increasing function of x for every x. But $f(0)=3$, hence $f(x)\geqslant 3$ when $x\geqslant 0$. Hence $f(x)$ is positive for all positive x.

Ex. 2. *If $0\leqslant x\leqslant \frac{1}{2}\pi$, prove that* $\dfrac{2}{\pi}\leqslant\dfrac{\sin x}{x}\leqslant 1.$

Let $\qquad f(x)=(\sin x)/x$, then

$$f'(x)=\frac{x\cos x-\sin x}{x^2}=\frac{(x-\tan x)\cos x}{x^2}.$$

When x is a positive acute angle, we know that $x<\tan x$. Also in the given range $\cos x$ and x^2 are both positive. Hence $f'(x)\leqslant 0$ when $0\leqslant x\leqslant\frac{1}{2}\pi$. Hence $f(x)$ is a decreasing function of x throughout the interval $[0, \frac{1}{2}\pi]$. Hence its greatest value occurs at $x=0$ and least at $x=\frac{1}{2}\pi$. When

$$x\to 0,\ f(x)=(\sin x)/x\to 1, \text{ and } f(\tfrac{1}{2}\pi)=2/\pi.$$

Hence we get the inequalities

$$\frac{2}{\pi}\leqslant\frac{\sin x}{x}\leqslant 1 \text{ where } 0\leqslant x\leqslant\tfrac{1}{2}\pi.$$

Ex. 3. *Prove that*

$$x-\tfrac{1}{2}x^2<\log(1+x)<x-\frac{x^2}{2(1+x)} \textit{ for } x>0.$$

Let $f(x)=\log(1+x)-x+\frac{1}{2}x^2$, then,

$$f'(x)=\frac{1}{1+x}-1+x=\frac{1-(1-x^2)}{1+x}=\frac{x^2}{1+x}>0$$

when $x>0$. Hence $f(x)$ is an increasing function of x for all positive values of x. But $f(0)=0$, hence $f(x)>0$ for $x>0$. Therefore

$$\log(1+x)-x+\tfrac{1}{2}x^2>0$$

or
$$x-\tfrac{1}{2}x^2<\log(1+x) \text{ for } x>0.$$

Similarly we can show that

$$\log(1+x)<x-\frac{x^2}{2(1+x)} \text{ for } x>0.$$

Combining these two results, we get

$$x-\tfrac{1}{2}x^2<\log(1+x)<x-\frac{x^2}{2(1+x)} \text{ for } x>0.$$

EXAMPLES XVI

1. Find the range of values of x for which the function
$$x^3-6x^2-36x+7$$
increases with x. *(Lucknow, 1949)*

2. If $y=2x^3-9x^2+12x-6$, find the range of values of x for which y is increasing and that for which it is decreasing.

3. Show that the function $x \sin x+\cos x+\cos^2 x$ continually diminishes as x increases from 0 to $\pi/2$.

4. If $\phi(x)=(x-1)e^x+1$, show that $\phi(x)$ is possitive for all positive values of x. *(Calcutta, 1943)*

5. Show that $\sin x$ lies between $x-\frac{1}{6}x^3$ and $x-\frac{1}{6}x^3+\frac{1}{120}x^5$.

6. If $x>0$ prove that :

   (*i*) $x-\log(1+x)>\frac{1}{2}x^2/(1+x)$.

   (*ii*) $x>\log(1+x)>x/(1+x)$.

7. If x is not equal to zero, prove that :

   (*i*) $x/(1+x)<1-e^{-x}<x$ for $x>-1$.

   (*ii*) $x<e^x-1<x/(1-x)$ for $x<1$.

8. If $y=2x-\tan^{-1} x-\log\{x+\sqrt{(1+x^2)}\}$, show that y continually increases as x changes from zero to positive infinity. *(Calcutta, 1942)*

5·3 Derivative as a rate measurer. Throughout the Differential Calculus we are connected with the growth or variation of related quantities. If one quantitity grows or varies or changes, we wish to know the *rate of growth* or *change* of any other related quantity. In other words, we are concerned with the rate of change of one quantity relative to another with which it is connected by some given relation.

Let $y=f(x)$ be any function of x; if h be the increase in the value of x, then $f(x+h)-f(x)$ represents the increase in the value of y or $f(x)$, and the ratio $\{f(x+h)-f(x)\}/h$ measures the average rate of increase in $f(x)$ as x increases from x to $x+h$. If h be taken smaller and smaller, this ratio will measure more and more approximately the rate of increase of $f(x)$ relative to x at the particular value of x under consideration. Therefore, in the limit, when $h\to 0$, $f'(x)$ measures the rate of change of $f(x)$ relative to x. Hence *the derivative* $\dfrac{dy}{dx}$ *measures the rate of change of y with respect to x.*

5·31 Rate of change with respect to time. Application to Mechanics. If a quantity x be a function of the time t, then $\dfrac{dx}{dt}$ measures the rate of change of x with respect to the time t. This is the most important case in practice.

As a simple application we consider the case of a particle moving in a straight line. Let the position of the particle at time t be fixed by measuring its distance x from an origin O in the straight line. At time $t+\delta t$, let the distance of the particle from O be $x+\delta x$. Then, the total space described in time δt is δx and the average rate of description of space or, in other words, the average speed is equal to $\delta x/\delta t$. As $\delta t \to 0$, *the rate of description of space at time t, i.e., the velocity at time t is represented by the derivative* $\frac{dx}{dt}$.

Hence if v denotes the velocity at time t, then

$$v=\frac{dx}{dt}. \tag{1}$$

Similar considerations show that if f stands for the acceleration at time t, *i.e.*, the rate of change of velocity at time t, then

$$f=\frac{dv}{dt}. \tag{2}$$

Since $v=dx/dt$, we can write

$$f=\frac{d^2x}{dt^2} \tag{3}$$

and also

$$f=\frac{dv}{dt}=\frac{d}{dx}(v).\ \frac{dx}{dt}=v\frac{dv}{dx}. \tag{4}$$

Equations (2), (3) and (4) may be used according as we are given a relation between v and t, x and t, or x and v.

Ex. 1. *Find the rate of change of the volume of a circular cylinder of radius r and height h when the redius varies.*

If V be the volume, then $V=\pi r^2 h$ and therefore, the rate of change of V with respect to r is $\frac{dV}{dr}=2\pi rh$.

Ex. 2. *Water is poured at a constant rate into a conical glass, which is filled in 2 minutes, the height of the cone being 12 in. At what rate, in inches per min., is the surface of the water rising (i) when the glass is filled to half its height, (ii) when half the liquid has been poured in ?*

Let α be the semi-vertical angle of the conical glass, then its volume is $\frac{1}{3}\pi\ (12)^3 \tan^2\alpha$ and this volume of water is poured in at a uniform rate in 2 minutes. If h inches be the height of water in the glass at time t minutes, then

$$\tfrac{1}{3}\pi h^3 \tan^2\alpha=\tfrac{1}{2}t.\tfrac{1}{3}\pi.12^3.\tan^2\alpha,$$

whence $$h=12(t/2)^{\frac{1}{3}} \text{ inches}.$$

Hence the rate at which the surface of the water is rising at time t minutes

$$=\frac{dh}{dt}=12\times\tfrac{1}{3}\times\tfrac{1}{2}(t/2)^{-2/3}$$
$$=2(t/2)^{-2/3} \text{ inches per min.}$$

(*i*) When the glass is filled to half its height, the quantity of liquid poured in is only one-eighth of the whole. Hence $t=\frac{1}{4}$ min., and so

$$\frac{dh}{dt}=2(\tfrac{1}{8})^{-\frac{2}{3}}=8 \text{ inches per min.}$$

(*ii*) When half the liquid has been poured in, $t=1$ min. and so

$$\frac{dh}{dt}=2(\tfrac{1}{2})^{-\frac{2}{3}}=2^{\frac{5}{3}} \text{ inches per min.}$$

Ex. 2. *A particle moves in a straight line and its distance s in feet from a fixed point O in the line at time t is given by $s=t(t-1)^2$. Find its velocity and acceleration on each occasion when it passes through O.*

The particle passes through O when $s=0$, *i.e.*, when $t(t-1)^2=0$ or when $t=0$ or 1 sec. If v and f denote the velocity and acceleration respectively at time t seconds, then

$$v=\frac{ds}{dt}=(3t^2-4t+1)\text{ft./sec.}$$

$$f=\frac{dv}{dt}=(6t-4)\text{ft./(sec)}^2$$

(*i*) When $t=0$, $v=1$ ft. per sec. and $f=-4$ ft. per sec. per sec.

(*ii*) When $t=1$, $v=0$ and $f=2$ ft. per sec. per sec.

EXAMPLES XVII

1. Find the rate at which the following vary with respect to a change in the radius :

(*i*) the area of a circle of radius r,

(*ii*) the total suface of a cylinder of radius r and height h,

(*iii*) the curved surface of a cone of radius r and height h,

(*iv*) the volume of a sphere of radius r.

2. The frequency n of vibration of a string of diameter D, length L and specific gravity σ, stretched with a force T, is given by

$$n=\frac{1}{DL}\sqrt{\left(\frac{gT}{\pi\sigma}\right)}.$$

Find the rate of change of the frequency when D, L, σ and T are varied singly.

3. A triangle has two of its vertices at $(-a, 0)$ and $(a, 0)$, and the third (x, y) moves along the line $y=mx$. If A be its area, show that $\frac{dA}{dx}=ma$.

4. Find the value of θ for which the angle θ changes twice as fast as its sine.

5. If the side of an equilateral triangle increases uniformly

at the rate of 0·2 inches per second, at what rate is the area increasing when the side is 10 inches ?

6. An inverted cone has a depth of 10 cms. and a base of radius 5 cms. Water is poured into it at the rate of $1\frac{4}{5}$ c.c. per min. Find the rate at which the level of the water in the cone is rising when the depth is 4 cms. (*Madras, 1943*)

7. A point moves so that its distance s from a fixed origin at time t is expressed by

$$s = ae^{-bt}\sin(wt+\alpha)\ ;$$

find the velocity of the point at any time t, and show that it becomes separated by equal intervals π/w.

8. If a point moves in a straight line and its distance s from a fixed point in the straight line be a quadratic function of the time t, then show that its acceleration is a constant.

9. If in the previous question s^2 is a quadratic function of t, show that the acceleration varies as $1/s^3$ except in a particular case.

10. A particle moves in a straight line such that the distance travelled from a fixed point in any time t is given by $s=a\sin(\sqrt{\mu}t)$. Find its velocity and acceleration when it is at a distance s from the fixed point. Also find its distance from the fixed point when (*i*) its velocity is zero, (*ii*) its acceleration is zero.

11. A rod AB, of constant length, slides with its ends A and B on two fixed perpendicular lines OX and OY respectively. Show that when A moves towards O, B moves away from O and *vice versa* and that their velocities are inversely proportional to their distances from O.

If $AB=10$ ft., $OA=8$ ft. and A is moving away from O with a velocity of 2 ft. per sec., find the velocity of B.

12. A light is a ft. above and directly over a horizontal path on which a man b ft. high is walking at a speed of x miles an hour away from the light. Find (*i*) the velocity of the end of his shadow, (*ii*) the rate at which his shadow is increasing.

13. A man, 5 ft. tall, is walking towards a lamp-post at uniform rate of 88 yards per minute. The lamp on the post is at a height of 15 feet above the ground. At what rate is the length of the man's shadow changing ? Is it increasing or decreasing ?

(*Agra, 1942*)

14. A man on a wharf, 20 feet above the level, pulls in a rope to which a boat is attached at the rate of 4 feet per second. At what rate is the boat approaching the shore, when there is still 25 feet of rope out ? (*Agra, 1944*)

5·4 Application to small errors and approximate evaluations. Suppose that $y=f(x)$ is a function of x and the value of y is calculated from the observed value of x by means of this relation. Suppose further that there is a small error δx in the observed value of x, and δy is the consequent small error in the

calculated value of y, *i.e.*, the true values of x and y are $x+\delta x$ and $y+\delta y$ respectively, then

$$\delta y=f(x+\delta x)-f(x)=f'(x)\,\delta x+\varepsilon.\delta x$$

by (1) of Art. 2·4, where $\varepsilon\to 0$ as $\delta x\to 0$. Now if δx be very small, then the term $\varepsilon.\delta x$ is smaller still and may be neglected in comparison with the term $f'(x)\delta x$ unless $f'(x)=0$. Hence, when $f'(x)\neq 0$, the error in the value of y is given *approximately* by the relation

$$\delta y=f'(x)\,\delta x. \tag{1}$$

It may be emphasised that this is not an exact equation. The symbol '$\doteq$' is very often used in place of '$=$' to indicate that the relation is true approximately. This approximate relation (1) may be obtained from the exact relation

$$dy=f'(x)\,dx$$

by replacing the differentials by the corresponding increments.

The case when $f'(x)=0$ will be considered in the second volume.

The quantity $\delta y/y$ which measures the error per unit in the value of y is called the *relative error* or the *proportional error* and is obtained most easily by logarithmic differentiation. The *percentage error* is equal to the relative error multiplied by one hundred.

If the values of a function $f(x)$ and its derivative $f'(x)$ are known for a value x of the independent variable, then the relation (1) can be used to calculate approximately the small increment δy in the value of y corresponding to a small increment δx in the value of x. We can thus calculate approximately the value $y+\delta y$ of y corresponding to the value $x+\delta x$ of x.

Ex. 1. *Using differentials calculate $\sqrt{99}$ approximately.*

$$\text{Let } y=\sqrt{x}, \text{ then } \delta y\doteq\frac{1}{2\sqrt{x}}\delta x.$$

If we take $x=100$, and $\delta x=-1$, then

$$y=\sqrt{100}=10, \qquad \delta y\doteq-\frac{1}{2\times 10}$$

$$\therefore \qquad y+\delta y=\sqrt{(x+\delta x)}=\sqrt{(99)}$$

$$=10-\frac{1}{20}=9{\cdot}95 \text{ app.}$$

Ex. 2. *The area S of a triangle is calculated by measuring b, c and A. If there be an error δA in the measurement of A, show that the relative error in area is given by*

$$\frac{\delta S}{S}=\cot A\,\delta A.$$

The area S is given by the formula

$$S=\tfrac{1}{2}bc\sin A.$$

Forming the differentials *w.r.* to A, we get approximately

$$\delta S = \tfrac{1}{2}\, bc \cos A . \delta A$$

$$\therefore \qquad \frac{\delta S}{S} = \cot A . \delta A \text{ approximately.}$$

EXAMPLES-XVIII

Using differentials calculate approximately :

1. $\sqrt{(9999)}$. 2. $(342)^{1/3}$. 3. $(257)^{1/4}$. 4. $(513)^{1/3}$.

5. In a tangent galvanometer, the tangent of the deflection of the needle is proportional to the current, show that the relative error in the value of the current due to an error in the reading of the deflection is least when the deflection is 45°.

6. An ellipse has semi-axes a and b. Prove that the relative errors in the area A due to errors δa and δb in the sides are given by

$$\frac{\delta A}{A} = \frac{\delta a}{a} \text{ and } \frac{\delta A}{A} = \frac{\delta b}{b} \text{ respectively.}$$

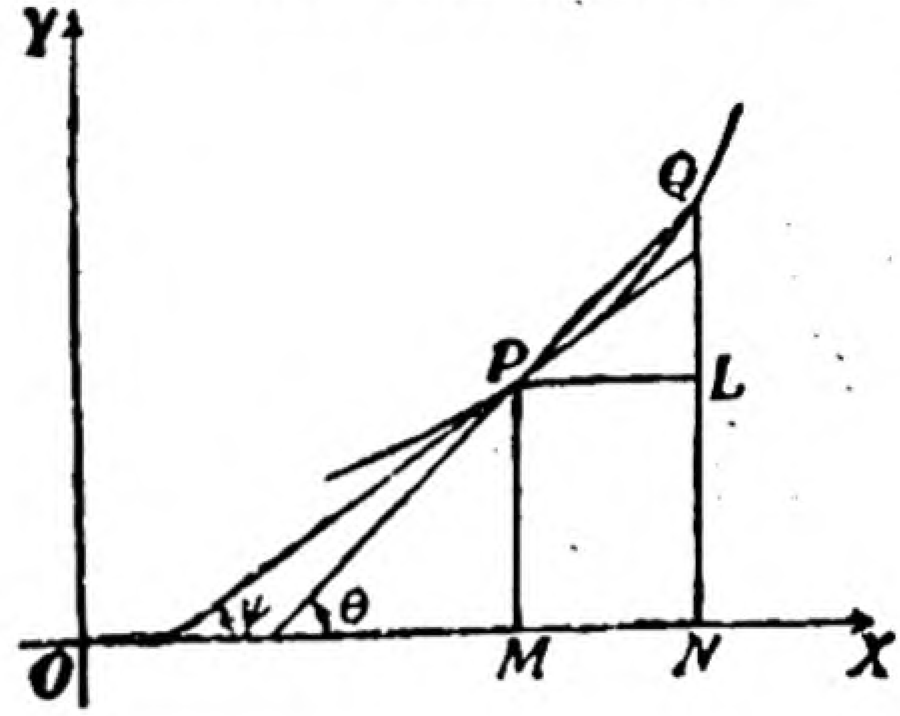
7. A triangle ABC is determined from a, b, A ; prove that if A receives an increment δA, the area S receives an increment δS where

$$\frac{\delta S}{S} = -\frac{\cos C}{\sin A \cos B}\delta A.$$

CHAPTER VI

TANGENTS AND NORMALS TO PLANE CURVES

6·1 Geometrical significance of the derivative: Let $y=f(x)$ be any function of x. Draw its graph and let $P(x, y)$, $Q(x+\delta x, y+\delta y)$ be any two points on it. Draw PM and QN perpendiculars to OX and $PL \perp NQ$. Then



$$PL = MN = x + \delta x - x = \delta x$$

and
$$LQ = NQ - MP$$
$$= y + \delta y - y = \delta y.$$

If the secant PQ makes an angle θ with the x-axis, then $\theta = \angle LPQ$, and therefore

$$\tan\theta = \frac{LQ}{PL} = \frac{\delta y}{\delta x}.$$

Thus the ratio of the increments of y and x is equal to the tangent of the angle which the secant makes with the x-axis.

Suppose now that the graph is continuous in the neighbourhood of P and possesses a unique tangent at P. Then, by the definition of a tangent, when $\delta x \to 0$, Q approaches P and the limit of the secant PQ is the tangent at P. Let the tangent at P make an angle ψ with the x-axis, then

$$\tan\psi = \text{Lt} \tan\theta = \text{Lt}\frac{\delta y}{\delta x} = \frac{dy}{dx} = f'(x).$$

Hence $f'(x)$ *represents the slope of the tangent to the curve* $y=f(x)$ *at the point* $P(x, y)$.

Conversely, if $f'(x)$ exists, the slope of the secant PQ approaches a definite limiting value, namely $f'(x)$, and therefore the secant PQ has a definite limit when $Q \to P$. Hence, if $f'(x)$ exists, the curve has a unique tangent at P. Hence *the curve* $y=f(x)$ *has a definite tangent at* $P(x, y)$ *if, and only if,* $f'(x)$ *exists at* P.

6·11 Equation of the tangent. Let (x_1, y_1) be a point on the curve, $y=f(x)$, then the slope of the tangent to the curve at the point (x_1, y_1) is, by the preceding article, the value of $\dfrac{dy}{dx}$ at (x_1, y_1). If this value be denoted by $\dfrac{dy_1}{dx_1}$ then from Coordinate Geometry, the equation of the tangent at (x_1, y_1) is

$$y - y_1 = \frac{dy_1}{dx_1}(x - x_1) \tag{1}$$

As the symbol y_1 is very often used to denote the derivative of y *w.r.* to x, we shall use the symbols X and Y to denote the current co-ordinates x and y of a point and use x and y to denote the co-ordinates of the point of contact (instead of x_1 and y_1) in the case of the tangent and the foot of the normal in the case of the normal. In the equation to the curve, however, we shall continue using x, y to denote the current coordination of a point on the curve. With this notation, the equation of the tangent to the curve $y=f(x)$ at (x, y) may be written as

$$Y-y=\frac{dy}{dx}(X-x). \tag{1'}$$

6·12. Equation of the normal. By definition the normal at a point of a curve is the line through the point perpendicular to the tangent at that point. Hence the slope of the normal at the point (x, y) of the curve $y=f(x)$ is the negative reciprocal of the slope of the tangent at the point and is therefore $-1/\frac{dy}{dx}$. Therefore the equation of the normal at the point (x, y) is

$$Y-y=-\frac{1}{\frac{dy}{dx}}(X-x)$$

or

$$(X-x)+\frac{dy}{dx}(Y-y)=0. \tag{2}$$

Cor. 1. If $\frac{dy}{dx}$ is zero at the point (x, y), then the tangent at the point is parallel to the x-axis and its equation is $Y-y=0$. The normal is parallel to the y-axis and its equation is $X-x=0$.

Cor. 2. If $\frac{dy}{dx}$ is infinite at the point (x, y), then the tangent is parallel to the y-axis and the normal to the x-axis and their equations are $X-x=0$ and $Y-y=0$ respectively.

Cor. 3. If the equation of a curve is given in the parametric form

$$x=f(t),\ y=\varphi(t)$$

then, at the point 't' on the curve,

$$\frac{dy}{dx}=\frac{dy}{dt}\bigg/\frac{dx}{dt}=\frac{\varphi'(t)}{f'(t)},$$

and therefore the slopes of the tangent and normal are $\varphi'(t)/f'(t)$ and $-f'(t)/\varphi'(t)$ respectively.

Hence the equation of the tangent at 't' is

$$y-\varphi(t)=\frac{\varphi'(t)}{f'(t)}\{x-f(t)\}$$

or
$$\frac{y-\varphi(t)}{\varphi'(t)}=\frac{x-f(t)}{f'(t)}, \tag{3}$$
and the equation of the normal is
$$y-\varphi(t)=-\frac{f'(t)}{\varphi'(t)}\{x-f(t)\}$$
or
$$\{x-f(t)\}f'(t)+\{y-\varphi(t)\}\varphi'(t)=0. \tag{4}$$

Ex. 1. *Find the equations of the tangent and normal to the curve* $y=x^3$ *at the point* $(2, 8)$.

Differentiating *w.r.* to x, we have $\dfrac{dy}{dx}=3x^2$.

At $(2, 8)$, $\dfrac{dy}{dx}=3.2^2=12$.

$\therefore$ Slope of tangent$=12$.

Hence the equation of the tangent at $(2, 8)$ is
$$Y-8=12(X-2) \text{ or } Y=12x-16.$$
Again, equation of the normal at $(2, 8)$ is
$$Y-8=-\tfrac{1}{12}(X-2) \text{ or } X+12Y-98=0.$$

Ex. 2. *Find the equation of the tangent and normal at any point of the ellipse given by the equations :*

(*i*) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1.$ (*ii*) $x=a\cos\theta, y=b\sin\theta.$

(*i*) We have $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1=0$. Differentiating *w.r.* to x,
$$\frac{2x}{a^2}+2\frac{y}{b^2}\cdot\frac{dy}{dx}=0, \quad \therefore \quad \frac{dy}{dx}=-\frac{b^2x}{a^2y}.$$
$\therefore$ Equation of the tangent at (x, y) is
$$Y-y=-\frac{b^2x}{a^2y}(X-x),$$
or
$$\frac{xX}{a^2}+\frac{yY}{b^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}. \tag{1}$$

$\because$ (x, y) is a point on the curve, $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, hence the equation of the tangent may be written as
$$\frac{xX}{a^2}+\frac{yY}{b^2}=1. \tag{1'}$$

Again, the equation of the normal at (x, y) is
$$Y-y=+\frac{a^2y}{b^2x}(X-x),$$
or
$$\frac{Y-y}{a^2y}=\frac{X-x}{b^2x}. \tag{2}$$

(*ii*) Here $x=a\cos\theta, y=b\sin\theta$.

$\therefore \quad \frac{dx}{d\theta}=-a\sin\theta, \frac{dy}{d\theta}=b\cos\theta$, and $\frac{dy}{dx}=-\frac{b\cos\theta}{a\sin\theta}$.

$\therefore$ Equation of the tangent is

$$Y-b\sin\theta=-\frac{b\cos\theta}{a\sin\theta}(X-a\cos\theta),$$

or $$b\cos\theta X+a\sin\theta . Y=ab\cos^2\theta+ab\sin^2\theta=ab$$

or $$\frac{X\cos\theta}{a}+\frac{Y\sin\theta}{b}=1. \tag{3}$$

Again, equation of the normal is

$$Y-b\sin\theta=\frac{a\sin\theta}{b\cos\theta}(X-a\cos\theta)$$

or $$a\sin\theta . X-b\cos\theta . Y=(a^2-b^2)\sin\theta\cos\theta$$

or $$a\sec\theta . X-b\operatorname{cosec}\theta . Y=a^2-b^2. \tag{4}$$

Ex. 8. *Find the condition that the st. line*

$$x\cos\alpha+y\sin\alpha=p \tag{1}$$

may touch the curve $$\frac{x^m}{a^m}+\frac{y^m}{b^m}=1. \tag{2}$$

Let $P(x, y)$ be the point of contact. Differentiating the equation to the curve *w. r.* to x we have

$$m\frac{x^{m-1}}{a^m}+m\frac{y^{m-1}}{b^m}\cdot\frac{dy}{dx}=0, \quad \therefore \quad \frac{dy}{dx}=-\frac{b^m x^{m-1}}{a^m y^{m-1}}.$$

Hence the equation of the tangent at (x, y) is

$$Y-y=-\frac{b^m x^{m-1}}{a^m y^{m-1}}\cdot(X-x),$$

or $$\frac{Xx^{m-1}}{a^m}+\frac{Y.y^{m-1}}{b^m}=\frac{x^m}{a^m}+\frac{y^m}{b^m},$$

i.e., $$\frac{X.x^{m-1}}{a^m}+\frac{Y.y^{m-1}}{b^m}=1, \tag{3}$$

since (x, y) is a point on the curve.

But the equation of the tangent at (x, y) is given to be

$$X\cos\alpha+Y\sin\alpha=p. \tag{1}$$

$\therefore$ (1) and (3) represent the same straight line. Hence comparing coefficients of like terms, we get

$$\frac{x^{m-1}}{a^m\cos\alpha}=\frac{y^{m-1}}{b^m\sin\alpha}=\frac{1}{p}.$$

$$\therefore \quad x=\left(\frac{a^m\cos\alpha}{p}\right)^{1/(m-1)}, \; y=\left(\frac{b^m\sin\alpha}{p}\right)^{1/(m-1)}.$$

These are the co-ordinates of the point of contact.

Since this point lies on the given curve, we have

$$\frac{\left(\frac{a^m \cos \alpha}{p}\right)^{m/(m-1)}}{a^m}+\frac{\left(\frac{b^m \sin \alpha}{p}\right)^{m/(m-1)}}{b^m}=1.$$

or
$$(a \cos \alpha)^{m/(m-1)}+(b \sin \alpha)^{m/(m-1)}=p^{m/(m-1)}.$$

Note. Putting $m=2$, we get

$$a^2 \cos^2 \alpha+b^2 \sin^2 \alpha=p^2$$

as the condition that the line $x \cos \alpha+y \sin \alpha=p$ may touch the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,

EXAMPLES XIX

1. Find the equations of the tangent and normal at any point of the following curves :

(i) $y^2=4ax$. (ii) $a^2y=x^3$. (iii) $x^3+y^3=3axy$.

(iv) $x=at^2, y=2at$. (v) $x=a \sec \theta, y=b \tan \theta$.

(vi) $x=a(t+\sin t), y=a(1-\cos t)$. *(Andhra, 1947)*

2. (i) Find the equation of the tangent at the point $(a, 0)$ to the curve $x^4+a^2xy=a^4$.

(ii) Find the equation of the normal at (a, a) to the curve $x^2y^3=a^5$.

3. If a normal to the curve $x^{2/3}+y^{2/3}=a^{2/3}$ makes an angle θ with the x-axis, show that its equation is

$$y \cos \theta-x \sin \theta=a \cos 2\theta.$$ *(Panjab, 1958 ; All., '39)*

4. Find the points on the curve $y=(x-1)(x-2)(x-3)$ at which the tangents are parallel to the x-axis.

5. Show that the tangents at the points where the straight line $ax+hy=0$ meets the conic $ax^2+2hxy+by^2=1$ are parallel to the x-axis. *(Delhi, 1954)*

6. Find the equation of the tangent to the parabola $y^2=4x+5$ parallel to the line $2x-y=3$.

7. Find the equation of the normal to $3x^2-y^2=14$ parallel to $x+3y=4$.

8. P_1, P_2 are any two points on the parabola $y=ax^2+bx+c$. M is the point of contact of the tangent to the parabola which is parallel to P_1P_2. Show that the abscissa of M lies midway between the abscissae of P_1 and P_2.

9. Show that the tangent to the curve $3xy^2-2x^2y=1$ at $(1, 1)$ meets the curve again at $(-\frac{16}{5}, -\frac{1}{20})$.

10. Prove that the equation of the tangent at the point $(4m^2, 8m^3)$ of the curve $y^2=x^3$ is $y=3mx-4m^3$ and that it meets the curve again at the point $(m^2, -m^3)$. Show that if $9m^2=2$, the tangent is also a normal to the curve.

11. Show that the condition that the line

$$x \cos \alpha + y \sin \alpha = p$$

may touch the curve $x^m y^n = a^{m+n}$ is

$$p^{m+n} m^m n^n = (m+n)^{m+n} a^{m+n} \cos^m \alpha \sin^n \alpha. \qquad \textit{(Panjab, 1951)}$$

12. If $x \cos \alpha + y \sin \alpha = p$ touches the curve

$$\left(\frac{x}{a}\right)^{n/(n-1)} + \left(\frac{y}{b}\right)^{n/(n-1)} = 1.$$

prove that $(a \cos \alpha)^n + (b \sin \alpha)^n = p^n$. *(Delhi, 1955)*

13. If $lx + my = 1$ touches the curve $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 1$, show that $(al)^{n/(n-1)} + (bm)^{n/(n-1)} = 1$.

14. Find the condition that the line $Ax + By = 1$ may be normal to the curve $a^{n-1} y = x^n$.

15. Prove that $x/a + y/b = 1$ touches the curve $y = be^{-x/a}$ at the point where the curve crosses the y-axis.

16. Prove that the curve $(x/a)^n + (y/b)^n = 2$ touches the straight line $x/a + y/b = 2$ at the point (a, b) whatever be the value of n. *(Agra, 1949)*

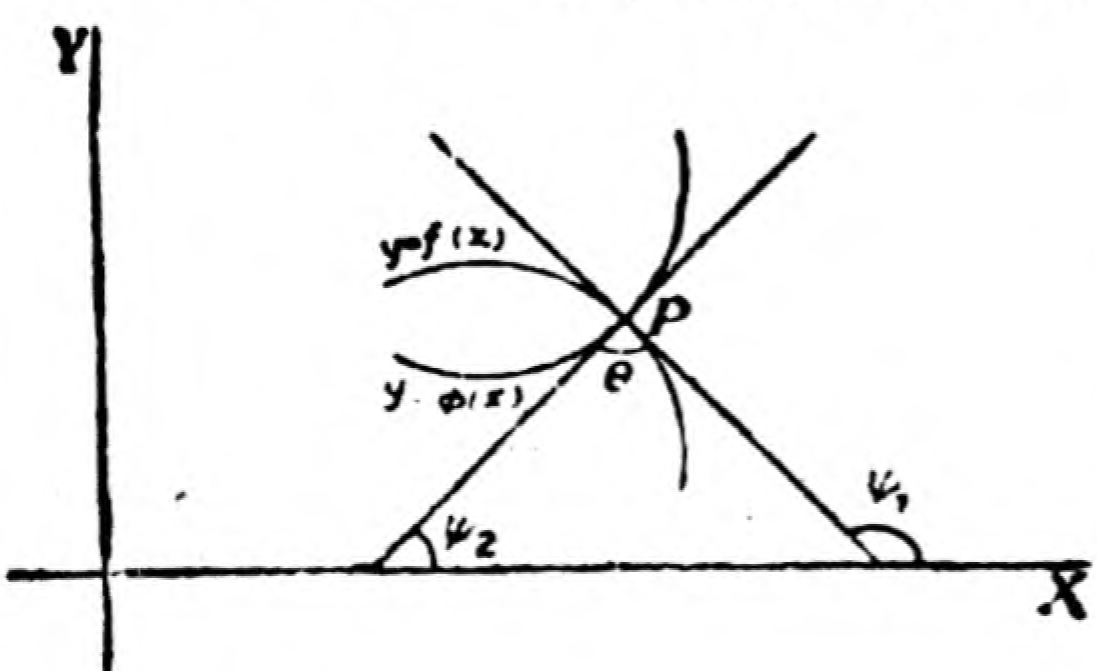
17. Prove that all points of the curve $y^2 = 4a\{x + a \sin(x/a)\}$ at which the tangent is parallel to the axis of x lie on a parabola. *(Panjab, 1959 ; Patna, '35)*

6 2 Angle of intersection of two curves. Consider two curves $y = f(x)$ and $y = \phi(x)$ intersecting at a point $P(x_1, y_1)$. The angle between the respective tangents to the curves at the point P is called the angle of intersection of the two curves.

Let m_1, m_2 be the slopes of the two tangents at P. Then by coordinate geometry, the angle θ from the line of slope m_2 to the line of slope m_1 is given by $\tan \theta = \dfrac{m_1 - m_2}{1 + m_1 m_2}$.

The supplement of this angle is $\tan^{-1} \dfrac{m_2 - m_1}{1 + m_1 m_2}$.

Now $m_1 = \tan \psi_1 = f'(x_1)$, $m_2 = \tan \psi_2 = \phi'(x_1)$,



$$\therefore \tan \theta = \frac{f'(x_1) - \phi'(x_1)}{1 + f'(x_1).\phi'(x_1)}$$

The two curves cut at right angles if

$$m_1 m_2 = -1,$$

i.e., if $f'(x_1)\ \phi'(x_1) = -1$, and we say that the curves cut **orthogonally**.

The two curves **touch** each other if

$$f'(x_1) = \phi'(x_1).$$

Ex. 1. *Find the angle of intersection of the parabola* $y^2=2x$ (1)
and the circle $x^2+y^2=8$. (2)

Substituting for y^2 from (1) in (2), we get

$$x^2+2x-8=0, \quad \text{or} \quad (x-2)(x+4)=0$$

$$\therefore \quad x=2 \quad \text{or} \quad -4.$$

When $x=2, y=\pm 2$. When $x=-4$, the corresponding values of y are imaginary and may be discarded. $\therefore$ The two points of intersection are $P(2, 2)$, $Q(2, -2)$.

Differentiating (1), we find

$$2y\frac{dy}{dx}=2, \quad \therefore \quad \frac{dy}{dx}=\frac{1}{y}. \tag{3}$$

Differentiating (2), we find

$$2x+2y\frac{dy}{dx}=0, \quad \therefore \quad \frac{dy}{dx}=-\frac{x}{y}. \tag{4}$$

At the point $P(2, 2)$, the slopes of the tangents, from (3) and and (4), are $\frac{1}{2}$ and -1. Hence the angle θ between the tangents at P is given by

$$\tan\theta=\frac{\frac{1}{2}-(-1)}{1+\frac{1}{2}(-1)}=3$$

$$\therefore \quad \theta=\tan^{-1}3=71°36' \text{ nearly.}$$

At the point $Q(2, -2)$, the slopes of the tangents, from (3) and (4), are $-\frac{1}{2}$ and 1. $\therefore$ The angle of intersection θ' at Q is given by

$$\tan\theta'=\frac{1-(-\frac{1}{2})}{1+1.(-\frac{1}{2})}=3.$$

$$\therefore \quad \theta'=\tan^{-1}3=71°36' \text{ App.}$$

We observe that the angles of intersection at P and Q are equal. This could have been anticipated from considerations of symmetry.

Ex. 2. *Show that the curves* $ax^2+by^2=1$ *and* $lx^2+my^2=1$ *intersect at right angles if*

$$\frac{1}{a}-\frac{1}{b}=\frac{1}{l}-\frac{1}{m}.$$ *(Panjab, 1953)*

Let (x_1, y_1) be a point of intersection of the given curves, then

$$ax_1^2+by_1^2=1, \text{ and } lx_1^2+my_1^2=1.$$

Solving these equations for x_1^2 and y_1^2, we get

$$x_1^2=\frac{m-b}{am-bl}, \quad y_1^2=\frac{a-l}{am-bl}. \tag{1}$$

Differentiating the equation of the first curve, we get

$$2ax+2by\frac{dy}{dx}=9 \text{ or } \frac{dy}{dx}=-\frac{ax}{by}. \tag{2}$$

Differentiating the equation of the second curve, we get

$$2lx+2my\frac{dy}{dx}=0 \text{ or } \frac{dy}{dx}=-\frac{lx}{my}. \qquad (3)$$

Therefore the slopes of the tangents to the two curves at (x_1, y_1) are respectively,

$$-\frac{ax_1}{by_1} \text{ and } -\frac{lx_1}{my_1}.$$

The two tangents will cut at right angles if

$$\frac{alx_1^2}{bmy_1^2}=-1, \text{ i.e., if } alx_1^2=-bmy_1^2.$$

Substituting for x_1^2 and y_1^2 from (1), the required condition is

$$\frac{al(m-b)}{am-bl}=-\frac{bm(a-l)}{am-bl}$$

or

$$\frac{1}{a}-\frac{1}{b}=\frac{1}{l}-\frac{1}{m}.$$

EXAMPLES XX

1. Find the angle of intersection of the parabola $y^2=4x$ and the ellipse $8x^2+y^2-6y=0$.

2. Find the angle of intersection of the curves $x^2-y^2=a^2$, and $x^2+y^2=\sqrt{2}a^2$. *(Panjab, 1956 ; Patna, 40)*

3. The curves $yx^2=c_1$ and $xy=c_2$ pass through the point (3, 4). Find the angle at which they intersect.

4. Show that the ellipse $x^2+4y^2=8$ and the hyperbola $x^2-2y^2=4$ intersect orthogonally at four points.

5. Find the angle at which the curve

$$y=16x^3+4x^2-16x+5$$

cuts the x-axis.

6. Show that the curves

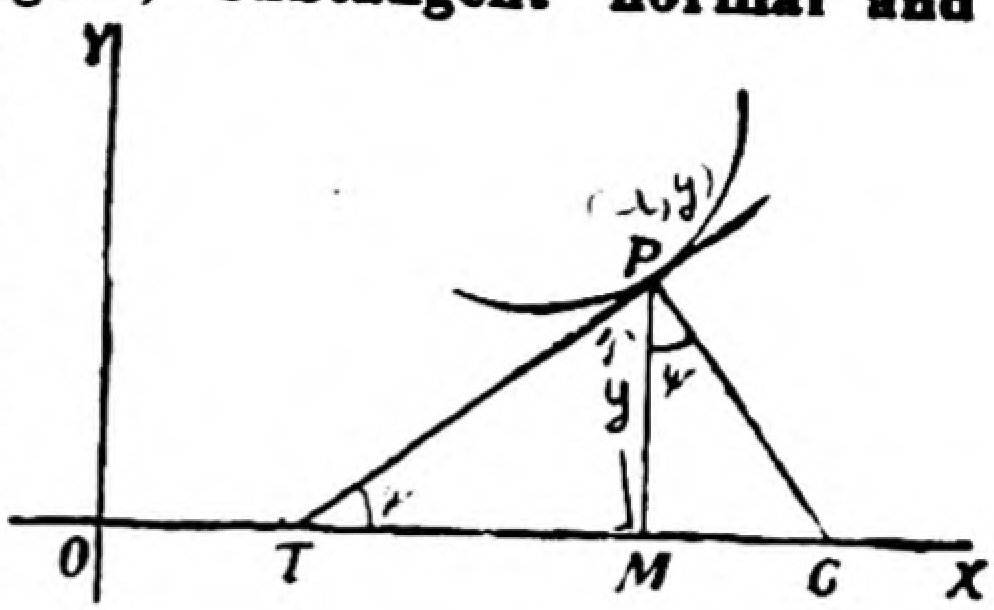
$$\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}=1 \text{ and } \frac{x^2}{a^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}=1$$

intersect at right angles.

7. Show that the curves $x^3-3xy^2=-2$ and $3x^2y-y^3=2$ cut orthogonally.

8. Show that the parabolas $x^2=ay$ and $y^2=2ax$ intersect on the curve $x^3+y^3=3axy$. Also find the angles between each pair at the points of intersection.

6·3 Length of the tangent, subtangent normal and subnormal. Let $P(x, y)$ be any point on the given curve and let the tangent and normal at P meet the x-axis in T and G respectively. Draw the ordinate $MP(=y)$ through P. Then PT is called the length of the tangent, PG the length of the normal at P. TM, the projection of the tangent on the x-axis is called the length of the subtangent and MG, the projection of the normal on the x-axis, is called the length of the subnormal.



Let the tangent at P make an angle ψ with the x-axis, then $\angle MPG$ is also ψ. If we write y_1 instead of $\dfrac{dy}{dx}$, then

$$\tan\psi=\frac{dy}{dx}=y_1,\ \sin\psi=\frac{y_1}{\sqrt{(1+y_1^2)}}\ \text{and}\ \cos\psi=\frac{1}{\sqrt{(1+y_1^2)}}.$$

From the figure, we have

(i) the tangent $PT=y\operatorname{cosec}\psi=y\sqrt{(1+y_1^2)}/y_1$,

(ii) the normal $PG=y\sec\psi=y\sqrt{(1+y_1^2)}$,

(iii) the subtangent $TM=y\cot\psi=y/y_1$,

(iv) the subnormal $MG=y\tan\psi=yy_1$.

In the above figure, y is positive and ψ is acute so that $\tan\psi$, $\sin\psi$ and $\cos\psi$ are all positive. Hence the expresssions (i) –(iv) are all positive. Even if y, $\dfrac{dy}{dx}$, or both, are negative, the above expressions give the numerical values of the various lengths. The student should draw diagrams indicating the different cases.

Ex. 1. *Show that the length of the subtangent is constant for the curve $y=a^x$.*

Differentiating the equation to the curve *w.r.* to x, we have

$$y_1=\frac{dy}{dx}=a^x\,.\log a=y\log a.$$

Subtangent $=y/y_1=1/\log a$, which is constant.

Ex. 2. *Find the length of the normal at any point P of the rectangular hyperbola $x^2-y^2=a^2$ and show that it is equal to the distance of P from the origin.*

Differentiating the equation to the curve *w.r.* to x, we have

$$2x-2y\frac{dy}{dx}=0.\qquad\therefore\ \frac{dy}{dx}=\frac{x}{y}.$$

Length of the normal

$$=y\sqrt{\left\{\left(\frac{dy}{dx}\right)^2+1\right\}}=y\cdot\sqrt{\left\{\left(\frac{x}{y}\right)^2+1\right\}}=\sqrt{(x^2+y^2)}$$

=distance of (x, y) from the origin.

EXAMPLES XXI

1. If the tangent to the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ at any point on it cuts the axes OX and OY at P and Q respectively, prove that $OP+OQ=a$. *(Agra, 1946)*

2. (*a*) In the catenary $y=a\cosh(x/a)$, prove that the length of the normal intercepted between the curve and the axis of x is y^2/a.

4 1970 (*b*) Show that the portion of the tangent to the curve $x^{2/3}+y^{2/3}=a^{2/3}$ which is intercepted between the axes is of constant length. *(Panjab, 1941)*

3. In the tractrix

$$x=a(\cos t+\log\tan\tfrac{1}{2}t),\ y=a\sin t,$$

prove that the portion of the tangent intercepted between the curve and the axis of x is of constant length. *(Rajputana, 1949 ; Panjab, '60)*

4. (*a*) Show that the tangent to the curve

$$\frac{x+\sqrt{(a^2-y^2)}}{a}=\log\frac{a+\sqrt{(a^2-y^2)}}{y}$$

is a constant and equal to a.

(*b*) Show that in the curve

$$x-a+\sqrt{(b^2-y^2)}=b\log\{b+\sqrt{(b^2-y^2)}\}$$

the sum of the subtangent and subnormal is constant.

5. Show that in the curve $y=a\log(x^2-a^2)$, the sum of the tangent and subtangent varies as the product of the co-ordinates of the point. *(Delhi, 1951)*

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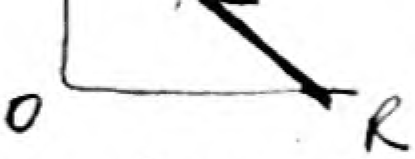
6. Prove that in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the length of the normal varies inversely as the perpendicular from the origin on the tangent. *(Delhi, 1950 ; Panjab, 1953)*

7. Prove that for the catenary $y=c\cosh(x/c)$, the perpendicular dropped from the foot of the ordinate upon the tangent is of constant length. *(Utkal, 1949)*

8. Show that in the exponential curve $y=be^{x/a}$, the subtangent is of constant length, and the subnormal varies as the square of the ordinate. *(Patna, 1937)*

9. Show that in the parabola $y^2=4ax$, (*i*) the subnormal is constant and (*ii*) the subtangent is bisected at the vertex. Hence

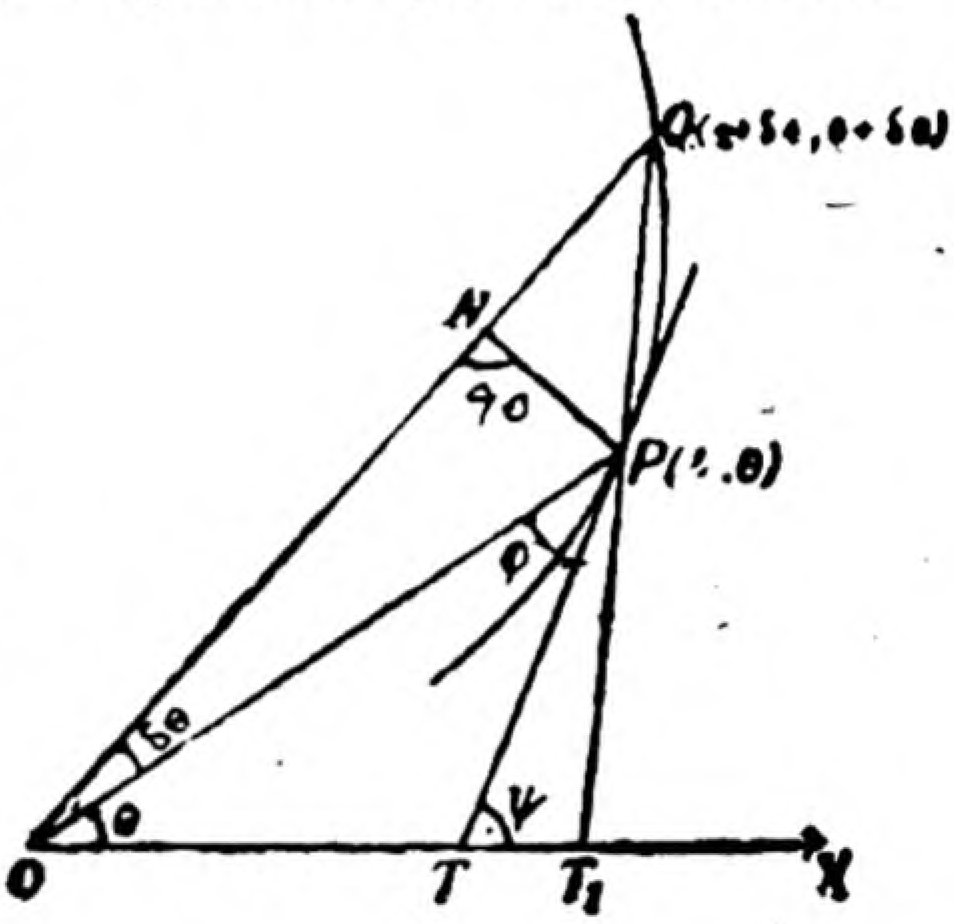


devise a method for constructing the tangent and normal at any point on the parabola.

10. In the curve $x^m y^n = a^{m+n}$, show that the subtangent at any point of the curve varies as the abscissa. (*Banaras, 1944*)

Also show that the portion of the tangent intercepted between the axes is divided in a constant ratio at the point of contact. (*Rajputana, 1950*)

6·4. Polar Co-ordinates. *To find the angle between the radius vector and the tangent.* Let $P(r, \theta)$ be any point on the given curve $f(r, \theta)=0$. Let $Q(r+\delta r, \theta+\delta\theta)$ be a neighbouring point on the curve. Draw PN perpendicular to OQ. Let the angle between the radius vector and the tangent be denoted by ϕ and let ψ denote the angle which the tangent at P makes with the positive direction of the initial line. Evidently ϕ is the limiting value of $\angle NQP$ when $Q \to P$.



From the rt. $\angle$d $\triangle$ ONP,

$$NP = OP \sin PON = r \sin \delta\theta \text{ and } ON = r \cos \delta\theta.$$

$$\because \quad NQ = OQ - ON = (r+\delta r) - r\cos\delta\theta$$
$$= \delta r + r(1-\cos\delta\theta) = \delta\theta + 2r\sin^2 \tfrac{1}{2}\delta\theta.$$

$$\therefore \quad \tan\phi = \underset{Q\to P}{\text{Lt}} \tan NQP = \underset{Q\to P}{\text{Lt}} \frac{NP}{NQ}$$

$$= \underset{\delta\theta\to 0}{\text{Lt}} \frac{r\sin\delta\theta}{\delta r + 2r\sin^2\tfrac{1}{2}\delta\theta}$$

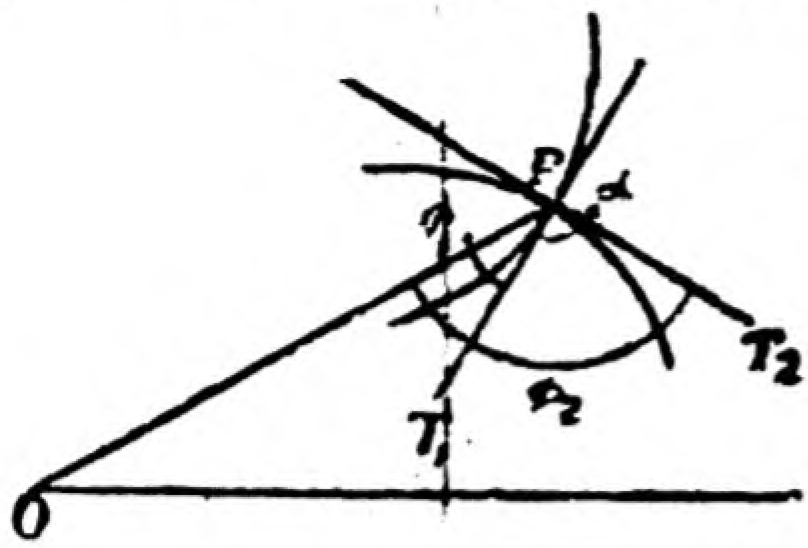
$$= \underset{\delta\theta\to 0}{\text{Lt}} \frac{r\dfrac{\sin\delta\theta}{\delta\theta}}{\dfrac{\delta r}{\delta\theta} + r\dfrac{\sin\tfrac{1}{2}\delta\theta}{\tfrac{1}{2}\delta\theta}\cdot\sin\tfrac{1}{2}\delta\theta}$$

$$= \frac{r.1}{\dfrac{dr}{d\theta} + r.1.0} = \frac{r}{\dfrac{dr}{d\theta}}.$$

Hence $\mathbf{\tan\phi = r\dfrac{d\theta}{dr}}$

1, 2, 7, 6, 8

Cor. Angle of intersection of two curves. Let two curves $r=f_1(\theta)$ and $r=f_2(\theta)$ intersect in P. Let ϕ_1, ϕ_2 be the angles between the common radius vector OP and the tangents PT_1, PT_2 to the two curves respectively. Then the angle of intersection



$$\alpha=|\phi_1-\phi_2|$$

For orthogonal intersection, $\alpha=\frac{1}{2}\pi$, and

$\therefore$ $\tan\phi_1=\tan(\phi_2+\frac{1}{2}\pi)=-\cot\phi_2=-1/\tan\phi_2$.

Hence the value of $r\dfrac{d\theta}{dr}$ for one curve will be the negative reciprocal of the value of $r\dfrac{d\theta}{dr}$ for the second curve.

The two curves touch if $\alpha=0$ and therefore $\tan\phi_1=\tan\phi_2$. Hence in this case the values of $r\dfrac{d\theta}{dr}$ for the two curves will be equal.

Ex. 1. *Show that in the equiangular spiral $r=ae^{\theta\cot\alpha}$, the tangent is inclined at a constant angle to the radius vector.* (*Lucknow, 1949*)

Differentiating the given equation *w.r.* to θ, we have

$$\frac{dr}{d\theta}=ae^{\theta\cot\alpha}.\cot\alpha=r\cot\alpha.$$

$\therefore$
$$r\frac{d\theta}{dr}=\tan\alpha \text{ or } \tan\phi=\tan\alpha.$$

$\therefore$
$$\phi=\alpha, \text{ a constant.}$$

Hence the tangent is inclined at a constant angle to the radius vector. It is on account of this property that the curve is called the equiangular spiral.

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Ex. 2. *Show that the curves*

$$r^n=a^n\sin n\theta \ldots(i) \text{ and } r^n=b^n\cos n\theta \qquad \ldots(ii)$$

intersect orthogonally. (*Panjab, 1952 S*)

Let (r_1, θ_1) be a point of intersection. Then

$$r_1^{\,n}=a^n\sin n\theta_1=b^n\cos n\theta_1 \qquad \ldots(iii)$$

From (i), taking logarithms, we get

$$n\log r=n\log a+\log\sin n\theta.$$

Differentiating *w.r.* to θ, we get

$$\frac{n}{r}\cdot\frac{dr}{d\theta}=\frac{1}{\sin n\theta}\cdot\cos n\theta.n,$$

$\therefore$ $$r\frac{d\theta}{dr}=\tan n\theta, \text{ or } \tan\phi_1=\tan n\theta.$$

$$\phi_1=n\theta_1 \text{ at the point } (r_1, \theta_1). \qquad \ldots(iv)$$

From (*ii*), taking logarithms, we get

$$n\log r=n\log b+\log\cos n\theta.$$

Differentiating *w.r.* to θ_1

$$\frac{n}{r}\cdot\frac{dr}{d\theta}=\frac{1}{\cos n\theta}.(-\sin n\theta).n.$$

$\therefore$ $$r\frac{d\theta}{dr}=-\cot n\theta=\tan\left(n\theta+\frac{\pi}{2}\right)$$

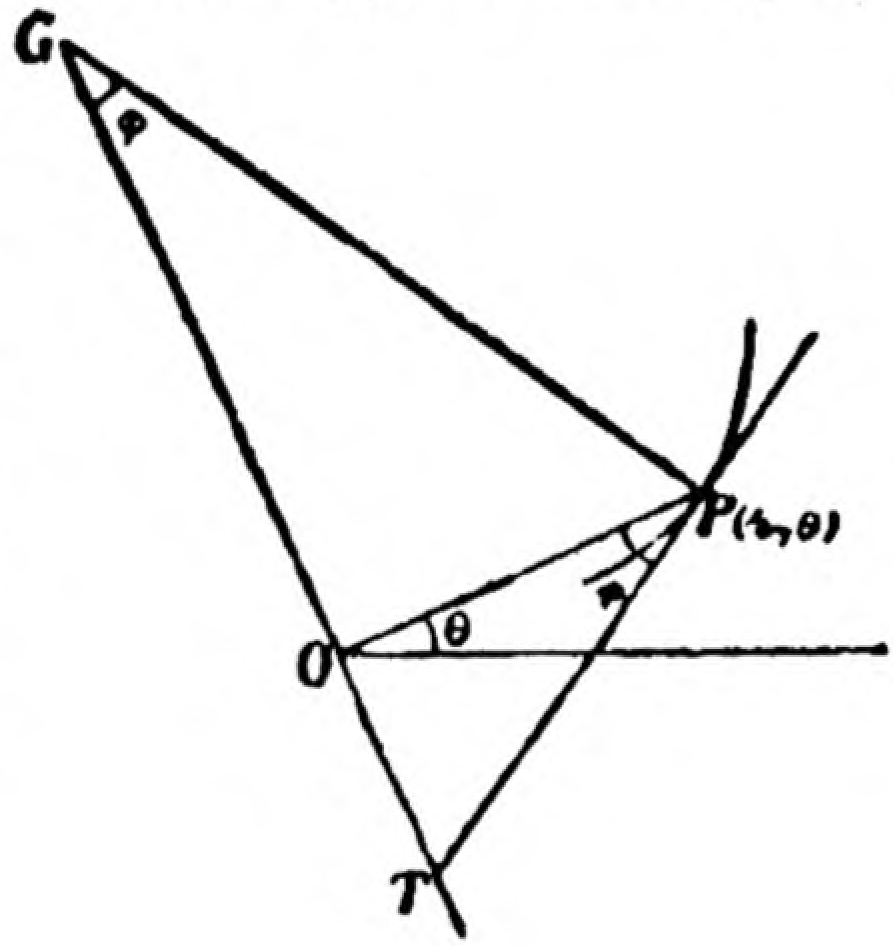
or $$\tan\phi_2=\tan(n\theta+\pi/2)$$

$$\phi_2=n\theta_1+\pi/2 \text{ at the point } (r_1, \theta_1).$$

Hence the angle of intersection $=\phi_2-\phi_1=\pi/2$.

$\therefore$ The two curves intersect orthogonally.

§ 41 The polar tangent, polar normal, polar subtangent and polar subnormal. Let $P(r, \theta)$ be any point on the curve $f(r, \theta)=0$. Through the origin O, draw GOT perpendicular to the radius vector OP meeting the tangent PT in T and the normal PG in G. Then the segments PT and PG are called the polar tangent and the polar normal respectively and the segments OT and OG are called the polar subtangent and the polar subnormal repectively. From the figure :



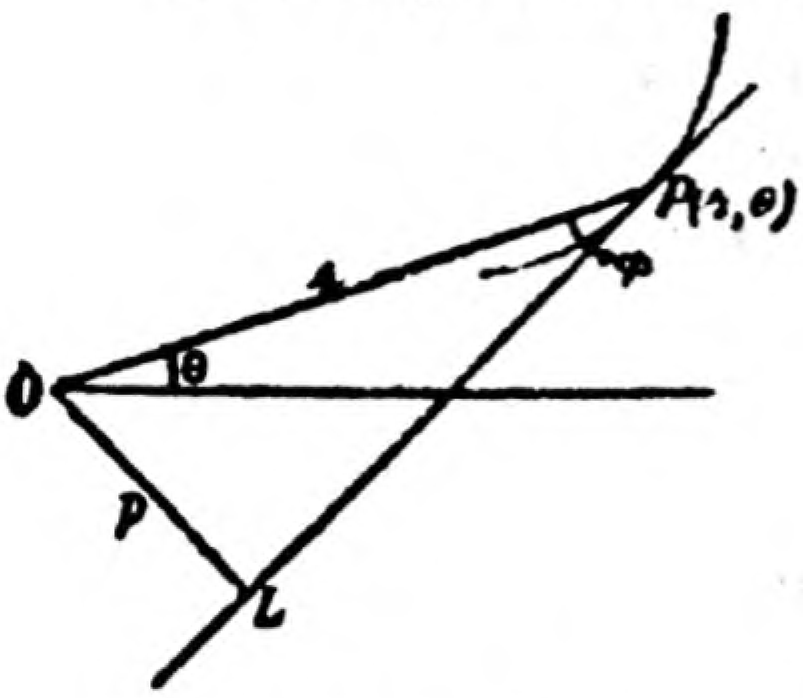
$$\text{Polar tangent}=PT=r\sec\phi$$
$$=r\sqrt{(1+\tan^2\phi)}$$
$$=r\sqrt{\left\{1+\left(r\frac{d\theta}{dr}\right)^2\right\}}$$

$$\text{Polar normal}=PG=r\operatorname{cosec}\phi=r\sqrt{(1+\cot^2\phi)}$$
$$=r\sqrt{\left\{1+\left(\frac{1}{r}\,\frac{dr}{d\theta}\right)^2\right\}}=\sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}$$

$$\text{Polar subtangent}=OT=OP\tan\phi=r.r\frac{d\theta}{dr}=r^2\,\frac{d\theta}{dr}.$$

$$\text{Polar subnormal}=OG=OP\cot\phi=r.\frac{1}{r}\cdot\frac{dr}{d\theta}=\frac{dr}{d\theta}.$$

6·42 Perpendicular from the pole on the tangent. Let p be the length of the perpendicular OL from the pole O on the tangent at $P(r, \theta)$; then from the right-angled $\triangle OLP$ we have



$$\mathbf{p = r \sin \phi,} \tag{1}$$

$$\text{and } \therefore \frac{1}{p^2} = \frac{1}{r^2 \sin^2\phi} = \frac{1}{r^2} \operatorname{cosec}^2 \phi$$
$$= \frac{1}{r^2}\left[1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right]$$

$$\text{Hence } \mathbf{\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2.} \tag{2}$$

6·43 It is often convenient to use $u = 1/r$, the reciprocal of the radius vector r instead of r itself. The formulae of Arts. 6·41 and 6·42 can then be expressed in terms of u.

If $\quad u = 1/r, \quad$ then $\quad r = 1/u,$

and so $\quad \dfrac{du}{d\theta} = -\dfrac{1}{r^2}\dfrac{dr}{d\theta}, \quad \dfrac{dr}{d\theta} = -\dfrac{1}{u^2}\dfrac{du}{d\theta}.$

Hence, we have

the polar subtangent $= r^2 \dfrac{d\theta}{dr} = -\dfrac{d\theta}{du}.$

the polar subnormal $= \dfrac{dr}{d\theta} = -\dfrac{1}{u^2}\dfrac{du}{d\theta},$

and $\quad \mathbf{\dfrac{1}{p^2}} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2 = \mathbf{u^2 + \left(\dfrac{du}{d\theta}\right)^2}.$

Ex. *For the cardioid $r = a(1-\cos\theta)$, prove that (i) $\phi = \frac{1}{2}\theta$. (ii) $2ap^2 = r^3$ and (iii) the polar subtangent $= 2a \sin^2 \frac{1}{2}\theta \tan \frac{1}{2}\theta$.*

(Panjab, 1950)

Differentiating the equation of the curve w r. to θ,

$$\frac{dr}{d\theta} = a \sin\theta \text{ and so } r\frac{d\theta}{dr} = \frac{a(1-\cos\theta)}{a\sin\theta} = \tan \tfrac{1}{2}\theta.$$

$$\therefore \qquad \tan\phi = \tan \tfrac{1}{2}\theta \quad \text{or} \quad \phi = \tfrac{1}{2}\theta. \tag{1}$$

(*ii*) We have $\quad p = r\sin\phi = r \sin \frac{1}{2}\theta \qquad$ [by (1)

$$= a(1-\cos\theta)\sin\tfrac{1}{2}\theta = 2a\sin^3\tfrac{1}{2}\theta, \tag{2}$$

and $$r = a(1-\cos\theta) = 2a\sin^2\tfrac{1}{2}\theta. \tag{3}$$

Hence eliminating θ between (2) and (3),

$$2ap^2 = r^3$$

(*iii*) The polar subtangent

$$= r^2\frac{d\theta}{dr} = \frac{a^2(1-\cos\theta)^2}{a\sin\theta} = 2a\sin^2\tfrac{1}{2}\theta\tan\tfrac{1}{2}\theta.$$

EXAMPLES XXII

Find the angle between the radius vector and the tangent in each of the following curves :

1. $r=a(1+\sin\theta)$ at $\theta=\frac{1}{6}\pi$. **2.** $r=a\operatorname{cosec}^2\frac{1}{2}\theta$ at $\theta=\frac{1}{3}\pi$.

3. $r^2=a^2\cos 2\theta$ at $\theta=\frac{1}{6}\pi$. **4.** $r=a\theta$ at any point.

5. $r^m=a^m(\cos m\theta-\sin m\theta)$ at $\theta=0$. *(Panjab, 1952)*

360 Find the angle of intersection of the following curves :

6. The circles $r=\frac{1}{2}$ and $r=\cos\theta$.

7. The circle $r=a\cos\theta$ and the cardioid $r=a(1-\cos\theta)$. 357

8. $r=a\theta$ and $r=a/\theta$. **9.** $r=a\sin 2\theta$ and $r=a\cos 2\theta$.

10. Show that the circle $r=b$ cuts the curve $r^2=a^2\cos 2\theta+b^2$ at an angle $\tan^{-1}(a^2/b^2)$.

11. Prove that the curves $r=a(1+\cos\theta)$ and $r=a(1-\cos\theta)$ cut orthogonally on the line $\theta=\frac{1}{2}\pi$.

363 **12.** Show that the tangents drawn at the ends of chords of the cardioid $r=a(1+\cos\theta)$ which pass through the pole are perpendicular to each other.

Find the lengths of the polar subtangent, subnormal, tangent and normal for the following curves :

13. $r=a(1+\cos\theta)$ at $\theta=\tan^{-1}\frac{3}{4}$. **14.** $r=a\sin 2\theta$ at $\theta=\frac{1}{6}\pi$.

15. In the curve $r=a\theta$, show that (*i*) $\tan\phi=\theta$, (*ii*) the polar subnormal is constant.

16. In the curve $r=a/\theta$, show that (*i*) $\tan\phi=-\theta$, (*ii*) the polar subtangent is constant.

17. Prove that in the parabola Page 358

$$\frac{2a}{r}=1-\cos\theta$$

(1) $\phi=\pi-\frac{1}{2}\theta$, (2) $p=a\operatorname{cosec}\frac{1}{2}\theta$ and (3) the polar subtangent $=2a\operatorname{cosec}\theta$. *(Panjab, Sept. 1949)*

18. Show that for the curve

$$\theta=\cos^{-1}r-\sqrt{(1-r^2)}/r,$$

the length of the polar tangent is constant.

19. Show that the locus of the extremity of the polar subnormal of the curve $r=f(\theta)$ is $r=f'(\theta-\frac{1}{2}\pi)$. *(Lucknow, 1947)*

Deduce that the locus for the equiangular spiral $r=ae^{m\theta}$ is another equiangular spiral

20. Prove that the locus of the extremity of the polar subtangent of the curve $1/r=f(\theta)$ is $(1/r)+f'(\theta+\frac{1}{2}\pi)=0$.

Hence show that the locus of the extremity of the polar subtangent of the curve

$$r(m+n\tan\tfrac{1}{2}\theta)=1+\tan\tfrac{1}{2}\theta \text{ is } (m-n)r=2(1+\cos\theta).$$

(Allahabad, 1943)

6·5 Pedal equations. The ralation between p, the perpendicular from the origin on the tangent to a curve and r, the radius vector through the point of contact of the tangent, is called the pedal equation of the curve. The pedal equation is also sometimes called the "p-r" equation of the curve.

6·51 *Given the cartesian equatian of a curve, to find the pedal equation.*

Let the cartesian equation of the curve be

$$y=f(x). \qquad \text{...(1)}$$

Equation to the tangent at any point (x, y) is

$$Y-y=f'(x)(X-x) \quad \text{or} \quad Xf'(x)-Y=xf'(x)-y.$$

The length of the perpendicular from the origin (0, 0) to the tangent is given by

$$p=\pm\frac{xf'(x)-y}{\sqrt{\{[f'(x)]^2+1\}}} \quad \text{or} \quad p^2=\frac{\{xf'(x)-y\}^2}{[f'(x)]^2+1} \qquad \text{...(2)}$$

Also

$$r^2=x^2+y^2 \qquad \text{...(3)}$$

Eliminating x and y between (1), (2) and (3), the eliminant, a relation between p and r, is the required pedal equation of the curve.

Ex. *Show that the pedal equation of the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ is } a^2+b^2-r^2=\frac{a^2b^2}{p^2}. \qquad \textit{(Banaras, 1947)}$$

The tangent at (x, y) is

$$\frac{Xx}{a^2}+\frac{Xy}{b^2}=1.$$

If p be the length of the perpendicular from the origin on the tangent, then

$$p=\frac{1}{\sqrt{\left(\frac{x^2}{a}+\frac{y^2}{b^4}\right)}} \quad \text{or} \quad \frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{p^2} \qquad (1)$$

Also $x^2+y^2=r^2.$ (2)

Equations (1) and (2) may be rewritten as

$$\frac{1}{a^2}\cdot\left(\frac{x^2}{a^2}\right)+\frac{1}{b^2}\cdot\left(\frac{y^2}{b^2}\right)-\frac{1}{p^2}=0$$

and

$$a^2\left(\frac{x^2}{a^2}\right)+b^2\left(\frac{y^2}{b^2}\right)-r^2=0.$$

whence

$$\frac{x^2/a^2}{\frac{b^2}{p^2}-\frac{r^2}{b^2}}=\frac{y^2/b^2}{\frac{r^2}{a^2}-\frac{a^2}{p^2}}=\frac{1}{\frac{b^2}{a^2}-\frac{a^2}{b^2}}.$$

$\therefore$ Substituting for x^2/a^2 and y^2/b^2 in the equation of the ellipse, we get

$$\frac{b^2}{p^2}-\frac{r^2}{b^2}+\frac{r^2}{a^2}-\frac{a^2}{p^2}=\frac{b^2}{a^2}-\frac{a^2}{b^2}$$

or
$$r^2\left(\frac{1}{b^2}-\frac{1}{a^2}\right)=\frac{a^2}{b^2}-\frac{b^2}{a^2}-\frac{1}{p^2}(a^2-b^2)$$

or
$$r^2\,\frac{a^2-b^2}{a^2b^2}=\frac{a^4-b^4}{a^2b^2}-\frac{a^2-b^2}{p^2}.$$

$\therefore$
$$a^2+b^2-r^2=\frac{a^2b^2}{p^2}$$

which is the required pedal equation.

Otherwise, we know from geometry that if CP and CD be conjugate semi-diameters of the ellipse and p be the length of the perpendicular from C on the tangent at P, then

(*i*) $CP^2+CD^2=a^2+b^2$, (*ii*) $p.CD=ab$.

Eliminating CD between these two relations and replacing CP by r, we get the required pedal equation.

$$r^2+\frac{a^2b^2}{p^2}=a^2+b^2 \quad \text{or} \quad a^2+b^2-r^2=\frac{a^2b^2}{p^2}$$

6·52 *Given the polar equation of a curve, to find the pedal equation.*

Let the equation of the curve be
$$f(r,\theta)=0. \tag{1}$$

Also, we have
$$\frac{1}{p^2}=\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2. \tag{2}$$

Eliminating θ between (1) and (2), we get a relation between p and r which is the required pedal equation.

Otherwise, we have
$$\tan\phi=r\frac{d\theta}{dr}, \tag{3}$$

and
$$p=r\sin\phi. \tag{4}$$

Eliminating θ and ϕ between (1), (3) and (4), we get the required pedal equation.

Ex. *Find the pedal equation of the parabola*
$$\frac{2a}{r}=1-\cos\theta. \tag{1}$$

Differentiating *w.r.* to θ, we get
$$-\frac{2a}{r^2}\,\frac{dr}{d\theta}=\sin\theta \quad \text{or} \quad \frac{dr}{d\theta}=-\frac{r^2\sin\theta}{2a} \tag{2}$$

$\therefore$
$$\frac{1}{p^2}=\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2=\frac{1}{r^2}+\frac{\sin^2\theta}{4a^2} \tag{3}$$

From (1), $\cos^2\theta=\left(1-\dfrac{2a}{r}\right)^2$,

and from (3), $\sin^2\theta=4a^2\left(\dfrac{1}{p^2}-\dfrac{1}{r^2}\right)$.

Hence eliminating θ by addition, we get

$$\left(1-\frac{2a}{r}\right)^2+4a^2\left(\frac{1}{p^2}-\frac{1}{r^2}\right)=1,$$

which, on simplification, gives

$$p^2=ar$$

as the required pedal equation.

Otherwise, $\tan\phi=r\dfrac{d\theta}{dr}=-r.\dfrac{2a}{r^2.\sin\theta}=-\dfrac{2a}{r\sin\theta}$

$$=-\frac{1-\cos\theta}{\sin\theta} \qquad \text{[by (1)}$$

$$=-\tan\tfrac{1}{2}\theta=\tan(\pi-\tfrac{1}{2}\theta)$$

$\therefore$ $\phi=\pi-\frac{1}{2}\theta$ and $p=r\sin\phi=r\sin\frac{1}{2}\theta$. (4)

Also from (1), $\dfrac{2a}{r}=2\sin^2\frac{1}{2}\theta$. (5)

Eliminating $\sin\frac{1}{2}\theta$ between (4) and (5), the pedal equation is

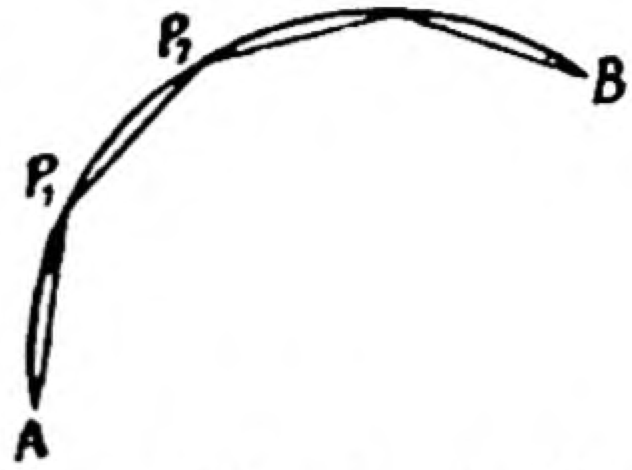
$$p^2=ar.$$

EXAMPLES XXIII

Find the pedal equations of the following curves :

1. $y^2=4a(x+a)$.
2. $x=a\cos^3\theta,\ y=a\sin^3\theta$.
3. $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$.
4. $\dfrac{(x-ae)^2}{a^2}+\dfrac{y^2}{b^2}=1$.
5. $r=a(1-\cos\theta)$. *(Agra, 1948 ; Panjab. '59)*
6. $r^m=a^m\cos m\theta$. *(Panjab, 1959)*
7. $r=ae^{\theta\cot\alpha}$
8. $r=a^\theta$.
9. $r^2=a^2\cos 2\theta$. *(Panjab, 1942)*
10. Find the pedal equation of the conic
$$\frac{a(e^2-1)}{r}=1+e\cos\theta\ (e>1). \qquad \textit{(Panjab, 1952)}$$
11. In the curve
$$x=(a+b)\cos\theta-b\cos\{(a+b)/b\}\theta,$$
$$y=(a+b)\sin\theta-b\sin\{(a+b)/b\}\theta,$$
show that
$$p=(a+2b)\sin\frac{a\theta}{2b},\ \psi=\frac{a+2b}{2b}\theta,\ p=(a+2b)\sin\frac{a\psi}{a+2b}$$
and $$(r^2-a^2)(a+2b)^2=4b(a+b)p^2.$$
12 Show that the pedal equation of a circle of radius a with respect to any point on its circumference is $r^2=2ap$.

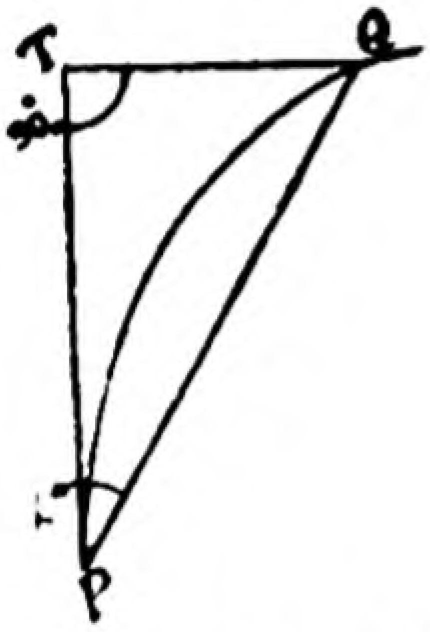
Derivative of the arc



6·6 Length of an arc. Let the arc AB of a curve be divided into a number of small arcs by the points $P_1, P_2, \ldots, P_{n-1}$, taken in order as a point traverses the arc AB from A to B. Let S_n denote the sum of the chords AP_1, $P_1P_2, \ldots, P_{n-1}B$.

If S_n tends to a limit s as the number of points of division tends to infinity such that each pair of consecutive points tends to coincide along the curve, then s is defined as the length of the arc AB. The analytical discussion of the existence of such a limit and other allied topics is beyond the scope of this book and will not be entered into. We shall, therefore, more or less depend upon an intuitive knowledge of the length of an arc.

6·61 Ratio of an arc to chord. If PQ be an arc of a curve supposed to be everywhere concave towards its chord, we prove geometrically that



$$\underset{Q\to P}{\text{Lt}} \frac{\text{arc } PQ}{\text{chord } PQ}=1.$$

Draw $QT \perp PT$, the tangent to the arc at P. It is assumed that Q is sufficiently near P so that the perpendicular QT does not meet the arc in any other point between Q and T. We further assume that

$$TP+TQ>\text{arc } PQ>\text{chord } PQ. \tag{1}$$

Let $\angle QPT=\alpha$, so that $TP=PQ\cos\alpha$, $TQ=PQ\sin\alpha$. Then (1) gives

$$PQ(\cos\alpha+\sin\alpha)>\text{arc } PQ>\text{chord } PQ.$$

$$\therefore \quad \cos\alpha+\sin\alpha>\frac{\text{arc } PQ}{\text{chord } PQ}>1. \tag{2}$$

As Q travels along the curve and tends to coincide with P, $\alpha\to 0$, $\therefore$ $\sin\alpha\to 0$ and $\cos\alpha\to 1$, so that the first member of the inequalities (2)$\to 1$. Hence the ratio $\dfrac{\text{arc } PQ}{\text{chord } PQ}$, lying as it does between unity and a number which tends to unity, also tends to unity as $Q\to P$.

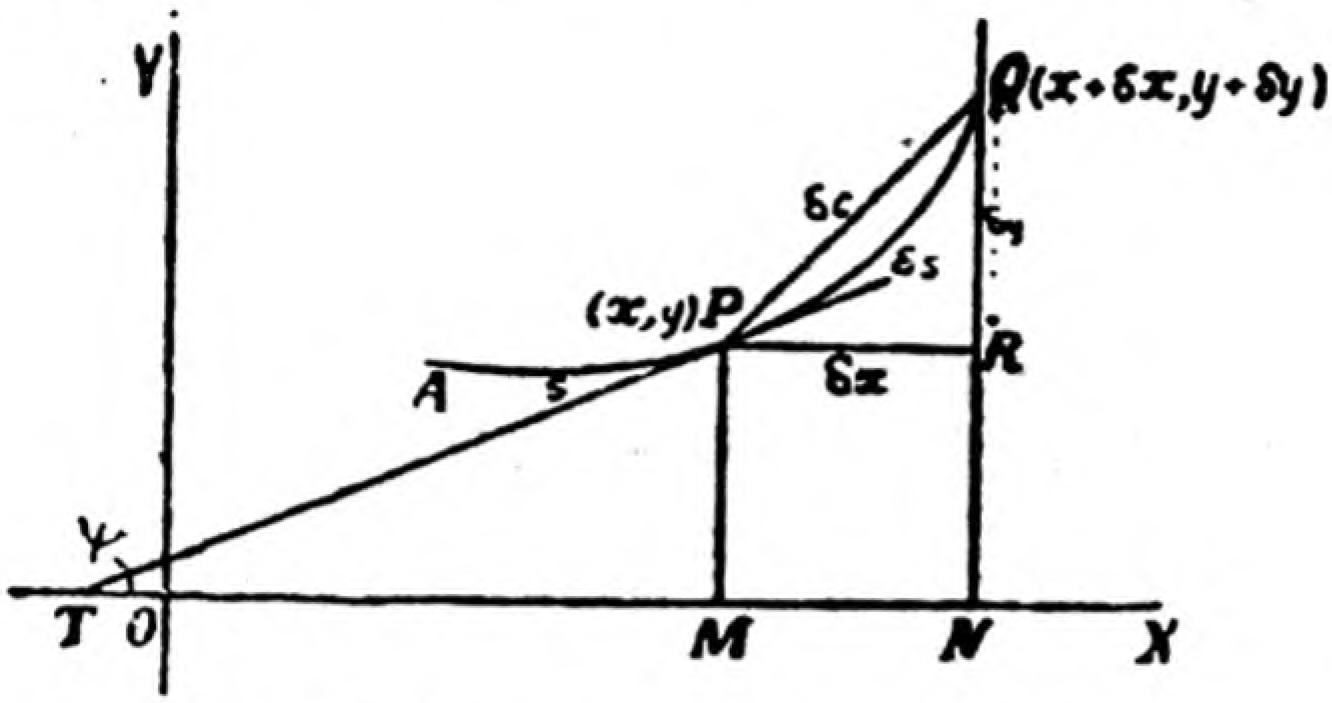
$$\therefore \quad \underset{Q\to P}{\text{Lt}} \frac{\text{arc } PQ}{\text{chord } PQ}=1.$$

Symbolically, if δc denote the chord PQ and δs the arc PQ we have

$$\underset{\delta c\to 0}{\text{Lt}} \frac{\delta s}{\delta c}=1 \quad \text{or} \quad \frac{ds}{dc}=1.$$

We may thus take δs and δc as equivalent infinitesimals.

6·7 Derivative of the arc. Cartesian co-ordinates. Consider the graph of a function $y=f(x)$ as shown in the diagram. Let A be a fixed point on the gragh and let $P(x, y)$ be any point on it. Let the length of the arc AP be s. Evidently s is a function of x or y and we may find the derivative of s *w. r. t.* x or y by the usual method. Let δx be a small increment in the value of x, δy the corresponding increment in y and δs the increment in s. Thus if Q is the point on the curve whose co-ordinates are $(x+\delta x, y+\delta y)$, arc $PQ=\delta s$.



Let the chord PQ be denoted by δc. Then from the right-angled $\triangle\ PQR$,

$$\delta c^2=\delta x^2+\delta y^2,$$

or

$$\left(\frac{\delta c}{\delta x}\right)^2=1+\left(\frac{\delta y}{\delta x}\right)^2.$$

$$\therefore \qquad \left(\frac{\delta s}{\delta x}\right)^2=\left(\frac{\delta s}{\delta c}\right)^2\cdot\left(\frac{\delta c}{\delta x}\right)^2=\left(\frac{\delta s}{\delta c}\right)^2\left\{1+\left(\frac{\delta y}{\delta x}\right)^2\right\}.$$

Proceeding to the limit as $\delta x\to 0$ and observing that $\text{Lt}\dfrac{\delta s}{\delta c}=1$, we have

$$\left(\frac{ds}{dx}\right)^2=1+\left(\frac{dy}{dx}\right)^2 \text{ or } \frac{ds}{dx}=\pm\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}.$$

If s increases as x increases (as in the figure), $\dfrac{ds}{dx}$ is positive and, therefore, taking the plus sign before the radical,

$$\frac{ds}{dx}=\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}$$

The — sign is to be taken before the radical if s decreases as x increases. This will be so if the point A is taken on the other side of P in the figure.

Cor. 1. The differential ds of the arc s is given by

$$ds=\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}dx.$$

Cor. 2. Regarding s as a function of y, we may show that

$$\frac{ds}{dy}=\sqrt{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}}$$

and that the differential ds of the arc s may be expressed as

$$ds=\sqrt{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}}dy.$$

Cor. 3. If the parametric equations of the curve be given so that x and y are functions of a third variable t, we have

$$\left(\frac{\delta c}{\delta t}\right)^2=\left(\frac{\delta x}{\delta t}\right)^2+\left(\frac{\delta y}{\delta t}\right)^2$$

and proceeding to the limit, after slight modifications, we get

$$\frac{ds}{dt}=\sqrt{\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right\}}$$

and the differential $ds=\sqrt{\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right\}}dt.$

6·71 *If ψ be the inclination to the x-axis of the tangent to a curve at a point $P(x, y)$, prove that*

(*i*) $\cos\psi=\frac{dx}{ds}$, (*ii*) $\sin\psi=\frac{dy}{ds}$.

Using the notation and the figure of the last article, we have from the right-angled triangle PQR,

$$\cos QPR=\frac{\delta x}{\delta c},\ \sin\ QPR=\frac{\delta y}{\delta c}.$$

When $Q\to P$, QP approaches the tangent at P so that $\angle QRP\to\psi$.

$$\text{Hence }\cos\psi=\underset{Q\to P}{\text{Lt}}\cos QPR=\underset{\delta c\to 0}{\text{Lt}}\frac{\delta x}{\delta c}.$$

$$=\text{Lt}\left(\frac{\delta x}{\delta s}\cdot\frac{\delta s}{\delta c}\right)=\text{Lt}\frac{\delta x}{\delta s}\,.\,\text{Lt}\frac{\delta s}{\delta c}$$

$$=\frac{dx}{ds}\,.\,1=\frac{dx}{ds}.$$

$$\text{Similarly, }\sin\psi=\underset{Q\to P}{\text{Lt}}\sin\ QPR=\underset{\delta c\to 0}{\text{Lt}}\frac{\delta y}{\delta c}.$$

$$=\text{Lt}\left(\frac{\delta y}{\delta s}\cdot\frac{\delta s}{\delta c}\right)=\text{Lt}\frac{\delta y}{\delta s}\,.\,\text{Lt}\frac{\delta s}{\delta c}$$

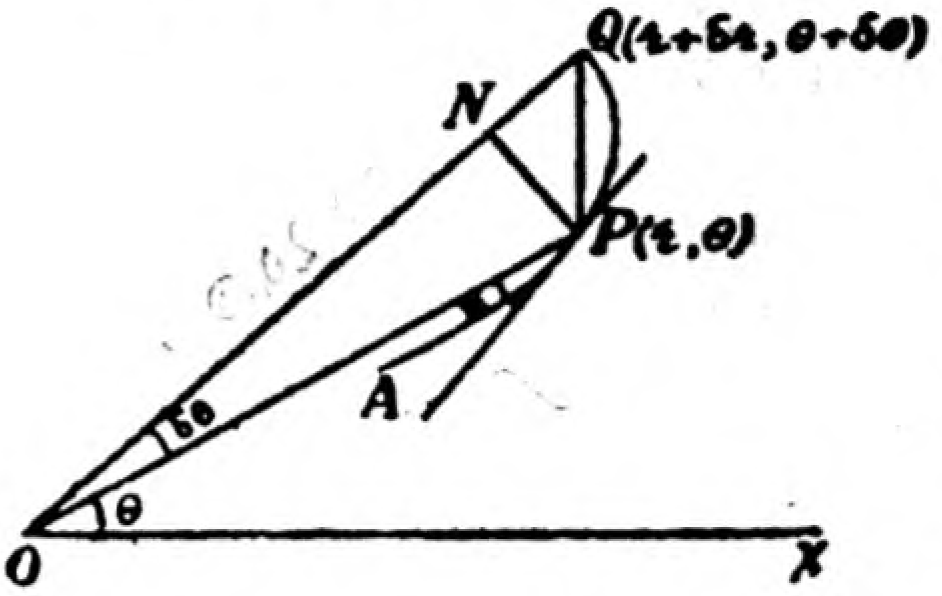
$$=\frac{dy}{ds}\,.\,1=\frac{dy}{ds}.$$

6·8. Derivative of the arc. Polar co-ordinates. *To prove that for the curve $f(r,\theta)=0$,*

(*i*) $\frac{ds}{d\theta}=\sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}$, (*ii*) $\frac{ds}{dr}=\sqrt{\left\{1+r^2\left(\frac{d\theta}{dr}\right)^2\right\}}$

Consider the graph of the function $f(r, \theta)=0$. Let A be a fixed point on the graph and let $P(r, \theta)$ be any other point on it. Let arc $AP=s$. Evidently s is a function of r and θ and we may find the derivative of s *w. r.* to r or θ.

Let $\delta\theta$ be a small increment in the value of θ, δr the corresponding increment in r and δs the increment in s. Thus, if Q is the point on the curve whose co-ordinates are $(r+\delta r, \theta+\delta\theta)$, arc $PQ=\delta s$. Let the chord $PQ=\delta c$. Also $OP=r$, $\angle PON=\delta\theta$. Draw $PN \perp OQ$. From the right-angled $\triangle ONP$,



$$NP=r \sin \delta\theta,\ ON=r \cos \delta\theta.$$

$$\therefore \quad NQ=OQ-ON=(r+\delta r)-r \cos \delta\theta=\delta r+r(1-\cos \delta\theta)$$
$$=\delta r+2r \sin^2 \tfrac{1}{2}\delta\theta.$$

Again, from the right-angled $\triangle PNQ$,

$$PQ^2=NP^2+NQ^2,$$

i.e.,
$$\delta c^2=r^2 \sin^2 \delta\theta+(\delta r+2r \sin^2 \tfrac{1}{2}\delta\theta)^2.$$

$$\therefore \quad \left(\frac{\delta c}{\delta\theta}\right)^2=r^2\left(\frac{\sin \delta\theta}{\delta\theta}\right)^2+\left(\frac{\delta r}{\delta\theta}+r\cdot\frac{\sin \frac{1}{2}\delta\theta}{\frac{1}{2}\delta\theta}\cdot \sin \frac{\delta\theta}{2}\right)^2 \qquad (1)$$

Also $$\operatorname*{Lt}_{\delta\theta\to 0} \frac{\delta c}{\delta\theta}=\operatorname*{Lt}_{\delta\theta\to 0}\left(\frac{\delta c}{\delta s}\cdot\frac{\delta s}{\delta\theta}\right)=1\cdot\frac{ds}{d\theta}=\frac{ds}{d\theta}.$$

$\therefore$ We have from (1) on proceeding to the limit as $\delta\theta\to 0$,

$$\left(\frac{ds}{d\theta}\right)^2=r^2.1+\left(\frac{dr}{d\theta}+r.1.0\right)^2$$

$$=r^2+\left(\frac{dr}{d\theta}\right)^2$$

$$\therefore \quad \frac{ds}{d\theta}=\pm\sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}.$$

Assuming that s increases as θ increases (as in the figure), so that $\frac{ds}{d\theta}$ is positive, we have

$$\frac{ds}{d\theta}=\sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}$$

The — sign has to be taken before the radical if s decreases as θ increases, which will be so if A is on the other side of P.

Cor. 1. The differential ds of the arc is given by

$$ds=\sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}d\theta.$$

Cor. 2. If s be considered as a function of r, we have similarly,

$$\frac{ds}{dr}=\sqrt{\left\{1+r^2\left(\frac{d\theta}{dr}\right)^2\right\}}$$

and the differential ds of the arc may be expressed as

$$ds=\sqrt{\left\{1+r^2\left(\frac{d\theta}{dr}\right)^2\right\}}dr.$$

Cor. 3. If ϕ be the angle between the radius vector and the tangent to a curve at the extremity of the radius vector, then

$$(i)\ \sin\phi=r\frac{d\theta}{ds}, \qquad (ii)\ \cos\phi=\frac{dr}{ds}.$$

From $\triangle PNQ$,

$$\sin PQN=\frac{PN}{PQ}=\frac{r\sin\delta\theta}{\delta c}$$

$$=r.\frac{\sin\delta\theta}{\delta\theta}\cdot\frac{\delta\theta}{\delta s}\cdot\frac{\delta s}{\delta c}.$$

As $Q\to P$, $\angle PQN\to\phi$. Hence

$$\sin\phi=\underset{Q\to P}{\text{Lt}}\sin PQN$$

$$=\text{Lt}\left[r.\frac{\sin\delta\theta}{\delta\theta}\cdot\frac{\delta\theta}{\delta s}\cdot\frac{\delta s}{\delta c}\right]$$

$$=r.\text{Lt}\frac{\sin\delta\theta}{\delta\theta}.\text{Lt}\frac{\delta\theta}{\delta s}.\text{Lt}\frac{\delta s}{\delta c}$$

$$=r.1.\frac{d\theta}{ds}.1=r\frac{d\theta}{ds}$$

Again, $$\cos PQN=\frac{NQ}{PQ}=\frac{\delta r+2r\sin^2\frac{1}{2}\delta\theta}{\delta c}$$

$$=\left[\frac{\delta r}{\delta s}+r.\frac{\sin\frac{1}{2}\delta\theta}{\frac{1}{2}\delta\theta}\cdot\frac{\delta\theta}{\delta s}.\sin\frac{\delta\theta}{2}\right]\frac{\delta s}{\delta c}.$$

Proceeding to the limit as $Q\to P$,

$$\cos\phi=\left[\frac{dr}{ds}+r.1.\frac{d\theta}{ds}.0\right].1=\frac{dr}{ds}.$$

EXAMPLES XXIV

1. Find $\frac{ds}{dx}$ for the following curves :

(i) $y^2=4ax$. (ii) $y=c\cosh(x/c)$.

(iii) $x^{2/3}+y^{2/3}=a^{2/3}$. (iv) $y^2(2a-x)=x^3$.

2. Find $\frac{ds}{dt}$ for the following curves :

(i) $x=a \cos t, y=b \sin t$. (ii) $x=a \cos^3 t, y=a \sin^3 t$,
(iii) $x=a(1+\cos t), y=a(t+\sin t)$.

3. In the curve $y=a \log \sec (x/a)$, prove that

$$\frac{ds}{dx}=\sec \frac{x}{a}, \quad \frac{ds}{dy}=\operatorname{cosec} \frac{x}{a} \text{ and } x=a\psi.$$

4. Show that for the curve

$$\frac{x+\sqrt{(a^2-y^2)}}{a}=\log \frac{a+\sqrt{(a^2-y^2)}}{y},$$

$\frac{ds}{dx}$ is inversely proportional to the ordinate.

5. Find $\frac{ds}{d\theta}$ for the following curves :

(i) The cardioid $r=a(1+\cos \theta)$.

(ii) The equiangular spiral $r=ae^{\theta \cot \alpha}$.

(iii) The lemniscate $r^2=a^2 \cos 2\theta$.

6. In the curve $r^m=a^m \cos m\theta$, prove that

$$\frac{ds}{d\theta}=a \sec^{\frac{m-1}{m}} m\theta. \qquad \textit{(Agra, 1951)}$$

7. Show that in the curve $2s=y^2$, $\frac{dy}{dx}=\frac{1}{\sqrt{(y^2-1)}}$.

8. Show that in the curve $y^2=c^2+s^2$, the perpendicular from the foot of the ordinate upon the tangent is of constant length. *(Patna, 1945)*

9. In any curve, prove that

(i) $p=r^2 \frac{d\theta}{ds}$.

(ii) $\sqrt{(r^2-p^2)}=r \frac{dr}{ds}$. *(Panjab, 1953 S)*

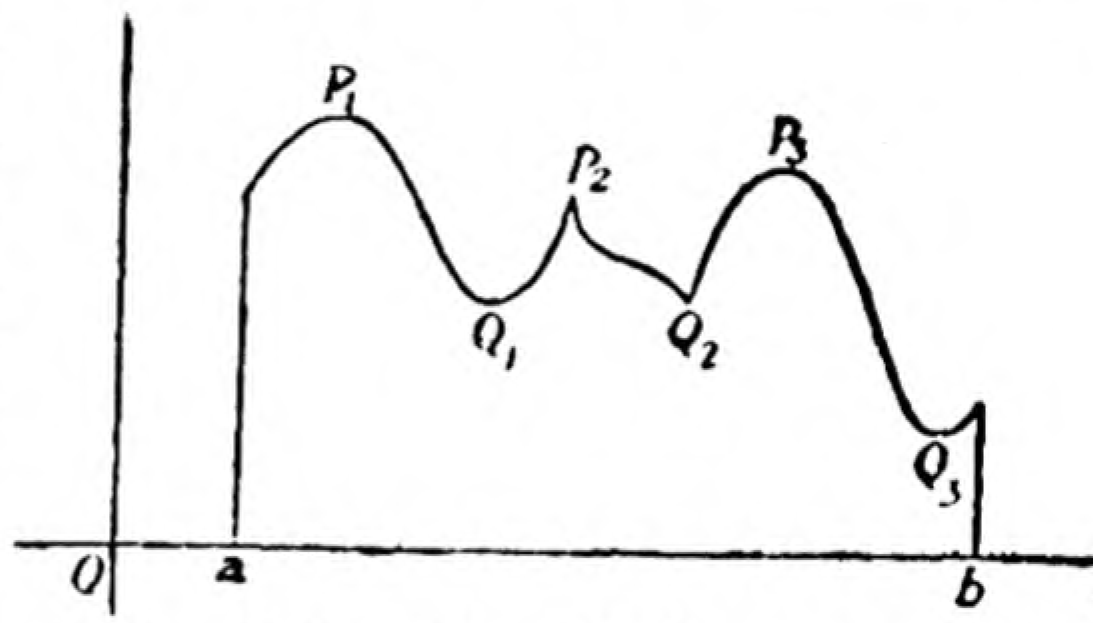
CHAPTER VII

MAXIMA AND MINIMA

7·1. Let the graph of the function $y=f(x)$ in the interval $a \leqslant x \leqslant b$ be as in the figure below. We assume that $f(x)$ is continuous in this interval and possesses a continuous derivative at all points except P_2 and Q_2.

Each of the points P_1, P_2, P_3 is the highest point on the graph in its own immediate neighbourhood. In other words, $f(x)$ has a greater value at P_1 than at any other point sufficiently near P_1, and similarly for P_2 and P_3. In symbols, if a_1, is the abscissa of P_1, then

$$f(x) < f(a_1) \text{ for all}$$



x sufficiently close to a_1, and similarly for P_2 and P_3. Points like P_1, P_2, P_3 are called **points of maxima** on the curve and the corresponding ordinates $f(a_1), f(a_2), f(a_3)$ are called **maximum values** of $f(x)$.

Again, each of the points Q_1, Q_2, Q_3 is the lowest point on the graph in its own immediate neighbourhood. $f(x)$ has a smaller value at Q_1 than at any other point sufficiently near Q_1. In symbols, if b_1 is the abscissa of Q_1, then

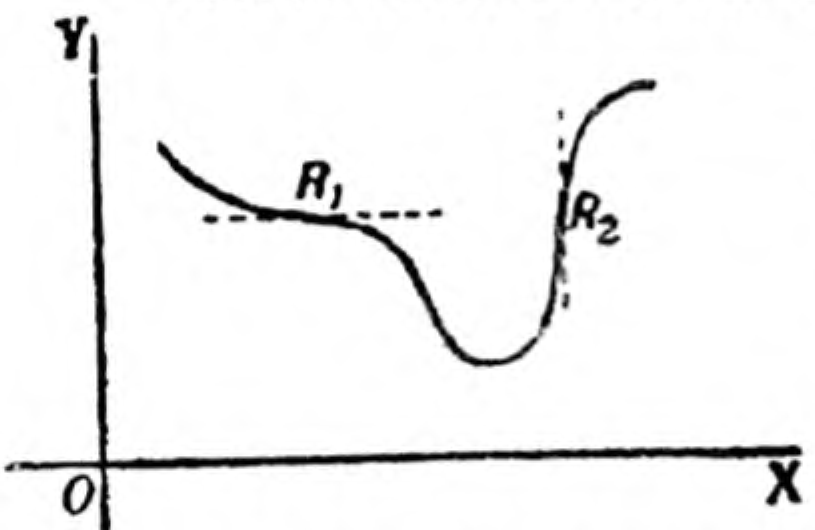
$$f(x) > f(b_1)$$

for all x sufficiently close to b_1. Similar statements hold for Q_2 and Q_3. Points like Q_1, Q_2, Q_3, are called **points of minima** on the curve and the corresponding ordinates $f(b_1)$, etc. are called **minimum values** of $f(x)$.

7·11. Let us consider the behaviour of the function (or its graph) in the immediate neighbourhood of points of maximum or minimum values. Immediately to the left of the points P_1, P_2, P_3, the function is increasing and, therefore, $f'(x)$ is positive, and immediately to the right of these points, the function is decreasing and, therefore, $f'(x)$ is negative. Thus we observe that in passing through maximum points of a curve, the derivative changes sign from plus to minus. If the derivative varies continuously, as at P_1 and P_3, then it must attain the value zero in passing from positive to negative values. Hence at these points, the derivative is zero and the tangent is parallel to the x-axis. At

P_2 the derivative does not exist and is, therefore, discontinuous. It may even tend to $+\infty$ or $-\infty$ according as a_2 is appioached from the left or from the right. Again, we observe that immediately to the left of the points Q_1, Q_2, Q_3 the function is decreasing and, therefore, $f'(x)$ is negative and immediately to the right of these points, the function is increasing and, therefore, $f'(x)$ is positive. Thus in passing through points of minimum value, $f'(x)$ changes sign from minus to plus and if $f'(x)$ varies continuously it must vanish as at Q_1 and Q_3 and the tangent is parallel to the x-axis. At Q_2 the derivative does not exist and is, therefore, discontinuous. It may even tend to $-\infty$ or $+\infty$ according as b_2 is approached from the left or from the right.

From the above discussion it follows that *a necessary condition for $f(x)$ to have a maximum or minimum value at a point is that $f'(x)=0$ or is discontinuous at the point.* That *the condition is not sufficient* can be easily seen from the graph opposite. At R_1 the tangent is parallel to the x-axis and, therefore, $f'(x)$ is zero; at R_2, the tangent is parallel to the y-axis and $f'(x)\to\infty$; yet at neither of these points, the function has an *extreme* (minimum or maximum) value. Thus the vanishing of the derivative or the existence of an infinite discontinuity or any other discontinuity thereof does not assure us about the existence of an extreme value. What does assure us about the existence of an extreme value is the fact that the derivative must change sign in passing through such a value. This derivative itself may be zero or infinite at the point or may not exist at the point.



7·2. The above discussion is based on geometrical considerations. We now give formal definitions followed by an analytical discussion.

Def. *A function $f(x)$ is said to have a* **maximum value** *at $x=c$ if there exists a positive quantity ε such that*

$$f(c\pm h)<f(c)$$

for all positive values of $h<\varepsilon$, and a **minimum value** *if*

$$f(c\pm h)>f(c)$$

for all positive $h<\varepsilon$.

Maximum and minimum values are also called **extremum, turning,** or **stationary** values. It is to be observed that at an extremum, the differences

$$f(c+h)-f(c) \quad \text{and} \quad f(c-h)-f(c) \tag{1}$$

are both of the same sign, negative at a maximum and positive at a

minimum. If the two differences (1) are of opposite signs, then $f(x)$ has neither a maximum nor a minimum at $x=c$.

Remark. It should be noted that the definition of an extremum value does not imply that maximum or minimum value of a function is its greatest or least value in its entire domain of definition. The maximum value is simply the greatest value in the immediate neighbourhood of the point of maximum. Similarly, the minimum value is the least in the immediate neighbourhood of the point of minimum. In fact, a minimum value may sometimes be greater than a maximum value For example, the graph of $y=\sec x$ in the interval $-\frac{1}{2}\pi \leqslant x \leqslant \frac{3}{2}\pi$ has a minimum at $x=0$ and a maximum at $x=\pi$. The minimum value 1 is greater than the maximum value -1.

7·21. Theorem. *If $f'(x)$ exists in $a \leqslant x \leqslant b$ and $f(x)$ has an extremum at $x=c$, where $a<c<b$, then $f'(c)=0$.*

Let $f(x)$ have a maximum at $x=c$, then, by definition, there exists a positive number ε such that

$$f(c+h)-f(c)<0 \quad \text{and} \quad f(c-h)-f(c)<0$$

for all positive values of $h<\varepsilon$. Hence

$$\frac{f(c+h)-f(c)}{h}<0 \text{ and } \frac{f(c-h)-f(c)}{-h}>0.$$

Since these inequalities are true however small h be, we have, on proceeding to the limits as $h \to 0$ and remembering that $f'(c)$ exists,

$$f'(c) \leqslant 0 \text{ and also } f'(c) \geqslant 0.$$

Hence $f'(c)=0$.

Similarly, if $f(x)$ has a minimum at $x=c$, then $f'(c)=0$.

Cor. *The only points where $f(x)$ can have extreme values are those at which the derivative is zero or does not exist.*

7·3 The following theorem gives a set of necessary and sufficient conditions for the existence of an extreme value of a continuous function.

Theorem. *$f(x)$ has an extremum at $x=c$ if and only if $f'(x)$ changes sign as x passes through the value c, the value $f(c)$ being a maximum or a minimum according as the change of sign is from plus to minus or from minus to plus.*

(*i*) Let $f'(x)$ change sign from plus to minus as x passes through the value c.

Then since $f'(x)$ is positive for $x<c$ but near c, an interval $(c-\delta, c)$ exists throughout which $f(x)$ is an increasing function.

Again, since $f'(x)$ is negative for $x>c$ but near c, an interval $(c, c+\delta)$ exists throughout which $f(x)$ is a decreasing function.

Thus $f(c)$ is greater than every value of the function in the neighbourhood of c. Hence $f(c)$ is a maximum.

(*ii*) Let $f'(x)$ change sign from minus to plus as x passes through the value c.

Then since $f'(x)$ is negative for $x<c$ but near c, an interval $(c-\delta, c)$ exists throughout which $f(x)$ is a decreasing function.

Again, since $f'(x)$ is positive for $x>c$ but near c, an interval $(c, c+\delta)$ exists throughout which $f(x)$ is an increasing function.

Thus $f(c)$ is less than every value of the function in the neighbourhood of c. Hence $f(c)$ is a minimum.

(*iii*) If $f'(x)$ does not change sign as x passes through c, $f'(x)$ is positive or negative in every neighbourhood of the form $(c-\delta, c+\delta)$. Hence $f(x)$ is either monotonically increasing or monotonically decreasing throughout this neighbourhood. Hence $f'(x)$ cannot have an extremum at $x=c$.

It should be observed that the proof does not depend upon the existence or continuity of $f'(x)$ at $x=c$, and, therefore, the theorem can be applied even when $f'(x)$ does not exist.

Hence, in practice, to find the maximum and minimum values of a function $f(x)$, we have to find all such values of x for which $f'(x)=0$ or is not defined and then test each of these values for a possible extremum.

Ex. 1. *Find the maximum and minimum values of*

$$x^3+15x^2+48x+7.$$

Let $f(x)=x^3+15x^2+48x+7$, then

$$f'(x)=3x^2+30x+48=3(x+2)(x+8).$$

$$f'(x)=0 \text{ gives } x=-2 \text{ or } -8.$$

There are no points for which $f'(x)$ does not exist.

(*i*) For x slightly <-8, $f'(x)=3(-)\,(-)=+$ive,
and for x slightly >-8, $f'(x)=3(-)\,(+)=-$ive.

Hence $f'(x)$ changes sign from $+$ive to $-$ive as x passes through the value -8. Hence $x=-8$ makes $f(x)$ a maximum and the maximum value

$$f(-8)=(-8)^3+15(-8)^2+48(-8)+7=71.$$

(*ii*) For x slightly <-2, $f'(x)=3(-)(+)=-$ive,
and for x slightly >-2, $f'(x)=3(+)(+)=+$ive.

Hence $f'(x)$ changes sign from $-$ive to $+$ive as x passes through the value $x=-2$. Hence $x=-2$ makes $f(x)$ a minimum and the minimum value

$$f(-2)=(-2)^3+15(-2)^2+48(-2)+7=-37.$$

Ex. 2. *Locate points of extreme values on the curve*

$$y^3=(x-1)^2(x+2).$$

Here $\qquad y=(x-1)^{2/3}(x+2)^{1/3},$

$$\therefore \qquad \frac{dy}{dx}=\frac{1}{3}\cdot(x-1)^{2/3}(x+2)^{-2/3}+\tfrac{2}{3}(x-1)^{-1/3}(x+2)^{1/3}$$

$$=\frac{x-1+2(x+2)}{3(x-1)^{1/3}(x+2)^{2/3}}=\frac{x+1}{(x-1)^{1/3}(x+2)^{2/3}}.$$

Now $\dfrac{dy}{dx}=0$ when $x=-1$ and it tends to infinity when $x=1$ or -2. We have, therefore, to examine these three points for extreme values.

(*i*) Consider the point $x=-1$.

For x slightly <-1, $x+1<0$, $(x-1)^{1/3}<0$, $(x+2)^{2/3}>0$,

$\therefore \qquad \dfrac{dy}{dx}$ is positive.

For x slightly >-1, $x+1>0$, $(x-1)^{1/3}<0$, $(x+2)^{2/3}>0$,

$\therefore \qquad \dfrac{dy}{dx}$ is negative.

Thus $\dfrac{dy}{dx}$ changes sign from positive to negative. Hence $x=-1$ gives a maximum, and the maximum value $=4^{1/3}$.

(*ii*) Next consider the point $x=1$.

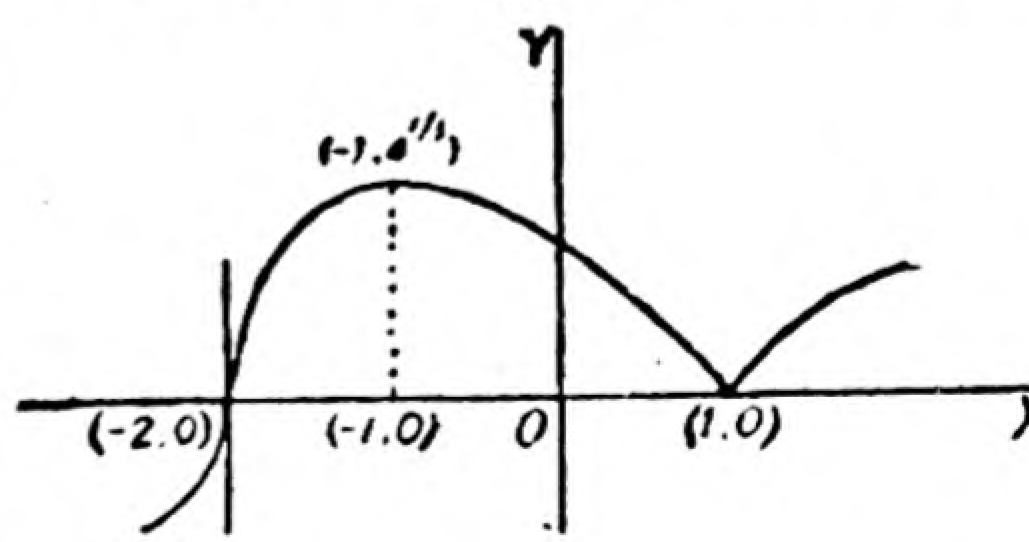
For x slightly <1, $\dfrac{dy}{dx}$ is negative ; and

for x slightly >1, $\dfrac{dy}{dx}$ is positive.

Consequently, $\dfrac{dy}{dx}$ changes sign from negative to positive. Hence $x=1$ gives a minimum, and the minimum value $=0$.

(*iii*) Lastly, the point $x=-2$.

It is easily seen that for x slightly <-2, $\dfrac{dy}{dx}$ is positive and for x slightly >-2, $\dfrac{dy}{dx}$ is again positive. Since $\dfrac{dy}{dx}$ does not change sign, $x=-2$ does not give an extreme value.



The graph is as shown in the diagram.

7 4. Geometrical meaning of the sign of the second derivative. Points of Inflexion. Let $y=f(x)$ be a twice differentiable function of x and let ψ be the angle that the tangent makes with the

x-axis. We measure ψ as the positive or negative acute angle made by the tangent, drawn in the sense of x increasing, with the positive direction of the x-axis.

If $f''(x)$ is positive, then $f'(x)=\tan\psi$ is increasing. But the derivative of $\tan\psi$, $\sec^2\psi$, is always positive, and, therefore, ψ and $\tan\psi$ increase or decrease together. Hence when $f''(x)$ is +ive, ψ is increasing and, therefore, as x increases, the tangent to the curve rotates in the counterclockwise sense. It follows that at a point where $f''(x)>0$ the curve lies above the tangent on either side of the point of contact and has its concavity towards the positive direction of the y-axis [Fig. (*i*)].

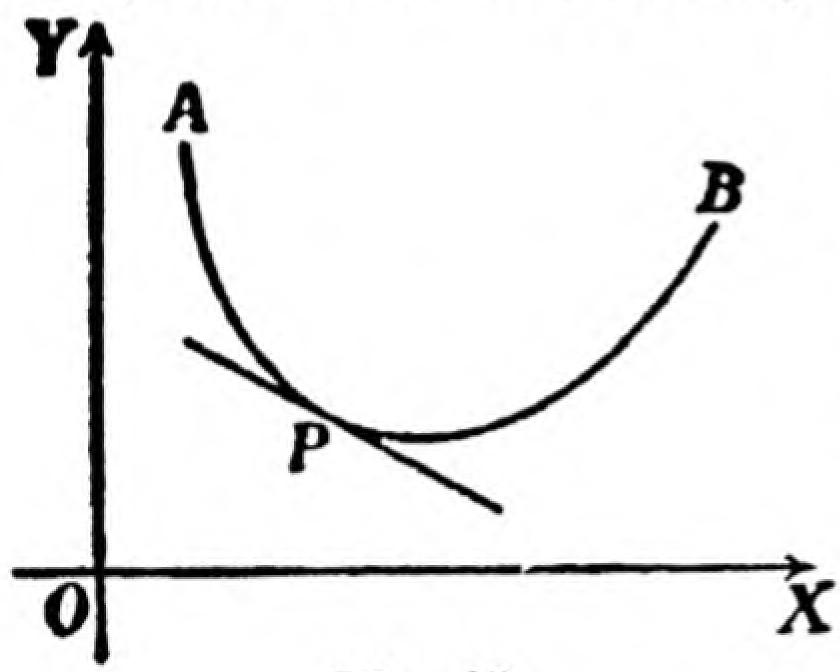


Fig. (*i*)

On the other hand, if $f''(x)$ is negative, $f'(x)=\tan\psi$ is decreasing, and therefore ψ is decreasing. Hence as x increases, the tangent to the curve rotates in the clockwise sense. Thus when $f''(x)<0$, the curve lies below the tangent on either side of the point of contact and is convex towards the positive direction of the y-axis [Fig (*ii*)].

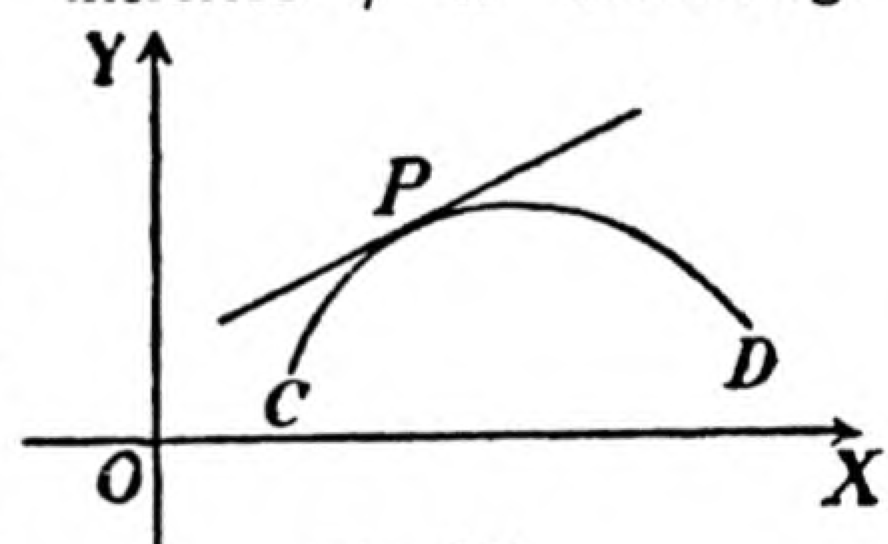


Fig. (*ii*)

Hence, *the curve* $y=f(x)$ *is* **concave** *or* **convex** *in the positive y-direction at a point according as* $f''(x)$ *is positive or negative at the point.*

If $f''(x)$ is zero at a point $x=c$ and changes sign as x passes through the value c, then the tangent to the curve changes its sense of rotation from clockwise to anticlockwise or *vice versa* as x passes through the value c. Therefore the curve lies above the tangent at $x=c$ on one side of the point of contact and below it on the other side [Fig. (*iii*)]. In other words, as the point $x=c$ is crossed the curve changes from concave to convex or *vice versa*. Such a point is called a **point of inflexion** on the curve.

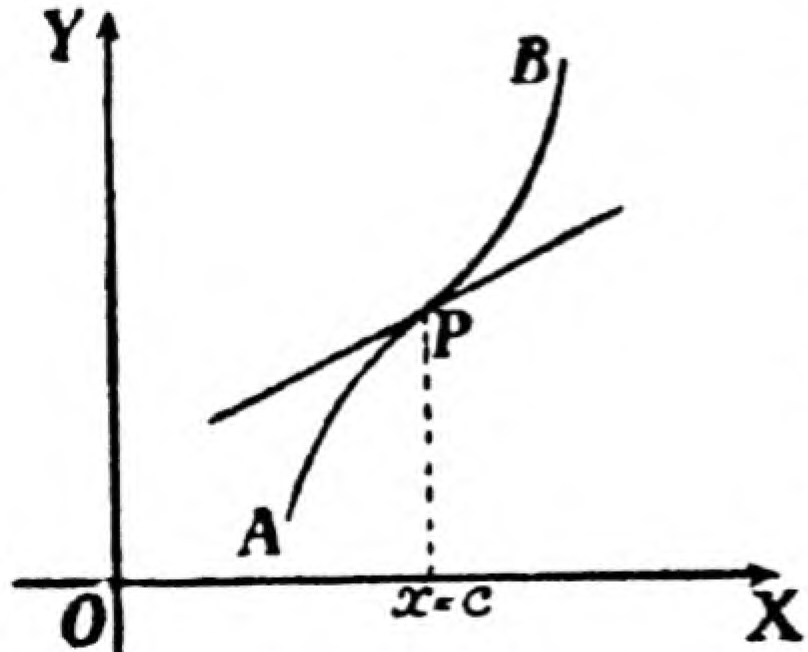


Fig. (*iii*)

The tangent at a point of inflexion crosses the curve. It should be observed that there is no inflexion if $f'(x)$ vanishes at $x=c$ without changing sign as x passes through the value c,

7·41. It follows from the above discussion and a reference to the figure of Art. 7·1, that *if $f(x)$ be a twice differentiable function, then $f''(c)$ is negative if $f(x)$ has a maximum at $x=c$ and $f''(c)$ is positive if $f(x)$ has a maximum at $x=c$.*

7·42. Alternative criteria for the extreme values. We now give two theorems which enable us to discuss the existence of the extreme values of a twice differentiable function in terms of the sign of the second derivative.

Theorem 1. *A function $f(x)$ has a* **maximum value** *at $x=c$ if $f'(c)=0$ and $f''(c)<0$.*

Since $f'(c)=0$, by Cor. to Th. of Art. 7·21, $x=c$ is a value where $f(x)$ can possibly be maximum or minimum. We have to show that $f(x)$ has a maximum at $x=c$. Since $f''(c)<0$, it follows that $f'(x)$ is a decreasing function of x at $x=c$. But $f'(c)=0$. Hence $f'(x)$ changes sign from positive to negative as x passes through the value c. Hence, by the Th. of Art. 7·3, $f(x)$ has a maximum at $x=c$.

Theorem 2. *A function $f(x)$ has a* **minimum value** *at $x=c$ if $f'(c)=0$ and $f''(c)>0$.*

This may be proved in a manner similar to the proof of Theorem 1.

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Ex. 1. *Show that $x^5-5x^4+5x^3-1$ has a maximum when $x=1$, a minimum when $x=3$, and neither when $x=0$.* (*Panjab, 1954*)

Let $f(x)=x^5-5x^4+5x^3-1$,

then $f'(x)=5x^4-20x^3+15x^2=5x^2(x-1)(x-3)$,

$f''(x)=20x^3-60x^2+30x=10x(2x^2-6x+3)$.

$f'(x)=0$ gives $x=0$, 1, or 3.

Hence the possible points of maxima and minima are amongst these three.

(*i*) At $x=1, f''(1)=-10<0$.

Hence $f(x)$ has a maximum at $x=1$ and the maximum value $=f(1)=0$.

(*ii*) At $x=3, f''(3)=90>0$.

Hence $f(x)$ has a minimum at $x=3$ and the minimum value $=f(3)=-28$.

(*iii*) At $x=0$, $f''(x)=0$ and, therefore, the test breaks down. However, we can easily see that $f'(x)$ does not change sign as x passes through the value 0 and therefore $f(x)$ has neither a maximum nor a minimum for $x=0$. In fact, the curve $y=f(x)$ has an inflexion at $x=0$ since $f''(0)=0$ and $f''(x)$ changes sign as x passes through the value 0.

Ex. 2. *Examine the function* $\dfrac{\log x}{x}$ *for maximum and minimum values.*

Let $f(x)=\dfrac{\log x}{x}$, then

$$f'(x)=\frac{1-\log x}{x^2} \text{ and } f''(x)=\frac{-1+2(1-\log x)}{x^3}.$$

$f'(x)=0$ gives $1-\log x=0$ or $\log x=1$, so that $x=e$, and $f''(e)=-1/e^3$ whiuh is negative. Hence $f(x)$ has a maximum at $x=e$ and the maximum value$=f(e)=1/e$.

Ex. 3. *Examine the function* $\sin x+\cos x$ *for extrema values.*

Let $f(x)=\sin x+\cos x$,

then $\quad f'(x)=\cos x-\sin x.$

$\quad f''(x)=-\sin x-\cos x.$

$f'(x)=0$ gives $\tan x=1$ whence $x=n\pi+\frac{1}{4}\pi$, where n is any integer or zero. We have

$$f''(n\pi+\tfrac{1}{4}\pi)=-[\sin(n\pi+\tfrac{1}{4}\pi)+\cos(n\pi+\tfrac{1}{4}\pi)]$$
$$=(-1)^{n+1}(\sin\tfrac{1}{4}\pi+\cos\tfrac{1}{4}\pi)=(-1)^{n+1}\sqrt{2},$$

and $\quad f(n\pi+\frac{1}{4}\pi)=\sin(n\pi+\frac{1}{4}\pi)+\cos(n\pi+\frac{1}{4}\pi)=(-1)^n\sqrt{2}.$

When n is zero or an even integer, then $f''(n\pi+\frac{1}{4}\pi)$ is negative and, therefore, $x=n\pi+\frac{1}{4}\pi$ makes $f(x)$ a maximum with the maximum value $+\sqrt{2}$.

When n is an odd integer, then $f''(n\pi+\frac{1}{4}\pi)$ is positive and therefore $x=n\pi+\frac{1}{4}\pi$ makes $f(x)$ a minimum with the mimimum value $-\sqrt{2}$.

EXAMPLES XXV

Examine the following functions for extreme values :

1. $2x^3+3x^2+4.$
2. $2x^3-24x^2+42x+10.$
3. $3x^5-25x^3+60x.$
4. $2x^3-21x^2+36x-20.$ (*Calcutta, 1936*)
5. $(x-3)^5(x+1)^4.$
6. $(x-2)^6(x-3)^5.$
7. $(1-x+x^2)/(1+x-x^2).$
8. $x/(x^2+a^2).$
9. How do you explain the fact that though

$$y=2x^3-9x^2+12x+1$$

can be made as large as we please by taking x to be large enough, the function has a maximum value when $x=1$.

Discuss the maximum and minimum values of :

10. $\sin x\cos^3 x.$
11. $\sin^3 x\cos^4 x.$
12. $\sin^n x\sin nx.$
13. $a\sin^2\theta+b\cos^2\theta.$
14. $a^2\sec^2\theta+b^2\operatorname{cosec}^2\theta.$

15. Show that $\sin x(1+\cos x)$ is a maximum when $x=\frac{1}{3}\pi$.
(Agra, 1950)

16. Show that the function $4\cos\theta+\cos 2\theta$ is maximum or minimum when $\cos\theta$ is maximum or minimum.

17. Find the maximum and the minimum values of $(1-x)^2e^x$.
(Panjab, 1953)

18. Prove that $x^{1/x}$ has a maximum at $x=e$.

19. Prove that the function

$$m_1(x-x_1)^2+m_2(x-x_2)^2+\ldots+m_n(x-x_n)^2$$

is minimum when x is given by

$$(m_1+m_2+\ldots+m_n)x=m_1x_1+m_2x_2+\ldots+m_nx_n.$$

20. Show that the successive maximum values of

$$y=ae^{-kx}\sin px$$

form a series in geometrical progression and that these maxima lie on the curve

$$y=ae^{-kx}\sin\alpha \text{ where } k\tan\alpha=p.$$

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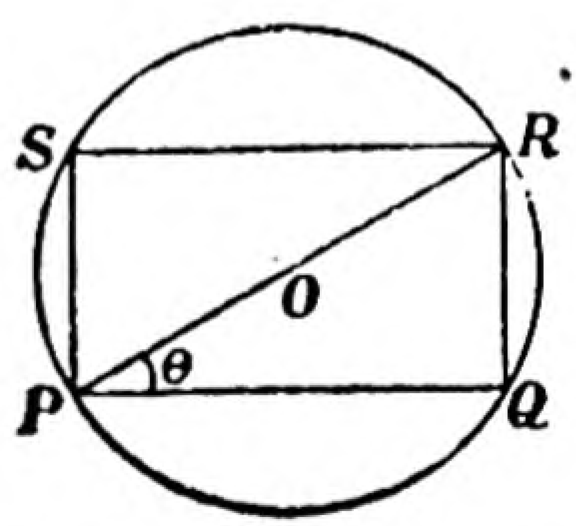
7 5. Application to problems. We now solve some problems to illustrate the application of the preceding theory o the solution of practical problems of maxima and minima.

Ex. 1. *Prove that the rectangle of maximum area that can be inscribed in a circle is a square.*



Let a be the radius of the circle and PQRS the rectangle inscribed in the circle, then PR is a diameter of the circle. Let $\angle RPQ=\theta$ and let A be the area of the rectangle. Then

$$PQ=PR\cos\theta=2a\cos\theta,$$
$$QR=PR\sin\theta=2\,a\sin\theta,$$

and $\therefore$ $A=PQ.PR=4a^2\sin\theta\cos\theta=2a^2\sin 2\theta.$

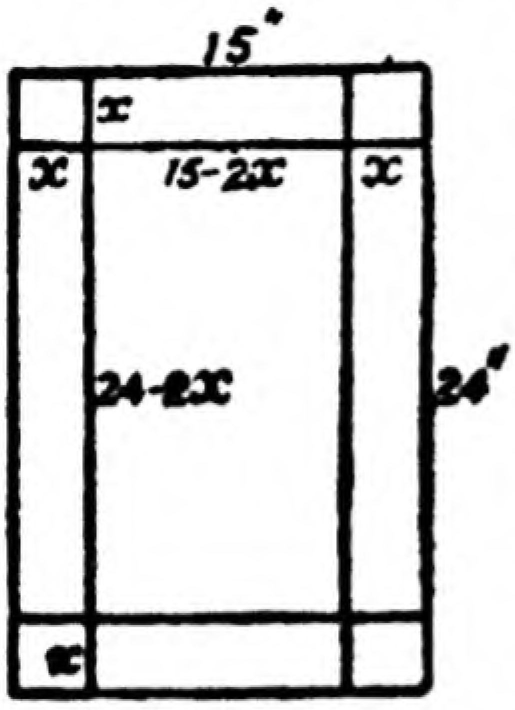
We have to find the maxima and minima of A as a function of θ.

$$\frac{dA}{d\theta}=4a^2\cos 2\theta, \qquad \frac{d^2A}{d\theta^2}=-8a^2\sin 2\theta.$$

$\frac{dA}{d\theta}=0$ gives $\cos 2\theta=0$, and since θ must be an acute angle, we have $\theta=\frac{1}{4}\pi$. This value of θ makes $\frac{d^2A}{d\theta^2}$ negative and, therefore, A maximum. Hence the maximum value of $A=2a^2\sin 2.\frac{1}{4}\pi$ $=2a^2$.

When $\theta=\frac{1}{4}\pi$, $PQ=QR=2a.1/\sqrt{2}=a\sqrt{2}$ and, therefore, the rectangle of maximum area inscribed in a circle is a square.

Ex. 2. *From a reetangular cardboard whose dimensions are $24'' \times 15''$, an open rectangular box is to be made by cutting ou tequal squares from the corners and folding up the sides. Find the volume of the largest box that can be so made.*



Let squares of side x'' be cut from the corners and let the folding up take place along the inner lines. The sides of the box thus formed are x'', $(15-2x)''$ and $(24-2x)''$ respectively. If V be the volume, then

$$V = x(15-2x)(24-2x) = 4x^3 - 78x^2 + 360x.$$

$$\frac{dV}{dx} = 12(x-3)(x-10), \quad \frac{d^2V}{dx^2} = 24x - 156.$$

$\frac{dV}{dx} = 0$ gives $x = 3''$ or $x = 10''$. The value $x = 10''$ is evidently inadmissible. When $x = 3''$, $\frac{d^2V}{dx^2}$ is negative and therefore V is a maximum for $x = 3''$. The max. value of $V = 3(15-6)(24-6) = 486$ cu. in.

Ex 3. *Prove that the minimum intercept made by the axes on the tangent to the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is } a+b. \qquad \textit{(Sagar, 1960)}$$

Let $P(a \cos \theta, b \sin \theta)$ be any point on the ellipse. The tangent at P is

$$\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1.$$

Intercepts made by the tangent on the axes are $a \sec \theta$; $b \operatorname{cosec} \theta$.

$\therefore$ the length of the portion of the tangent intercepted between the axes is

$$L = \sqrt{(a^2 \sec^2 \theta + b^2 \operatorname{cosec}^2 \theta)}.$$

L is maximum or minimum according as

$$f(\theta) = a^2 \sec^2 \theta + b^2 \operatorname{cosec}^2 \theta$$

is maximum or minimum.

Now $f'(\theta) = 2a^2 \sec^2 \theta \tan \theta - 2b^2 \operatorname{cosec}^2 \theta \cot \theta$,

and $f'(\theta) = 0$ at an extremum. Hence for an extremum

$$\frac{\sec^2 \theta \tan \theta}{\operatorname{cosec}^2 \theta \cot \theta} = \frac{b^2}{a^2}$$

or $$\tan^4 \theta = \frac{b^2}{a^2}, \text{ i.e., } \tan^2 \theta = \frac{b}{a}.$$

Also

$$f''(\theta) = 2a^2(\sec^4\theta + 2 \sec^2\theta \tan^2\theta) + 2b^2(\operatorname{cosec}^4\theta + 2 \operatorname{cosec}^2\theta \cot^2\theta),$$

which is essentially positive,

$\therefore$ $f(\theta)$ is a minimum when $\tan^2\theta = b/a$ and the minimum value

$$= a^2\left(1+\frac{b}{a}\right)+b^2\left(1+\frac{a}{b}\right)=a^2+2ab+b^2=(a+b)^2.$$

$\therefore$ the minimum value of $L=a+b$.

Ex. 4. *An open cylindrical can of given capacity is to be made from a metal sheet of uniform thickness. If no allowance is made for waste of material, what will be the most economical ratio of the radius to the height of the can?* (*Panjab, 1945*)

We are required to find when the surface area will be minimum when the volume is fixed.

Let r be the radius of the base and h the altitude of the can. Then, the surface area

$$S=\pi r^2+2\pi rh. \qquad \ldots(i)$$

Also, volume $\qquad V=\pi r^2 h \qquad \ldots(ii)$

where V is constant.

Here S is a function of r and h where r and h are connected by (ii). From (ii), $h=V/(\pi r^2)$. Substituting in (i) for h, we get

$$S=\pi r^2+\frac{2V}{r}. \qquad \ldots(iii)$$

Thus we have expressed S as a function of a single variable r. Differentiating (iii) *w.r.t.* r,

$$\frac{dS}{dr}=2\pi r-\frac{2V}{r^2}=\frac{2}{r^2}(\pi r^3-V).$$

At an extremum, $\dfrac{dS}{dr}=0$,

$\therefore$
$$\pi r^3-V=0, \quad \text{or} \quad r=(V/\pi)^{1/3}.$$

Again, $\qquad \dfrac{d^2S}{dr^2}=2\pi+\dfrac{4V}{r^3}$, which is positive.

$\therefore$ $\quad S$ is minimum for $r=(V/\pi)^{1/3}$.

and then
$$h=\frac{V}{\pi(V/\pi)^{2/3}}=\left(\frac{V}{\pi}\right)^{1/3}=r.$$

Consequently, the required proportions are that the base radius should equal the altitude.

EXAMPLES XXVI

1. Divide 24 into two parts such that

(i) the product of the two parts is maximum ;

(ii) the product of one by the square of the other is a maximum.

(iii) the product of one by the cube of the other is a maximum.

2. Prove that a rectangular field of given area and least perimeter is a square.

3. A rectangular area of 20 acres is to be fenced on three sides, the fourth facing a river. Find the dimensions so that the cast of fencing is least.

4. If the hypotenuse of a right-angled triangle is given, prove that the area is a maximum if the triangle is isosceles.

5. Find the isosceles triangle of maximum area whose perimeter is given to be one foot.

6. A cylindrical box, open at the top, has a bottom which is five times as expensive per unit area as the cylindrical side. If the volume of the box is given, prove that for the cost of construction to be least, the height is five times the radius of the base.

7. Show that the semi-vertical angle of the cone of maximum volume and given slant height is $\tan^{-1}\sqrt{2}$. *(Agra, 1951)*

8. An open rectangular tank, with a square base and vertical sides, is to be constructed of sheet metal to hold a given quantity of water. Show that the cost of the material will be the least when the depth is half the width. *(Aligarh, 1950)*

9. Show that the right circular cylinder of given surface and maximum volume is such that its height is equal to the diameter of its base. *(Panjab, 1960)*

10. Show that the semi-vertical angle of the right cone of given total surface and maximum volume is $\sin^{-1}\frac{1}{3}$. *(Panjab, 1956)*

11. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius a is $2a.\sqrt{3}$. *(Annamalai, 1949)*

12. Prove that the cone of maximum volume which can be inscribed in a sphere of radius r has an altitude $4r/3$.

13. Find the volume of the greatest cylinder which can be inscribed in a cone of height h and semi-vertical angle α. *(Delhi, 1955 ; Panjab, 1951)*

14. At a variable point P of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, a normal is drawn to the ellipse. Find the maximum distance of the normal from the centre of the ellipse. *(Panjab, 1943)*

15. A ladder is to be carried in a horizontal position round a corner formed of two streets a ft. and b ft. wide meeting at right angles. Prove that the length of the longest ladder that will pass round the corner without jamming is $(a^{2/3}+b^{2/3})^{3/2}$ ft.

MISCELLANEOUS EXAMPLES II

1. Examine the differentiability of the function

$$f(x)=x^m \sin(1/x) \text{ when } x\neq 0,\ m>0,$$
$$=0 \qquad\qquad \text{when } x=0,$$

at the point $x=0$. Determine m when $f'(x)$ is continuous at the origin. *(Delhi Hons., 1952)*

2. If $f(x)=0$ has two equal roots α each, show that $f'(x)=0$ has one root equal to α.

3. Show that $(\sin x)/x$ continuously decreases and $(\tan x)/x$ continuously increases in the interval $(0, \frac{1}{2}\pi)$. Deduce that

$$\frac{2}{\pi}<\frac{\sin x}{x}<1 \text{ for } 0<x<\tfrac{1}{2}\pi.$$

4. Prove that $\log\{x+\sqrt{(1+x^2)}\}$ is greater than $\tan^{-1}x$ for all positive values of x.

5. A man is walking at a uniform speed v towards the foot of a tower of height h; show that when he is at a distance x from the foot of the tower, the rate of increase of the perpendicular elevation of the top of the tower is $vh/(x^2+h^2)$.

6. A stone thrown into still water causes a series of concentric ripples. If the radius of the outer ripple is increasing at the rate of 5 ft. per second, how fast is the area of the disturbed water increasing when the outer ripple has a radius of 12 ft.? *(Panjab, 1946)*

7. Find the maximum and minimum values of

$a \sec x+b \operatorname{cosec} x,\ 0<a<b.$ *(Panjab, 1959)*

8. A lane runs at a right angle out of a road 18 ft. wide. How many feet wide is the lane if it is just possible to carry a pole 45 ft. long from the road into lane, keeping it horizontal? *(M.T.I., 1934)*

9. Show that $(\alpha-\alpha^{-1}-x)(4-3x^2)$ has just one maximum and just one minimum, and the difference between them is

$$\tfrac{4}{9}(\alpha+\alpha^{-1})^3.$$

What is the least value of this difference for different values of α? *(M.T.I., 1933)*

10. If $f(x)=\sin x+\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x$, divide the interval $(0, \pi)$ into sub-intervals in each of which $f(x)$ is increasing or decreasing indicating the sense of variation in each sub-interval. Prove that $f(x)>0$ for $0<x<\pi$, and find where $f(x)$ attains its greatest value in the interval $(0, \pi)$. *(M.T.I., 1944)*

11. Prove that the minimum radius vector of the curve $(a^2/x^2)+(b^2/y^2)=1$ is of length $(a+b)$. *(Delhi, 1957; Panjab, 1949)*

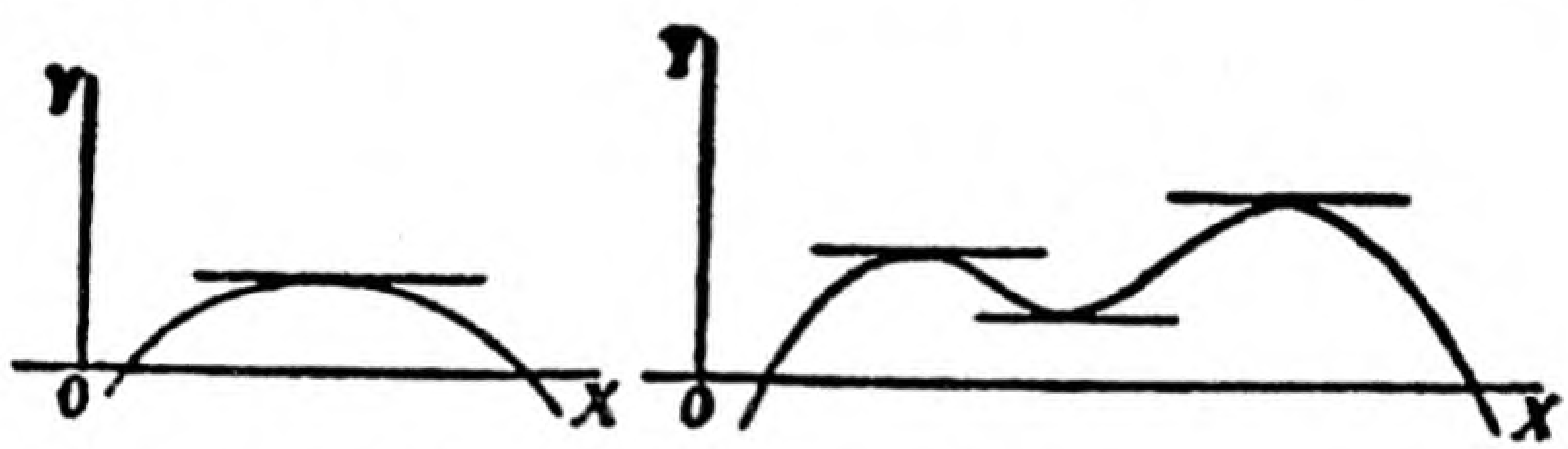
CHAPTER VIII

MEAN VALUE THEOREM

8·1. Rolle's Theorem. *If a function $f(x)$ is continuous in the closed interval $a \leqslant x \leqslant b$, possesses a derivative in the open interval $a < x < b$, and $f(a) = f(b) = 0$, then there is at least one value $x = c$ between a and b such that $f'(c) = 0$.*

We shall prove this theorem rigorously in Part III of this book. Here we shall only illustrate the theorem graphically.

If the graph of $y = f(x)$ be drawn between $x = a$ and $x = b$, then by the conditions of Rolle's theorem it is a continuous curve cutting the x-axis at $x = a$ and $x = b$ and having a unique tangent at all intermediary points. The theorem asserts that there is at least one point on the curve between the points $x = a$ and $x = b$ at which the tangent to the curve is parallel to the x-axis. This is illustrated in the figures given below.



8·11. Rolle's theorem can be easily extended to the case when $f(a) = f(b) \neq 0$. For, consider the function $F(x) = f(x) - f(a)$, them $F(x)$ satisfies all the conditions of Rolle's theorem if $f(x)$ does so. Hence $F'(x)$ vanishes for at least one value $x = c$ such thet $a < c < b$. But $F'(x) = f'(x)$, hence $f'(c) = 0$.

Since most curves that we meet with in practice are continuous and do possess a unique tangent at each point, therefore the Rolle's theorem may be stated simply as : 'Beeween any two equal ordinates of a curve, there is, in general, at least one point on the curve where the tangent is parallel to the x-axis'.

8·12. Algebraic interpretation of Rolle's Theorem. Let $f(x)$ be a polynomial in x and let $x = a$ and $x = b$ be two roots of the equation $f(x) = 0$. Then Rolle's Theorem asserts that at least one root of the equation $f'(x) = 0$ lies between a and b. Rolle stated his theorem only for the case of a polynomial.

8·13. Cases of breakdown of Rolle's Theorem. Rolle's Theorem breaks down when either (*i*) $f(a) \neq f(b)$, or (*ii*) $f(x)$ is not continuous in the closed interval $[a, b]$, that is, $f(x)$ is discontinuous either at $x = a$ or at $x = b$ or at some intermediary point, or (*iii*) the

derivative $f'(x)$ does not exist at some point lying between $x=a$ and $x=b$. These possibilities are illustrated graphically in the attached figures.

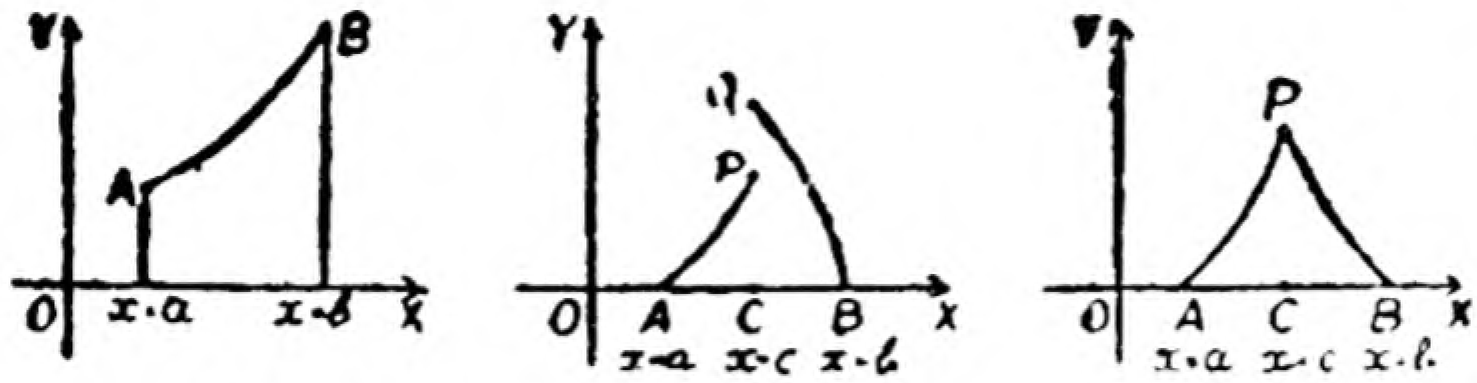


Fig. 1 Fig. 2. Fig. 3.

In Fig. 1, $f(b) > f(a)$ and the tangent to the curve AB is nowhere parallel to the x-axis.

In Fig. 2, the function $f(x)$ is discontinuous at $x=c$ where $a<c<b$, while $f(a)=f(b)=0$. The graph is, therefore, discontinuous at $x=c$, there being a jump from P to Q. There is no point on the curve where the tangent is parallel to the x-axis. Since $f(x)$ is discontinuous at $x=c$, the derivative at $x=c$ does not exist.

In Fig. 3, $f(a)=f(b)=0$ and $f(x)$ is continuous in the closed interval (a, b) but $f(x)$ has no unique derivative at $x=c$, where $a<c<b$. At the point P of the graph where $x=c$, the two portions AP and PB have different slopes and hence there is no unique tangent at P. Once again, there is no point on the curve where the tangent is parallel to the x-axis.

Note. If $f(x)$ is continuous in $[a, b]$ and is derivable in (a, b), it is said to satisfy Rolle's condition in $[a, b]$.

Ex. *Verify Rolle's theorem for the function* $f(x)=x^2-6x+8$ *in the interval* $[2, 4]$.

Here (*i*) $f(2)=0=f(4)$,

(*ii*) $f(x)$, being the sum of three continuous functions, is continuous in $[2, 4]$, and

(*iii*) $f'(x)$ exists in $(2, 4)$.

Hence the conditions of the theorem are satisfied and therefore there must be at least one point inside the interval $[2, 4]$ at which $f'(x)=0$.

Now $f'(x)=2x-6$. Putting $f'(x)=0$ we get

$$2x-6=0 \quad \text{or} \quad x=3.$$

This is a point inside the interval $(2, 4)$ and, therefore, the theorem is verified.

Art 8.2 **Mean Value Theorem.** *If $f(x)$ is continuous in the closed interval $a \leq x \leq b$ and possesses a derivative in the open interval $a<x<b$, then there exists a value $x=c$ such that*

$$f(b)-f(a)=(b-a)f'(c) \text{ where } a<c<b.$$

I (a)

Let $\quad f(b)-f(a)=(b-a)M$ or $f(b)=f(a)+(b-a)M. \quad (1)$

We shall show that there exists a value $x=c$ where $a<c<b$ such that $M=f'(c)$.

Consider the function

$$F(x)=f(x)+(b-x)M \tag{2}$$

obtained by replacing a by x on the right-hand side of (1). It should be observed that M as defined by (1) does not involve x Then

(*i*) $F(x)$, being the sum of two functions continuous in $a \leqslant x \leqslant b$, is itself continuous in $a \leqslant x \leqslant b$,

(*ii*) since $f'(x)$ exists in $a<x<b$ and $(b-x)M$ has a derivative for every x, therefore $F'(x)$ exists in $a<x<b$.

(*iii*) from (2),

$F(b)=f(b)$ and $F(a)=f(a)+(b-a)M=f(b)$ using (1),

therefore $F(b)=F(a)$.

Hence $F(x)$ satisfies all the conditions of Rolle's Theorem in $[a, b]$. Hence there exists a value $x=c$ where $a<c<b$ such that $F'(c)=0$. But, from (2),

$$F'(x)=f'(x)-M.$$

Hence, putting $x=c$, we have

$$0=F'(c)=f'(c)-M \quad \text{or} \quad M=f'(c).$$

Substituting for M in (1), we get

$$f(b)-f(a)=(b-a)f'(c) \quad \text{where } a<c<b.$$

This proves the theorem.

The above theorem is due to Lagrange. There is also another form of the theorem due to Cauchy. We shall consider this in Part III.

Since $b-a$ is the finite increment in the value of x in changing from $x=a$ to $x=b$ and $f(b)-f(a)$ measures the corresponding finite increment in the value of $f(x)$, the formula (1) may also be called the **theorem of finite increments.**

Alternative form of the Mean Value Theorem.

If in (1) we write $b-a=h$ or $b=a+h$, then since $a<c<b$, $c=a+\theta h$, where $0<\theta<1$, and (1) becomes

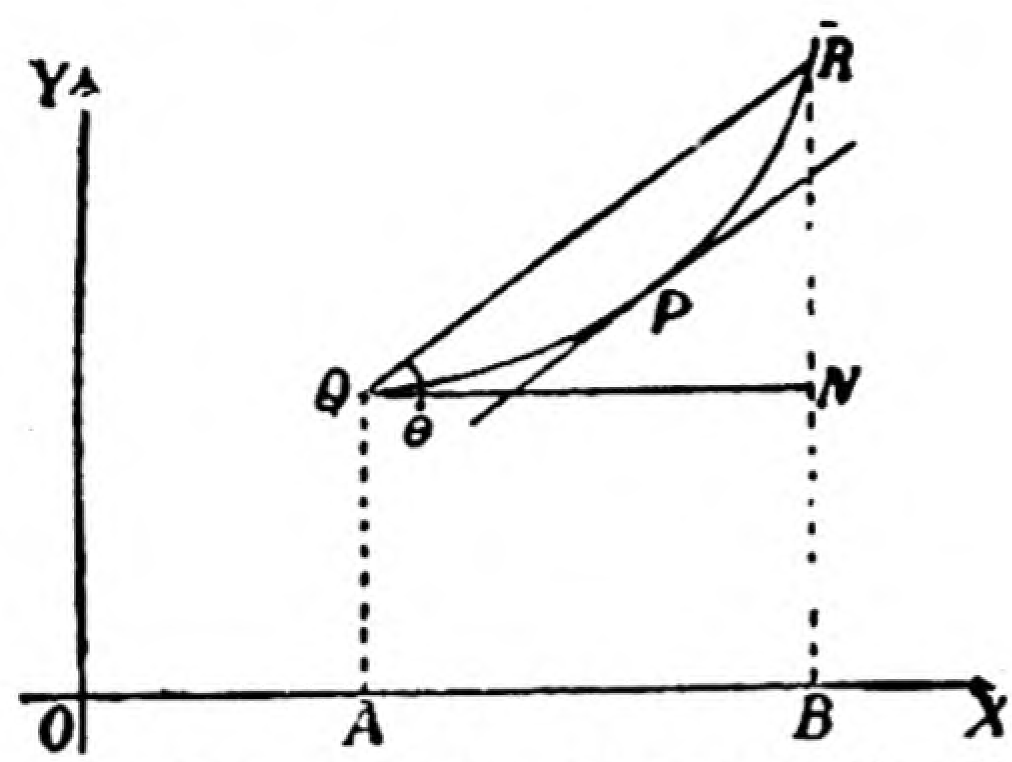
$$f(a+h)=f(a)+hf'(a+\theta h), \text{ where } 0<\theta<1, \tag{3}$$

a form which is very often used.

8·21. Geometrical significance of the Mean Value Theorem. Let Q and R be the points on the graph of $y=f(x)$ corresponding to $x=a$ and $x=b$. Then the co-ordinates of Q and R are $\{a, f(a)\}$ and $\{b, f(b)\}$. Draw QA and RB $\perp$s to OX and $QN \perp BR$. If θ be the angle which QR makes with OX, then

$$\tan\theta=\tan\angle NQR=\frac{NR}{QN}=\frac{f(b)-f(a)}{b-a}.$$

By formula (1) above this is equal to $f'(c)$, the slope of the tangent to the curve at $x=c$. Hence the mean value theorem asserts that there is some point P on the arc QR of the curve $y=f(x)$ the tangent at which is parallel to the chord QR.



8·22. Deductions from the Mean Value Theorem I. *If $f(x)$ be a function such that $f'(x)=0$ for all x in $[a, b]$, then $f(x)$ reduces to a constant in $[a, b]$.*

For, if x_1 and x_2 are any two values of x in the interval (a, b), then by the Mean Value Theorem

$$f(x_1)-f(x_2)=(x_1-x_2)f'(c)=0,$$

since $f'(x)=0$ for all x in $[a, b]$. Hence $f(x_1)=f(x_2)$. Since x_1, x_2 are any two values of x in $[a, b]$, it follows that $f(x)$ has the same value for each x in $[a, b]$ and, therefore, is a constant in $[a, b]$.

This result has a corollary which is of fundamental importance in the theory of indefinite integrals.

Cor. *If two functions $f(x)$ and $\varphi(x)$ have the same derivative for each x in $[a, b]$, then they differ by a constant in $[a, b]$.*

For the derivative of the function $f(x)-\varphi(x)$ is zero for each x in $[a, b]$ and, therefore, $f(x)-\phi(x)$ reduces to a constant in $[a, b]$. In other words, $f(x)$ and $\phi(x)$ differ by a constant in $[a, b]$.

II. *If the derivative $f'(x)$ is positive or zero in $[a, b]$, without however being always zero, then $f(b)>f(a)$.*

Let x be any value between a and b; then applying the Mean Value Theorem to the function $f(x)$ for the two intervals $[a, x]$ and $[x, b]$, we get

$$f(x)-f(a)=(x-a)f'(c_1)\geqslant 0,$$
$$f(b)-f(x)=(b-x)f'(c_2)\geqslant 0,$$

where c_1, c_2 are suitable values lying between a and x, and x and b, respectively. From these it follows that

$$f(b)\geqslant f(x)\geqslant f(a), \text{ i.e., } f(b)\geqslant f(a).$$

But $f(b)\neq f(a)$, for, if it were so, then would $f(x)=f(b)$, where x is arbitrary, and the function would reduce to a constant, which is contrary to the hypothesis that $f'(x)$ is not always zero in $[a, b]$. Hence $f(b)>f(a)$.

Similarly, *if $f'(x)$ is negative or zero in $[a, b]$, without however being always zero, then $f(b) > f(a)$*

In the first case we say that $f(x)$ is *increasing in the interval* $[a, b]$ and in the second case *decreasing in the interval* $[a, b]$.

Ex. 1. *Find the 'c' of the Mean Value theorem when*

$$f(x)=x(x-1)(x-2),\ a=0,\ b=\tfrac{1}{2}.$$

(Delhi Hons., 1951)

Here $f(x)=x^3-3x^2+2x$, $f'(x)=3x^2-6x+2$ and so the formula (I) § 8·2 gives

$$\tfrac{1}{2}(\tfrac{1}{2}-1)(\tfrac{1}{2}-2)-0=\tfrac{1}{2}(3c^2-6c+2)$$

or
$$12c^2-24c+5=0, \text{ whence } c=1\pm\tfrac{1}{6}\sqrt{(21)} = \frac{6-\sqrt{21}}{6}$$

Taking the minus sign, we get a value of c which lies between 0 and $\frac{1}{2}$ and is, therefore, the required value.

***Ex. 2.** *A twice differentiable function $f(x)$ is such that $f(a)=f(b)=0$, and $f(c)>0$, where $a<c<b$. Prove that there is at least one value ξ between a and b for which $f''(\xi)<0$.* (*Kashmir, 1960*)

Since $f''(x)$ exists in the interval $[a, b]$, $f'(x)$ and $f(x)$ are both continuous in the interval $[a, b]$.

Applying the Mean Value theorem to $f(x)$ for the intervals $[a, c]$ and $[c, b]$ respectively, there exist numbers ξ_1 and ξ_2 such that

$$f'(\xi_1)=\frac{f(c)-f(a)}{(c-a)} \qquad \text{with } a<\xi_1<c,$$

and
$$f'(\xi_2)=\frac{f(b)-f(c)}{b-a} \qquad \text{with } c<\xi_2<b.$$

Since $f(a)$ and $f(b)$ are given to be zero, we get

$$f'(\xi_1)=\frac{f(c)}{c-a},\ f'(\xi_2)=-\frac{f(c)}{b-c} \text{ with } a<\xi_1<c<\xi_2<b.$$

Again, applying the Mean Value theorem to $f'(x)$ in the interval $[\xi_1, \xi_2]$, we get

$$f''(\xi)=\frac{f'(\xi_2)-f'(\xi_1)}{\xi_2-\xi_1} \qquad \text{where } \xi_1<\xi<\xi_2.$$

Substituting for $f'(\xi_1)$ and $f'(\xi_2)$ from above, we get

$$f''(\xi)=-\frac{f(c)}{\xi_2-\xi_1}\left[\frac{1}{b-c}+\frac{1}{c-a}\right]$$

But $f(c)$, $\xi_2-\xi_1$, $b-c$, $c-a$ are all positive. Hence $f''(\xi)$ is negative.

EXAMPLES XXVII

1. State and prove Rolle's theorem. Deduce by considering

$$\varphi(x)=f(x)-f(a)-\frac{x-a}{c-a}\{f(c)-f(a)\}$$

that $f(c)-f(a)=(c-a)f'(\xi)$ where $a<\xi<c$. (*Panjab, 1937*)

2. Establish, under conditions to be stated, that

$$f(x)=f(a)+(x-a)f'\{a+\theta(x-a)\}$$

where θ is a positive proper fraction. (*Panjab, 1930*)

3. Discuss the applicability of Rolle's theorem to the function $f(x)=2+(x-1)^{2/3}$ in the interval $[0, 2]$. Illustrate your answer by a rough sketch. (*Panjab Hons. 1926 ; Bombay, 1936*)

4. Verify Rolle's theorem for the following functions :

(*i*) $f(x)=8x-x^2$ in $[0, 8]$. (*ii*) $f(x)=\sin x$ in $[0, 2\pi]$.

(*iii*) $f(x)=x^3-4x$ in $[-2, 2]$. (*iv*) $f(x)=x(x+3)e^{-x/2}$ in $[-3, 0]$.

5. Verify the Mean Value theorem for the following functions in the specified intervals :

(*i*) $f(x)=x^3-3x+2$ in $[-2, 3]$. [$c=\sqrt{(7/3)}$.

(*ii*) $f(x)=\log x$ in $[\frac{1}{2}, 2]$. [$c=1{\cdot}08$.

(*iii*) $f(x)=\cosh x$ in $[-1, 3]$. [$c=1{\cdot}5$.

6. In the theorem $f(b)-f(a)=(b-a)f'(c)$, if

$$f(x)=x^2-3x-1,\ a=1,\ b=3, \text{ find } c.$$

7. Find a point on the curve $y=x^3$ at which the tangent is parallel to the chord joining the points $(1, 1)$, $(2, 8)$ and thus verify the Mean Value theorem for the function $f(x)=x^3$.

*8. Prove that $\log(x+1)-\log x=1/c$, where $0<x<c<x+1$.

Deduce that $\frac{d}{dx}[x\{\log(x+1)-\log x\}]$ is positive, and hence that $(1+1/x)^x$ is an increasing function of x for $x>0$.

*9. Proceeding as in the preceding question, prove that

$$\frac{d}{dx}[(x+1)\{\log(x+1)-\log x\}] \text{ is negative.}$$

Deduce that $(1+1/x)^{x+1}$ is decreasing for $x>0$ and hence that $(1-1/x)^{-x}$ is decreasing for $x>1$.

*10. With the help of questions 8 and 9, prove that

$$(1+1/x)^x<(1-1/x)^{-x} \quad \text{for } x\geqslant 1,$$

and that $$2<(1+1/x)^x<4 \quad \text{for } x>1.$$

CHAPTER IX

TAYLOR'S THEOREM

9·1. The Taylor's theorem is a generalisation of the Mean Value theorem and its object is to expand $f(a+h)$ in ascending powers of h and the values of the derivatives of $f(x)$ $x=a$ in the form

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2\,!}f''(a)+\ldots\ldots+\frac{h^{n-1}}{(n-1)\,!}f^{(n-1)}(a)+R_n$$

The last term R_n on the R.H.S. is called the **remainder after n terms.** The value of R_n can be obtained in several different forms, the most common being that due to Lagrange.

9·11. Taylor's theorem with Lagrange's form of the remainder. *If $f(x)$ and all its derivatives up to the $(n-1)$th order are continuous in the closed interval $a\leqslant x\leqslant b$ and $f^{(n)}(x)$ exists in the open interval $a<x<b$,* then

$$f(b)=f(a)+(b-a)f'(a)+\frac{(b-a)^2}{2\,!}f''(a)+\ldots\ldots$$

$$+\frac{(b-a)^{n-1}}{(n-1)\,!}f^{(n-1)}(a)+\frac{(b-a)^n}{n\,!}f^{(n)}(c), \quad \ldots (1)$$

where $a<c<b$.

If we write $b=a+h$, *i.e.* $b-a=h$, then $c=a+\theta h$, where θ is a positive proper fraction, *i.e.*, $0<\theta<1$, and (1) takes the form

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2\,!}f''(a)+\ldots\ldots$$

$$+\frac{h^{n-1}}{(n-1)\,!}f^{(n-1)}(a)+\frac{h^n}{n\,!}f^{(n)}(a+\theta h), \quad \ldots (2)$$

where $0<\theta<1$.

We prove the theorem in the form (1) by using a method similar to that used to prove the Mean Value theorem in the last chapter. It may be observed that the Mean Value theorem is simply the Taylor's theorem with $n=1$, *i.e.*, with the remainder after one term.

Let

$$f(b)=f(a)+(b-a)f'(a)+\frac{(b-a)^2}{2\,!}f''(a)+\ldots\ldots$$

$$+\frac{(b-a)^{n-1}}{(n-1)\,!}f^{(n-1)}(a)+\frac{(b-a)^n}{n\,!}M, \quad \ldots (3)$$

We shall show that $M=f^{(n)}(c)$, where $a<c<b$.

Consider the function

$$F(x)=f(x)+(b-x)f'(x)+\frac{(b-x)^2}{2!}f''(x)+\dots$$

$$+\frac{(b-x)^{n-1}}{(n-1)!}f^{(n-1)}(x)+\frac{(b-x)^n}{n!}M, \quad \dots (4)$$

obtained by replacing a by x on the R.H S. of (3). Then :

(*i*) $F(x)$, being the sum of a finite number of continuous functions, is itself continuous in the closed interval $a \leqslant x \leqslant b$.

(*ii*) Each term on the R.H.S. of (4) possesses a derivative, for the highest order derivative of $f(x)$ obtained by the differentiation of the R.H.S. of (4) is $f^{(n)}(x)$ and this exists by hypothesis. Hence $F(x)$, being the sum of a finite number of derivable terms, is itself derivable in $a<x<b$.

(*iii*) From (4), $F(b)=f(b)$ and

$$F(a)=f(a)+(b-a)f'(a)+\dots\dots\dots+\frac{(b-a)^n}{n!}M$$

$$=f(b) \text{ using (3).}$$

Hence $F(a)=F(b)$.

Thus $F(x)$ satisfies all the conditions of Rolle's theorem and therefore a value $x=c$, where $a<c<b$, exists such that $F'(c)=0$. But by actual differentiation of (4),

$$F'(x)=\frac{(b-x)^{n-1}}{(n-1)!}\{f^{(n)}(x)-M\}.$$

Putting $x=c$, we get

$$0=F'(c)=\frac{(b-c)^{n-1}}{(n-1)!}\{f^{(n)}(c)-M\}$$

Since $b-c \neq 0$, $\therefore$ $f^{(n)}(c)-M=0$ or $M=f^{(n)}(c)$.

Substituting this value of M in (3) we obtain equation (1). This proves the theorem.

The last term

$$\frac{(b-a)^n}{n!}f^{(n)}(c) \text{ in (1) or } \frac{h^n}{n!}f^{(n)}(a+\theta h) \text{ in (2)}$$

is the **Lagrange's form of the remainder after n terms.**

If we change b into x in (1), we get

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\dots\dots\dots$$

$$+\frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a)+\frac{(x-a)^n}{n!}f^{(n)}(c), \quad (5)$$

where $a<c<b$. This gives the expansion of $f(x)$ in ascending powers of $(x-a)$ and the values of $f(x)$ and its derivatives at $x=a$.

9·12. Deduction of Taylor's theorem from the Mean Value theorem. If $\varphi(x)$ be continuous in the closed interval $[a, a+h]$ and $\varphi'(x)$ exist in the open interval $(a, a+h)$, then by the Mean Value theorem

$$\varphi(a+h)=\varphi(a)+h\varphi'(a+\theta h), \qquad 0<\theta<1,$$

Let $\varphi(x)=f(x)+(a+h-x)f'(x)+\dfrac{(a+h-x)^2}{2\,!}f''(x)+\ldots\ldots$

$$+\frac{(a+h-x)^{n-1}}{(n-1)\,!}f^{(n-1)}(x),$$

then $\varphi(a+h)=f(a+h),$

and $\varphi(a)=f(a)=hf'(a)+\dfrac{h^2}{2\,!}f''(a)+\ldots\ldots+\dfrac{h^{n-1}}{(n-1)\,!}f^{(n-1)}(a).$

Also $\varphi'(x)=\dfrac{(a+h-x)^{n-1}}{(n-1\,!)}f^{n}(x)$, other terms cancelling.

$$\therefore\quad \varphi'(a+\theta h)=\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)\,!}f^{(n)}(a+\theta h).$$

Hence $\varphi(a+h)=\varphi(a)+h\varphi'(a+\theta h)$ becomes

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2\,!}f''(a)+\ldots\ldots$$

$$+\frac{h^{n-1}}{(n-1)\,!}f^{(n-1)}(a)+\frac{h^n(1-\theta)^{n-1}}{(n-1)\,!}f^{(n)}(a+\theta h),$$

which is the Taylor's theorem with Cauchy's form of the remainder after n terms.

This proof is due to Vidya Chandra.

9·2. **Infinite Series.** Let $u_1, u_2, u_3, \ldots\ldots\ldots$ be any infinite sequence, then

$$u_1+u_2+u_3+\ldots+u_n+\ldots\ldots$$

is an **infinite series**, u_n being its nth term. The symbol $\sum\limits_{n=1}^{\infty} u_n$, or simply Σu_n, is used to denote the infinite series.

If we write

$s_1=u_1,\ s_2=u_1+u_2, \ldots,\ s_n=u_1+u_2\ldots+u_n$, and so on, then we obtain a new sequence (s_n), which is called the **sequence of nth partial sums** of the given series. We have the following

Definition. *The series Σu_n is said to* **converge, diverge** *or* **oscillate** *according as the sequence (s_n) of its nth partial sums converges, diverges, or oscillates as n tends to infinity.*

We are concerned only with convergent infinite series. If Σu_n be convergent, then, by definition, the sequence (s_n) of its nth partial

sums converges to a definite limit, say s. This number s is called the **sum** of the series, and we write

$$s = u_1 + u_2 + \ldots\ldots + u_n + \ldots\ldots$$

It should be noticed that this is not a sum in the ordinary sense of the word. The number of terms in the series being infinite, we cannot actually add them up as the process of addition would never come to an end. What is true is that if we take a larger and larger number of terms of the series, their sum approaches closer and closer to the value s.

If we write

$$R_n = u_{n+1} + u_{n+2} + \ldots\ldots = \sum_{r=n+1}^{\infty} u_r,$$

then R_n is called the **remainder after n terms** of the infinite series, and we have

$$\Sigma u_n = s_n + R_n.$$

If Σu_n converges to the sum s, then we can write

$$s = s_n + R_n.$$

When $n \to \infty$, $s_n \to s$, therefore it follows that $R_n \to 0$. Conversely, if $R_n \to 0$, then $s_n - s \to 0$ or, in other words, $s_n \to s$ and, therefore, the series converges to the sum s.

Hence the series converges if, and only if, $R_n \to 0$ as $n \to \infty$.

It follows from the above that if for any quantity s, we have an expansion of the form

$$s = u_1 + u_2 + \ldots\ldots + u_n + R_n,$$

where R_n is the remainder after n terms, then the expansion on the right may be continued to infinity provided $R_n \to 0$. We then get

$$s = u_1 + u_2 + \ldots\ldots + u_n + \ldots\ldots,$$

an infinite series expansion of s, such that s is the sum of the infinite series on the right in the sense defined above.

9·3. Taylor's Infinite Series. When $f(x)$ is derivable indefinitely, we can take n as large as we please. If, in addition, R_n tends to zero as n tends to infinity, then by the preceding article we can prolong the series in the Taylor's theorem indefinitely; the last term R_n vanishing in the limit, we get the expansion of $f(a+h)$ as an infinite series in ascending powers of h, namely,

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots + \frac{h^n}{n!} f^{(n)}(a) + \ldots \quad (6)$$

This is called **Taylor's series.**

Ex. 1. *Prove that*

$$\sec^{-1}(x+h) = \sec^{-1}x + \frac{h}{x(x^2-1)^{1/2}} - \frac{h^2}{2!}\,\frac{2x^2-1}{x^2(x^2-1)^{3/2}} + \ldots\ldots\ldots$$

(Panjab, 1943)

Here $f(x+h)=\sec^{-1}(x+h)$, $\therefore f(x)=\sec^{-1}x$, and so

$$f'(x)=\frac{1}{x(x^2-1)^{1/2}},\ f''(x)=\frac{-(2x^2-1)}{x^2(x^2-1)^{3/2}},\ \ldots\ldots\ldots$$

Hence by Taylor's theorem,

$$\sec^{-1}(x+h)=\sec^{-1}x+\frac{h}{x(x^2-1)^{1/2}}-\frac{h^2}{2\,!}\ \frac{2x^2-1}{x(x^2-1)^{3/2}}+\ldots\ldots\ldots$$

Ex. 2. *Expand* $\sin x$ *in ascending powers of* $(x-\frac{1}{4}\pi)$.

Let $f(x)=\sin x$, then

$$f'(x)=\cos x,\ f''(x)=-\sin x,\ f'''(x)=-\cos x,\ \ldots\ldots\ldots$$

$$\therefore\ f\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}},\ f'\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}},\ f''\left(\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}},$$

$$f'''\left(\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}},\ldots\ldots\ldots$$

By Taylor's theorem [(5) of Art. 9·11],

$$f(x)=f(a)+(x-a)\,f'(a)+\frac{(x-a)^2}{2\,!}f''(a)+\ldots\ldots\ldots$$

Taking $f(x)=\sin x$ and $a=\frac{1}{4}\pi$, we get

$$\sin x=\frac{1}{\sqrt{2}}+\left(x-\frac{1}{4}\right)\frac{1}{\sqrt{2}}+\frac{1}{2\,!}\left(x-\frac{1}{4}\pi\right)^2\left(-\frac{1}{\sqrt{2}}\right)$$

$$+\frac{1}{3\,!}\left(x-\frac{1}{4}\pi\right)^3\left(-\frac{1}{\sqrt{2}}\right)+\ldots\ldots\ldots$$

$$=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\left(x-\frac{1}{4}\pi\right)-\frac{1}{2\sqrt{2}}\left(x-\frac{1}{4}\pi\right)^2$$

$$-\frac{1}{6\sqrt{2}}\left(x-\frac{1}{4}\pi\right)^3+\ldots\ldots\ldots$$

EXAMPLES XXVIII

Assuming the possibility of expansion, prove the following :

1. $e^{x+h}=e^x\left[1+h+\frac{h^2}{2\,!}+\frac{h^3}{3\,!}+\ldots\ldots\right]$ *(Panjab I, 1956)*

2. $\tan^{-1}(x+h)=\tan^{-1}x+\frac{h}{1+x^2}-\frac{xh^2}{(1+x^2)^2}+\ldots\ldots\ldots$

3. $\sin^{-1}(x+h)=\sin^{-1}x+\frac{h}{\sqrt{(1-x^2)}}+\frac{x}{(1-x^2)^{3/2}}\cdot\frac{h^2}{2\,!}$
$+\frac{1+2x^2}{(1-x^2)^{5/2}}\cdot\frac{h^3}{3\,!}+\ldots\ldots\ldots$

4. $\log\sin(x+h)=\log\sin x+h\cot x-\frac{1}{2}h^2\operatorname{cosec}^2 x$
$+\frac{1}{3}h^3\cot x\operatorname{cosec}^2 x+\ldots\ldots\ldots$

5. $e^{a(x+h)}\sin m(x+h)=e^{ax}[\sin mx+hr\sin(mx+\phi)$
$+\frac{1}{2}h^2r^2\sin(mx+2\phi)+\ldots\ldots\ldots]$

where $r^2=a^2+m^2$ and $a\tan\phi=m$,

6. Expand (*i*) x^3 in powers of $(x-1)$,

(*ii*) x^n in powers of $(x-a)$,

(*iii*) $\log\sin x$ in powers of $(x-a)$.

(*ix*) $\sin x$ in powers of $(x-\frac{1}{2}\pi)$. (*Nagpur, 1911*)

9·4. Maclaurin's Theorem and Maclaurin's Series. If in (2) of Art. 9·11 we take $a=0$, $h=x$, we obtain

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2\,!}f''(0)+\ldots\ldots\ldots$$
$$+\frac{x^{n-1}}{(n-1)\,!}f^{(n-1)}(0)+\frac{x^n}{n\,!}f^{(n)}(\theta x), \quad (1)$$

where $0<\theta<1$. This is **Maclaurin's formula with Lagrange's form of the remainder after n terms.** The expansion (1) is subject to the condition that $f(x)$ and all its derivatives upto the $(n-1)$th order are continuous in the closed interval $[0, x]$ and $f^{(n)}(x)$ exists in the open interval $(0, x)$.

When $f(x)$ has derivatives of all orders and the remainder after n terms, $\frac{x^n}{n\,!}f^{(n)}(\theta x)$, tends to zero as $n\to\infty$, then the series (1) can be continued to infinity, and we obtain **Maclaurin's series**

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2\,!}f''(0)+\ldots\ldots.+\frac{x^n}{n\,!}f^{(n)}(0)+\ldots \quad (2)$$

In the following we shall denote $f^{(n-1)}(x)$ and $f^{(n)}(x)$ simply by $f^{n-1}(x)$ and $f^n(x)$.

9·5. Expansions of elementary functions.

I. e^x. Let $f(x)=e^x$, then $f^n(x)=e^x$, $f^n(0)=e^0=1$, and therefore Maclaurin's theorem with Lagrange's form of the remainder gives

$$e^x=1+x+\frac{x^2}{2\,!}+\frac{x^3}{3\,!}+\ldots\ldots\ldots+\frac{x^{n-1}}{(n-1)\,!}+\frac{x^n}{n\,!}e^{\theta x}. \quad (1)$$

Since $e^{\theta x}$ is finite for all finite values of x, and $x^n/n\,!\to 0$ as $n\to\infty$ for every x, therefore the remainder $R_n=(x^n/n\,!)\,e^{\theta x}\to 0$ as $n\to\infty$. Hence the finite series (1) can be continued to infinity and we get the Maclaurin's infinite series expansion of e^x as

$$e^x=1+x+\frac{x^2}{2\,!}+\frac{x^3}{3\,!}+\ldots\ldots\ldots+\frac{x^n}{n\,!}+\ldots\ldots\ldots \quad (2)$$

valid for all x,

Cor. Changing x into $x \log a$ in (2), we get

$$a^x = e^{x \log a} = 1 + x \log a + \frac{(x \log a)^2}{2\,!} + \ldots\ldots + \frac{(x \log a)^n}{n\,!} + \ldots (2)$$

valid for all x.

II. sin x. If $f(x) = \sin x$, then $f^n(x) = \sin(x + \frac{1}{2}n\pi)$, and so $f^n(0) = \sin \frac{1}{2}n\pi = 0, 1, 0,$ or -1 according as n is of the type $4m$, $4m+1$, $4m+2$, or $4m+3$ respectively. Thus $f(0)=0$, $f'(0)=1$, $f''(0)=0$, $f'''(0)=-1$, etc. If we take $n=2m+1$, then

$$f^{2m+1}(\theta x) = \sin\{\theta x + \tfrac{1}{2}(2m+1)\pi\} = (-1)^m \cos \theta x,$$

and Maclaurin's formula with Lagrange's form of the remainder after n terms gives

$$\sin x = x - \frac{x^3}{3\,!} + \frac{x^5}{5\,!} - \ldots\ldots + \frac{(-1)^{m-1} x^{2m-1}}{(2m-1)\,!}$$

$$+ \frac{(-1)^m x^{2m+1}}{(2m+1)\,!} \cos \theta x.$$

As for e^x, the remainder tends to zero as $n \to \infty$ and $\therefore m \to \infty$ for every finite x. Hence the expansion of $\sin x$ as an infinite series in ascending powers of x is

$$\sin x = x - \frac{x^3}{3\,!} + \frac{x^5}{5\,!} - \ldots\ldots\ldots + \frac{(-1)^m x^{2m+1}}{(2m+1)\,!} + \ldots\ldots \qquad (4)$$

valid for all x.

III. cos x. Proceeding as for $\sin x$, we get

$$\cos x = 1 - \frac{x^2}{2\,!} + \frac{x^4}{4\,!} - \ldots\ldots\ldots + \frac{(-1)^m x^{2m}}{(2m)\,!} + \ldots\ldots\ldots \qquad (5)$$

valid for all x.

IV. log (1 + x). Taking $f(x) = \log(1+x)$, we get

$$f^n(x) = (-1)^{n-1} \frac{(n-1)\,!}{(1+x)^n}, \text{ so that } f^n(0) = (-1)^{n-1}\,(n-1)\,!.$$

Putting $n = 1, 2, 3$, etc., we get $f'(0) = 1$, $f''(0) = -1$, $f'''(0) = 2\,!$, etc. Also $f(0) = \log 1 = 0$. Hence Maclaurin's formula gives

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots\ldots + \frac{(-1)^{n-2} x^{n-1}}{n-1} + R_n.$$

It is shown in higher books that $R_n \to 0$ as $n \to \infty$ provided $-1 < x \leqslant 1$. Hence we get the infinite series

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots + \frac{(-1)^{n-1} x^n}{n} + \ldots\ldots\ldots \qquad (6)$$

valid for $-1 < x \leqslant 1$. If we put $x = 1$ in (6), we get

$$\log 2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \ldots\ldots\ldots \qquad (7)$$

V. $(1+x)^m$. Taking $f(x)=(1+x)^m$, we get

$$f^n(x)=m(m-1)\ldots\ldots(m-n+1)(1+x)^{m-n}$$

so that $$f^n(0)=m(m-1)\ldots\ldots(m-n+1).$$

Hence $f(0)=1, f'(0)=m, f''(0)=m(m-1)$, etc. and Maclaurin's formula gives

$$(1+x)^m=1+mx+\frac{m(m-1)}{2\,!}x^2+\ldots\ldots+\frac{m(m-1)\ldots(m-n+1)}{n\,!}x^n+\mathrm{R}_n.$$

We shall assume that the remainder $\mathrm{R}_n \to 0$ as $n \to \infty$ provided $|x|<1$. Hence we may continue the series indefinitely provided $|x|<1$ and we thus get the **Binomial Theorem for any index**,

$$(1+x)^m=1+mx+\frac{m(m-1)}{2\,!}x^2+\ldots\ldots+\frac{m(m-1)\ldots(m-r+1)}{r\,!}x^r+\ldots \quad (8)$$

valid for $-1<x<1$.

In case m is a positive integer, then $f^m(x)=a$ constant, $f^{m+1}(x)$ and all subsequent derivatives vanish. Hence the expansion (8) stops with the $(m+1)$th term. We then get the **Binomial theorem for a positive integral index m** in the form

$$(1+x)^m=1+mx+\frac{m(m-1)}{2\,!}x^2+\ldots\ldots+\frac{m(m-1)\ldots(m-r+1)}{r\,!}x^r+\ldots$$
$$\ldots+mx^{m-1}+x^m. \quad (9)$$

The series on the *r.h.s.* of (9) being a finite series, the equation (9) is valid for all x.

It will be observed from the above examples that in each case, we first obtain the Maclaurin's finite series for the given function $f(x)$. Next, we try to find out the values of x for which the remainder after n terms, *viz.*, R_n, tends to zero. For these values of x, the finite series can be continued to infinity. The discussion of the behaviour of R_n as $n \to \infty$ is a matter of some difficulty except for the simplest functions. In the following examples, we assume the possibility of expansion in the form of an infinite series, *i.e.*, we assume that $\mathrm{R}_n \to 0$.

Ex. 1. *Apply Maclaurin's theorem to obtain the expansion of tan x as far as the term containing x^5.*

Here
$$\begin{aligned} f(x)&=\tan x, & f(0)&=0,\\ f'(x)&=\sec^2 x, & f'(0)&=1,\\ f''(x)&=2\sec^2 x\tan x, & f''(0)&=0,\\ f'''(x)&=2\sec^4 x+4\sec^2 x\tan^2 x, & f'''(0)&=2,\\ f^{iv}(x)&=16\sec^4 x\tan x+8\sec^2 x\tan^3 x, & f^{iv}(0)&=0,\\ f^{v}(x)&=16\sec^6 x+88\sec^4 x\tan^2 x & f^{v}(0)&=16.\\ &\quad+16\sec^2 x\tan^4 x, & & \end{aligned}$$

Hence by Maclaurin's theorem,

$$\tan x = 0 + x.1 + \frac{x^2}{2!}.0 + \frac{x^3}{3!}.2 + \frac{x^4}{4!}.0 + \frac{x^5}{5!}.16 + \ldots\ldots$$

$$= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \ldots\ldots$$

The calculation of coefficients in the Maclurin's expansion can sometimes be simplified by suitable transformations as illustrated in the next example.

Ex. 8. *Expand $e^{\sin x}$ in powers of x by Maclaurin's theorem as far as the term containing x^4.* *(Delhi, 1954)*

Let $f(x) = e^{\sin x}$, then

$f'(x) = e^{\sin x} \cos x = f(x) \cos x,$

$f''(x) = -f(x) \sin x + f'(x) \cos x,$

$f'''(x) = -f(x) \cos x - 2f'(x) \sin x + f''(x) \cos x,$

$f^{iv}(x) = f(x) \sin x - 3f'(x) \cos x - 3f''(x) \sin x + f'''(x) \cos x$

Putting $x = 0$, we get

$$f(0) = 1, f'(0) = 1, f''(0) = 1, f'''(0) = 0, f^{iv}(0) = -3.$$

Hence substituting in Maclaurin's series, we get

$$e^{\sin x} = 1 + x + \tfrac{1}{2}x^2 - \tfrac{1}{8}x^4 + \ldots\ldots$$

9·6. Calculation of the nth derivative. The practical difficulty in the application of Maclaurin's theorem lies in the calculation of $f^n(x)$. It is not possible, except in some simple cases, to express the nth derivative in a reasonable form and, consequently, the discussion of the remainder R_n offers great difficulties.

In many cases, however, we can calculate $f^n(0)$ with the help of Leibnitz's theorem as has been illustrated by a solved example in Art. 4·5, Part I. In such cases, the formal expansion of $f(x)$ by Maclaurin's infinite series can be written down and the infinite series thus obtained will, in general, represent the function $f(x)$ for such values of x for which it is convergent.

Ex. 1. *Expand $f(x) = \sin(m \sin^{-1} x)$ in a Maclaurin series.*

Let $$y = f(x) = \sin(m \sin^{-1} x), \quad (i)$$

then $$y_1 = \frac{m}{\sqrt{(1-x^2)}} \cos(m \sin^{-1} x). \quad (ii)$$

$$\therefore \quad (1-x^2) y_1^2 = m^2 \cos^2(m \sin^{-1} x) = m^2(1-y^2).$$

Differentiating again and removing the factor $2y_1$, we get

$$(1-x^2) y_2 - xy_1 + m^2 y = 0. \quad (iii)$$

Differentiating this n times by Leibnitz's theorem, we get

$$(1-x^2)y_{n+2} + n(-2x) y_{n+1} + n(n-1)(-1) y_n - xy_{n+1} - ny^n + m^2 y_n = 0,$$

i.e., $$(1-x^2) y_{n+2} - (2n+1)xy_{n+1} + (m^2 - n^2) y_n = 0. \quad (iv)$$

Putting $x=0$ in (*iv*), we get

$$(y_{n+2})_0=(n^2-m^2)(y_n)_0 \tag{v}$$

where $(y_n)_0$ stands for the value of y_n at $x=0$, that is, it stands for $f^n(0)$. (*v*) gives us a recurrence relation by means of which we can calculate the values of the successive derivatives at $x=0$ provided we know the values of y, y_1 and y_2 at $x=0$. Putting $x=0$ in (*i*), (*ii*) and (*iii*), we get

$$(y)_0=0,\ (y_1)_0=m, \text{ and } (y_2)_0=0.$$

Now putting $n=2, 4, 6, \ldots\ldots$ in (*v*), we see that

$$(y_4)_0=(2^2-m^2)(y_2)_0=0=(y_6)_0=\ldots=(y_{2n})_0.$$

Hence all even order derivatives vanish at $x=0$.

Next, put $n=1, 3, 5, \ldots\ldots$, we get

$$(y_3)_0=(1^2-m^2)\,(y_1)_0=m(1^2-m^2),$$
$$(y_5)_0=(3^2-m^2)(y_3)_0=m(1^2-m^2)(3^2-m^2),$$

and so on, the general value is

$$(y_{2n+1})_0=m(1^2-m^2)(3^2-m^2)\ldots\ldots\{(2n-1)^2-m^2\}.$$

Hence substituting in the Maclaurin's series

$$\sin\,(m\,\sin^{-1}x)=mx+\frac{m(1^2-m^2)}{3\,!}x^3+\frac{m(1^2-m^2)(3^2-m^2)}{5\,!}x^5+\ldots$$

If m be an odd integer, then the expression comes to a stop as soon as one of the factors in the numerator becomes zero, otherwise it extends to infinity.

Cor. 1. If we put $\sin^{-1}x=\theta$, so that $x=\sin\theta$, we get

$$\sin m\theta=m\sin\theta+\frac{m(1^2-m^2)}{3\,!}\sin^3\theta+\frac{m(1^2-m^2)(3^2-m^2)}{5\,!}\sin^5\theta+\ldots$$

Cor 2 Proceeding similaaly we can get

$$\cos m\theta=1-\frac{m^2}{2\,!}\sin^2\theta+\frac{m^2(m^2-2^2)}{4\,!}\sin^4\theta+\ldots$$

Ex. 3. *Expand* $e^{a\sin^{-1}x}$ *in Maclaurin's series.*

(*Panjab, Sept. 1950*)

Let $$y=f(x)=e^{a\sin^{-1}x}, \tag{i}$$

then $$y_1=\frac{a}{\sqrt{(1-x^2)}}e^{a\sin^{-1}x} \tag{ii}$$

$$\therefore\quad (1-x^2)y_1^2=a^2y^2.$$

Differentiating again and removing the factor $2y_1$, we get

$$(1-x^2)\,y_2-xy_1-a^2y=0. \tag{iii}$$

Putting $x=0$ in (*i*), (*ii*) and (*iii*), we get

$$(y)_0=1,\ (y_1)_0=a \text{ and } (y_2)_0=a^2.$$

Differentiating (*iii*) n times by Leibnitz's theorem, we have

$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+a^2)y_n=0.$$

Putting $x=0$, we get $(y_{n+2})_0=(n^2+a^2)(y_n)_0$.

Hence putting $n=1, 2, 3, \ldots\ldots$, we get

$$(y_3)_0=(1^2+a^2)(y_1)_0=a(1^2+a^2),$$
$$(y_4)_0=(2^2+a^2)(y_2)_0=a^2(2^2+a^2),$$
$$(y_5)_0=(3^2+a^2)(y_3)_0=a(1^2+a^2)(3^2+a^2), \text{ and so on.}$$

Hence substituting in Maclaurin's series

$$e^{a\sin^{-1}x}=1+ax+\frac{a^2}{2\,!}x^2+\frac{a(1^2+a^2)}{3\,!}x^3+\frac{a^2(2^2+a^2)}{4\,!}x^4+\ldots$$

Note. It should be observed that $e^{a\sin^{-1}x}$ can also be expanded as

$$e^{a\sin^{-1}x}=1+(a\sin^{-1}x)+\frac{(a\sin^{-1}x)^2}{2\,!}+\frac{(a\sin^{-1}x)^3}{3\,!}+\ldots\ldots$$

Equating the coefficients of a, a^2, etc. in the expansions, we get

$$\sin^{-1}x=x+\frac{1^2}{3\,!}x^3+\frac{1^2.3^2}{5\,!}x^5+\ldots\ldots$$

$$\frac{(\sin^{-1}x)^2}{2\,!}=\frac{x^2}{2\,!}+\frac{2^2}{4\,!}x^4+\frac{2^2.4^2}{6\,!}x^6+\ldots\ldots$$

and so on.

9·7. The formal expansion of a function can also be obtained by the methods illustrated in the solved examples below.

Ex. 1. *Expand* $\tan^{-1}x$ *in a series of ascending power of* x.

Let $y=\tan^{-1}x=a_0+a_1x+a_2x^2+\ldots\ldots+a_nx^n+a_{n+1}x^{n+1}+\ldots$

then $$y_1=\frac{1}{1+x^2} \text{ or } (1+x^2)y_1=1.$$

Also, differentiating the series term by term,

$$y_1=a_1+2a_2x+3a_3x^2+\ldots\ldots+na_nx^{n-1}+(n+1)a_{n+1}x^n+\ldots$$

Hence substituting in the relation $(1+x^2)y_1=1$, we get

$$(1+x^2)(a_1+2a_2x+3a_3x^2+\ldots\ldots+na_nx^{n-1}+(n+1)a_{n+1}x^n+\ldots)=1.$$

Equating the constant terms, $a_1=1$.

Equating the coefficients of x, $a_2=0$.

Equating the coefficients of x^n on both sides, $n\geqslant 2$,

$$(n+1)a_{n+1}+(n-1)a_{n-1}=0 \text{ or } a_{n+1}=-\frac{n-1}{n+1}a_{n-1}.$$

Putting $n=3, 5, 7, \ldots\ldots$, we see that

$$a_4=a_6=\ldots\ldots=0 \text{ since } a_2=0.$$

Putting $n=2, 4, 6, \ldots\ldots$

$$a_3=-\tfrac{1}{3}a_1=-\tfrac{1}{3},$$
$$a_5=-\tfrac{3}{5}a_3=\tfrac{1}{5},$$
$$a_7=-\tfrac{5}{7}\,a_5=-\tfrac{1}{7}, \text{ and so on.}$$

Also from the original relation, putting $x=0$, $a_0=0$. Hence

$$\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots\ldots$$

The expansion is valid only for $|x|\leqslant 1$.

Ex. 2. *Show that*

$$\log(1+e^x)=\log 2+\tfrac{1}{2}x+\tfrac{1}{8}x^2-\tfrac{1}{192}x^4+\ldots\ldots$$

(*Panjab, 1959*)

Assuming that the series involved are all 'well behaved' and all the operations used are permissible, we have

$$\log\,(1+e^x)=\log\left(2+x+\frac{x^2}{2\,!}+\frac{x^3}{3\,!}+\frac{x^4}{4\,!}+\ldots\ldots\right)$$
$$=\log 2\cdot(1+x/2+x^2/4+x^3/3!\cdot 2\ldots)$$
$$=\log 2+\log\left(1+\tfrac{1}{2}x+\tfrac{1}{4}x^2+\tfrac{1}{12}x^3+\tfrac{1}{48}x^4+\ldots\ldots\right)$$
$$=\log 2+\left(\tfrac{1}{2}x+\tfrac{1}{4}x^2+\tfrac{1}{12}x^3+\tfrac{1}{48}x^4+\ldots\ldots\right)$$
$$-\tfrac{1}{2}\left(\tfrac{1}{2}x+\tfrac{1}{4}x^2+\tfrac{1}{12}x^3+\ldots\right)^2+\tfrac{1}{3}\left(\tfrac{1}{2}x+\tfrac{1}{4}x^2+\ldots\ldots\right)^3$$
$$-\tfrac{1}{4}\left(\tfrac{1}{2}x+\ldots\ldots\right)^4+\ldots\ldots$$
$$=\log 2+\left(\tfrac{1}{2}x+\tfrac{1}{4}x^2+\tfrac{1}{12}x^3+\tfrac{1}{48}x^4+\ldots\ldots\right)$$
$$-\tfrac{1}{2}\left(\tfrac{1}{4}x^2+\tfrac{1}{16}x^4+2.\tfrac{1}{2}x.\tfrac{1}{4}x^2+2.\tfrac{1}{2}x.\tfrac{1}{12}.x^3+\ldots\ldots\right)$$
$$+\tfrac{1}{3}\left(\tfrac{1}{8}x^3+3.\tfrac{1}{4}x^2.\tfrac{1}{4}x^2+\ldots\ldots\right)-\tfrac{1}{4}\left(\tfrac{1}{16}x^4+\ldots\ldots\right)$$

(retaining terms upto and including x^4 only)

$$=\log 2+\tfrac{1}{2}x+\tfrac{1}{8}x^2-\tfrac{1}{192}x^4+\ldots\ldots$$

EXAMPLES XXIX

Assuming the possibility of expansion, obtain the following series :

1. (*i*) $e^{mx}=1+mx+\dfrac{m^2x^2}{2\,!}+\dfrac{m^3x^3}{3\,!}+\ldots\ldots$

(*ii*) $a^{mx}=1+(m\log a)x+\dfrac{(m\log a)^2x^2}{2\,!}+\dfrac{(m\log a)^3x^3}{3\,!}+\ldots\ldots$

(*iii*) $\log(1-x)=-x-\tfrac{1}{2}x^2-\tfrac{1}{3}x^3-\tfrac{1}{4}x^4+\ldots\ldots$

(*iv*) $\sin ax=ax-\dfrac{a^3x^3}{3\,!}+\dfrac{a^5x^5}{5\,!}-\ldots\ldots$

2. $e^{ax}\cos bx=1+ax+\dfrac{a^2-b^2}{2\,!}x^2+\dfrac{a(a^2-3b^2)}{3\,!}x^3+\ldots\ldots$

(*Panjab, 1961 S*)

3. $e^{ax}\sin bx=bx+abx^2+\dfrac{3a^2b-b^3}{3\,!}x^3+\ldots\ldots$

4. $\sin(m\sin^{-1}x)=mx+\dfrac{m(1^2-m^2)}{3\,!}x^3+\dfrac{m(1^2-m^2)(3^2-m^2)}{5\,!}x^5$

$+\ldots\ldots$

5. $\dfrac{e^x}{\cos x}=1+x+\dfrac{2x^2}{2\,!}+\dfrac{4x^3}{3\,!}+\ldots\ldots$

6. $\tan^{-1}x=x-\frac{1}{3}x^3+\frac{1}{5}x^5-\ldots\ldots$

7. $(1+x)^{1+x}=1+x+x^2+\frac{1}{2}x^3+\ldots\ldots$ *(Panjab)*

8. By Maclaurin's theorem or otherwise find the expansion of $y=\sin(e^x-1)$ upto and including the term in x^4. *(Panjab, 1949)*

9. Find the first five terms in the expansion of $\log(1+\sin x)$ in ascending powers of x. *(Panjab, 1947 ; Agra, 1950)*

10. Obtain by Maclaurin's theorem the coefficient of x^4 in the expansion of $\log \cos x$. Verify this algebraically by the use of the known expansion of $\cos x$ and $\log(1+x)$. *(Agra, 1942)*

11. $\log \sec x=\frac{1}{2}x^2+\frac{1}{12}x^4+\frac{1}{45}x^6+\ldots\ldots$ 705 *(Delhi, 1953)*

12. Obtain the first three terms in the expansion of $\log(1+\tan x)$. *(Panjab, 1955)*

13. Expand $\log\{1-\log(1-x)\}$ in powers of x by Maclaurin's theorem as far as the term containing x^3. *(Panjab, Sept. 1954)*

14. Obtain by Maclaurin's theorem the first four terms of the expansion of $e^{a \cos x}$ in ascending powers of x. *(Panjab, 1956)*

15. If $y=\frac{1}{2}(\sin^{-1}x)^2=a_0+a_1x+a_2\dfrac{x^2}{2\,!}+a_3\dfrac{x^3}{3\,!}+\ldots\ldots$

show that $a_{n+2}-n^2a_n=0$. Hence expand y.

16. If $y=\sin^{-1}x=a_0+a_1x+a_2x^2+a_3x^3+\ldots\ \ldots$, prove that

(i) $(1-x^2)\,y_2=xy_1$. (ii) $(n+1)(n+2)a_{n+2}=n^2a_n$, and

(iii) $\sin^{-1}x=x+\dfrac{1}{2}\,\dfrac{x^3}{3}+\dfrac{1.3}{2.4}\,\dfrac{x^5}{5}+\ldots\ldots$ *(Calcutta, 1947)*

17. Prove that

$$\{x+\sqrt{(1+x^2)}\}^n=1+nx+\frac{n^2x^2}{2\,!}+\frac{n(n^2-1^2)}{3\,!}x^3$$
$$+\frac{n^2(n^2-2^2)}{4\,!}x^4+\frac{n(n^2-1^2)(n^2-3^2)}{5\,!}x^5+\ldots$$

and deduce the expansions of

$$\log\{x+\sqrt{(1+x^2)}\} \quad\text{and}\quad \tfrac{1}{2}[\log\{x+\sqrt{(1+x^2)}\}]^2.$$

18. Can $\sin(1/x)$ or $\log x$ or $\cot x$ be expanded in ascending powers of x by Maclaurin's theorem ? Give reasons.

9·8. Applications of Taylor's theorem. I. Criteria for extreme values.

Let $f(x)$ be defined in the range $[a, b]$ and let

(i) $f^n(x)$ be continuous in $(c-\varepsilon, c+\varepsilon)$, $a<c<b$,

(ii) $f'(c)=f''(c)=\ldots=f^{n-1}(c)=0$, (iii) $f^n(c)\neq 0$,

then $f(x)$ *has an extreme value at* $x=c$ *if* n *is even, being a maximum if* $f^n(c)<0$ *and a minimum if* $f^n(c)>0$.

$f(x)$ *has no extreme value at* $x=c$ *if* n *is odd.*

By Taylor's theorem, $\Delta_1=f(c+h)-f(c)$

$$=hf'(c)+\frac{h^2}{2\,!}f''(c)+\ldots\ldots\frac{h^{n-1}}{(n-1\,!)}f^{n-1}(c)+\frac{h^n}{n\,!}f^n(c+\theta h)$$

$$=\frac{h^n}{n\,!}f^n(c+\theta h),\ \text{where } 0<\theta<1 \text{ and } h<\epsilon,$$

and $\Delta_2=f(c-h)-f(c)$

$$=-hf'(c)+\frac{h^2}{2\,!}f''(c)+\ldots\ldots+\frac{(-h)^{n-1}}{(n-1)\,!}f^{n-1}(c)+\frac{(-h)^n}{n\,!}f^n(c-\theta'h)$$

$$=(-1)^n\cdot\frac{h^n}{n\,!}f^n(c-\theta'h),\ \text{where } 0<\theta'<1 \text{ and } h<\epsilon.$$

$\because f^n(x)$ is continuous in $(c-\epsilon, c+\epsilon)$, $f^n(c+\theta h)$ and $f^n(c-\theta'h)$ have the same sign as $f^n(c)$ when h is sufficiently small.

For an extremum to exist at $x=c$, Δ_1 and Δ_2 must be of the same sign. Hence n must be even and then the sign of Δ_1 and Δ_2 is the same as that of $f^n(c)$. Hence $f(x)$ is maximum or minimum according as $f^n(c)$ is negative or positive provided n is even.

If n is odd, Δ_1 and Δ_2 are of opposite signs. Hence $f(x)$ cannot have an extremum in this case.

In particular if $n=2$, the conditions for the existence of a maximum and minimum are:

(i) $f'(c)=0$,

and (ii) if $f''(c)<0$, there is a maximum at $x=c$;

if $f''(c)>0$, there is a minimum at $x=c$.

Ex. *Show that* $x^5-5x^4+5x^3-1$ *has a maximum when* $x=1$, *a minimum when* $x=3$, *and neither when* $x=0$.

(*Panjab, 1954*; *Cal., 1949*)

Let $f(x)=x^5-5x^4+5x^3-1$.

then $f'(x)=5x^4-20x^3+15x^2=5x^2(x-1)(x-3)$,

$f''(x)=20x^3-60x^2+30x$,

$f'''(x)=60x^2-120x+30$.

$f'(x)=0$ gives $x^2(x-1)(x-3)=0$, whence $x=0$, 1, or 3. Hence the possible points or maxima and minima of $f(x)$ are amongst these three.

(i) At $x=0$, $f''(0)=0$, $f'''(0)=30$. Hence the first derivative that does not vanish is of an odd order. Hence $f(x)$ has neither a maximum nor a minimum at $x=0$.

(*ii*) At $x=1$, $f''(1)=20-60+30=-10$. Hence the first derivative that does not vanish is of an even order and its sign is negative. Hence $f(x)$ has a maximum at $x=1$ and the maximum value $f(1)=1-5+5-1=0$.

(*iii*) At $x=3$, $f''(3)=540-540+90=90$. Hence the first derivative that does not vanish is of an even order and its sign is positive. Hence $f(x)$ has a minimum at $x=3$ and the minimum value $f(3)=243-405+135-1=-28$.

Hence $f(x)$ has a maximum at $x=1$, a minimum at $x=3$ and neither at $x=0$.

The example illustrates also the important point that *a function $f(x)$ does not necessarily have an extremum at every point where $f'(x)=0$.*

II. Approximations.

Ex. 1. *Expand* sin $(x+h)$ *in ascending powers of h and hence calculate the value of* sin 31° *correct to four places of decimals.*

Let $f(x+h)=\sin(x+h)$;

$\therefore \quad f(x)=\sin x$.

Differentiating successively *w.r.t.* x, we get

$$f'(x)=\cos x,\ f''(x)=-\sin x,\ f'''(x)=-\cos x, \ldots\ldots$$

Hence by Taylor's theorem,

$$\sin(x+h)=\sin x+h\cos x-\frac{h^2}{2\,!}\sin x-\frac{h^3}{3\,!}\cos x+\ldots\ldots \tag{1}$$

Now $31°=30°+1°=\dfrac{\pi}{6}+\dfrac{\pi}{180}$ radians.

Hence putting $x=\dfrac{\pi}{6}$ and $h=\dfrac{\pi}{180}$ in (1), we get

$$\sin\left(\frac{\pi}{6}+\frac{\pi}{180}\right)=\sin\frac{\pi}{6}+\frac{\pi}{180}\cos\frac{\pi}{6}-\frac{1}{2\,!}\left(\frac{\pi}{180}\right)^2\sin\frac{\pi}{6}$$
$$-\frac{1}{3\,!}\left(\frac{\pi}{180}\right)^3\cos\frac{\pi}{6}-\ldots\ldots$$
$$=\cdot5000+\cdot01745\times\cdot8660-\tfrac{1}{2}(\cdot01745)^2\times\cdot5000-\ldots\ldots$$
$$=\cdot5000+\cdot01512-\cdot00007=\cdot5150.$$

Ex. 2. *Apply Taylor's theorem to evaluate* $f(2\cdot001)$ *to six places of decimals if* $f(x)=x^3-2x+5$.

By Taylor's theorem.

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2\,!}f''(x)+\frac{h^3}{3\,!}f'''(x)+\ldots\ldots$$

Setting $x=2$ and $h=\cdot002$, we get

$$f(2\cdot001)=f(2)+(\cdot001).f'(2)+\frac{1}{2\,!}(\cdot001)^2 f''(2)+\frac{1}{3\,!}(\cdot001)^3$$
$$f'''(2)+\ldots\ldots \tag{1}$$

Now $f(x)=x^3-2x+5 \quad \therefore \quad f(2)=9.$

$$\begin{aligned} f'(x)&=3x^2-2 & f'(2)&=10\\ f''(x)&=6x & f''(2)&=12\\ f'''(x)&=6 & f'''(2)&=6\\ f^{iv}(x)&=0=f^{v}(x)=\ldots\ldots & f^{iv}(2)&=0=f^{v}(2)=\ldots\ldots \end{aligned}$$

Hence from (1),

$$f(2{\cdot}001)=9+({\cdot}001)\times 10+\tfrac{1}{2}({\cdot}001)^2\times 12+\tfrac{1}{6}({\cdot}001)^3\times 6$$
$$=9{\cdot}010006.$$

***III. Newton's method. Approximate solution of equations.**

Let $c+h$ be a non-repeated root of $f(x)=0$ so that $f(c+h)=0$. If h is small, c can be considered as a fair approximation to the root.

We assume that $f(x)$ and its first two derivatives are continuous in every neighbourhood of c. We have, by Taylor's theorem

$$f(c+h)=f(c)+hf'(c)+\frac{h^2}{2\,!}f''(c+\theta h), \quad 0<\theta<1.$$

Taking h to be so small that the last term may be neglected, we get

$$0=f(c)+hf'(c) \text{ so that } h=-f(c)/f'(c).$$

Hence if c is an approximation to a root of $f(x)=0$, then a closer approximation to the root is

$$c-\frac{f(c)}{f'(c)}.$$

Denoting this value by c_1 and repeating the above process, we find that a still closer approximation to the root is

$$c_1-\frac{f(c_1)}{f'(c_1)}.$$

The process may be repeated to obtain further improvement in approximation.

Ex. *Find the positive root of* $x^3+x-3=0$ *correct to two places of decimals.*

Let $f(x)=x^3+x-3$.

Since there is only one change of sign, the equation has one and only one positive root.

Again $f(1)=-1, f(2)=7$. Thus the positive root lies between 1 and 2.

Now $f'(x)=3x^2+1$. Hence taking 1 as a first approximation, we have

$$\frac{f(1)}{f'(1)}=\frac{-1}{4}=-{\cdot}25.$$

Hence a second approximation is $1-(-\cdot 25)=1\cdot 25$.

Again $\dfrac{f(1\cdot 25)}{f'(1\cdot 25)}=\dfrac{1\cdot 9531+1\cdot 25-3}{3\times 1\cdot 5625+1}=\dfrac{\cdot 203}{5\cdot 6875}=\cdot 036.$

Hence a closer approximation is $1\cdot 25-\cdot 036=1\cdot 214$. A further approximation is easily seen not to affect the second place of decimal. **Hence** the root is $1\cdot 21$ correct to two places.

EXAMPLES XXX

Examine for **extreme** values :

1. x^4+2x^3-2x-1. **2.** $(x-1)^4(x-2)^3$.

3. Expand cos $(x+h)$ by Taylor's theorem and hence calculate the value of cos 61°. *(Panjab I, 1947)*

Use the above expansion to prove that

$$\cos(x+y)=\cos x\cos y-\sin x\sin y.$$

4. By suitable substitution in the Taylor's expansion of $\tan(x+h)$, calculate correct to four decimal places the value of tan 46° 48′ taking $\pi=3\cdot 14159$. *(Delhi, 1949)*

5. If $f(x)=x^3-2x^2-5x+11$, calculate $f(\frac{9}{10})$ by the application of Taylor's theorem for $f(x+h)$. *(Panjab I, 1954)*

6. Apply Taylor's theorem to calculate the value of $f(\frac{11}{10})$ if $f(x)=x^3+3x^2+15x-24$. *(Panjab I, 1951)*

7. Prove that a root of $2(x+\sin x)=3$ correct to four decimal places is $\cdot 7897$.

CHAPTER X

INDETERMINATE FORMS

10·1. The limit of the quotient $f(x)/F(x)$, when $x \to a$, is in general equal to $f(a)/F(a)$. The exception to this rule usually occurs when both $f(a)$ and $F(a)$ are equal to zero. In that case the fraction $f(x)/F(x)$ is said to assume the **indeterminate form 0/0** as x tends to a. Similarly, if $\text{Lt}\, f(x)=\infty$ and $\text{Lt}\, F(x)=\infty$ as $x \to a$, then the fraction $f(x)/F(x)$ assumes the indeterminate form ∞/∞. The other simple indeterminate forms are $0\times\infty$, $\infty-\infty$, 0^0, 1^∞ and ∞^0. It is the object of this chapter to evaluate the limits of such indeterminate forms. The limiting value of an indeterminate form is also sometimes called its **true value.**

It may be observed that it has to be specified clearly whether x tends to a in any manner whatsoever or it tends to a from the right only or from the left only. Thus, for example, $\text{Lt}\,(\tan x)/x = 1$ when $x \to 0$ in any manner, whereas $\text{Lt}\,(x \log x)=0$ when $x \to 0$ through positive values only, for the function $\log x$ is not defined for any negative value of x.

10·2. The form 0/0. L' Hospital's rule. The rule consists in substituting for the functions $f(x)$ and $F(x)$ their derivatives $f'(x)$ and $F'(x)$. L'Hospital stated this rule in a geometrical form only and is supposed to have borrowed it from John Bernouilli. We shall state the rule in two different forms, the second of which has the advantage that it lends itself to extensions to which the first does not.

Form I. *If $f(a)=F(a)=0$ and $f'(a)$ and $F'(a)$ both exist and are neither both zero nor both infinite, then the limit (finite or infinite) of $f(x)/F(x)$ as $x \to a$ is $f'(a)/F'(a)$, that is,*

$$\underset{x\to a}{\text{Lt}}\ \frac{f(x)}{F(x)}=\frac{f'(a)}{F'(a)}.$$

Put $x=a+h$, then $h \to 0$ as $x \to a$ and so

$$\underset{x\to a}{\text{Lt}}\ \frac{f(x)}{F(x)}=\underset{h\to 0}{\text{Lt}}\ \frac{f(a+h)}{F(a+h)}=\underset{h\to 0}{\text{Lt}}\ \frac{f(a+h)-f(a)}{F(a+h)-F(a)}$$

$$=\underset{h\to 0}{\text{Lt}}\ \frac{\dfrac{f(a+h)-f(a)}{h}}{\dfrac{F(a+h)-F(a)}{h}}=\frac{f'(a)}{F'(a)}$$

which establishes the rule. Here 'a' is supposed to be finite and the rule cannot be extended to the case when $x \to \infty$. Also if $F'(a)=0$, then the limit is infinite but the sign remains undetermined.

Assuming that the derivatives involved are all continuous, the rule can be generalised as follows :

If $f(x)$ and $F(x)$ vanish together with their first $(n-1)$ derivatives at $x=a$, while their nth derivatives are finite, continuous and not both zero at $x=a$, then

$$\operatorname*{Lt}_{x\to a}\frac{f(x)}{F(x)}=\frac{f^n(a)}{F^n(a)}.$$

Since both $f^n(x)$ and $F^n(x)$ are assumed continuous at $x=a$, then by the application of Taylor's theorem with Lagrange's form of the remainder, we get

$$f(a+h)=\frac{h^n}{n\,!}\,f^n(a+\theta h),\quad F(a+h)=\frac{h^n}{n\,!}\,F^n(a+\theta' h)$$

where $0<\theta<1$, $0<\theta'<1$. Hence

$$\operatorname*{Lt}_{x\to a}\frac{f(x)}{F(x)}=\operatorname*{Lt}_{h\to 0}\frac{f(a+h)}{F(a+h)}=\operatorname*{Lt}_{h\to 0}\frac{f^n(a+\theta h)}{F^n(a+\theta' h)}=\frac{f^n(a)}{F^n(a)}.$$

Form II. *If $f(x)$ and $F(x)$ both vanish for $x=a$, then*

$$\operatorname*{Lt}_{x\to a}\frac{f(x)}{F(x)}=\operatorname*{Lt}_{x\to a}\frac{f'(x)}{F'(x)},$$

provided that the latter limit (finite or infinite) is determinate.

The proof depends on Cauchy's form of the Mean Value theorem and we shall therefore assume the truth of the rule.

If $f'(x)/F'(x)$ is again of the form $0/0$, then this rule can be repeated so long as the indeterminacy lasts. Thus

$$\operatorname*{Lt}_{x\to a}\frac{f(x)}{F(x)}=\operatorname*{Lt}_{x\to a}\frac{f'(x)}{F'(x)}=\operatorname*{Lt}_{x\to a}\frac{f''(x)}{F''(x)}=\ldots\ldots$$

This rule can be easily extended to the case when $x\to\pm\infty$. To be definite, let $x\to+\infty$, then

$$\operatorname*{Lt}_{x\to+\infty}\frac{f(x)}{F(x)}=\operatorname*{Lt}_{y\to 0+}\frac{f(1/y)}{F(1/y)}=\operatorname*{Lt}_{y\to 0+}\frac{f'(1/y)\times(-1/y^2)}{F'(1/y)\times(-1/y^2)}$$

$$=\operatorname*{Lt}_{y\to 0+}\frac{f'(1/y)}{F'(1/y)}=\operatorname*{Lt}_{x\to+\infty}\frac{f'(x)}{F'(x)}.$$

It may be remarked that this rule is more useful than the first because we can cancel any common factors in the numerator and denominator or effect any other simplifications in the quotient $f'(x)/F'(x)$ before proceeding with a repeated application of the rule.

Ex. 1. *Evaluate* $\operatorname*{Lt}_{x\to 0}\dfrac{1-x+\frac{1}{2}x^2-e^{-x}}{x^3}.$

This is of the form $0/0$. Here

$$f(x)=1-x+\tfrac{1}{2}x^2-e^{-x},\qquad f'(x)=-1+x+e^{-x}.$$

$$f''(x)=1-e^{-x},\qquad f'''(x)=e^{-x},$$

so that $f(0)=f'(0)=f''(0)=0$ and $f'''(0)=1$.

Again $F(x)=x^3$, $F'(x)=3x^2$, $F''(x)=6x$, $F'''(x)=6$,

so that $\quad F(0)=F'(0)=F''(0)=0$ and $F'''(0)=6$.

Hence by the first rule

$$\underset{x\to 0}{\text{Lt}}\ \frac{f(x)}{F(x)}=\frac{f'''(0)}{F'''(0)}=\frac{1}{6}.$$

Ex. 2. *Find the limit as* $\theta\to 0$ *of* $\dfrac{\sin\theta-\theta\cos\theta}{\sin\theta-\theta}$.

(M.T.I, 1948)

By repeated application of the second rule, we get

$$\underset{\theta\to 0}{\text{Lt}}\ \frac{\sin\theta-\theta\cos\theta}{\sin\theta-\theta}=\underset{\theta\to 0}{\text{Lt}}\ \frac{\cos\theta-\cos\theta+\theta\sin\theta}{\cos\theta-1}.$$

$$=\underset{\theta\to 0}{\text{Lt}}\ \frac{\theta\sin\theta}{\cos\theta-1}=\underset{\theta\to 0}{\text{Lt}}\ \frac{\sin\theta+\theta\cos\theta}{-\sin\theta}$$

$$=\underset{\theta\to 0}{\text{Lt}}\ \frac{\cos\theta+\cos\theta-\theta\sin\theta}{-\cos\theta}=\frac{2}{-1}=-2.$$

EXAMPLES XXXI

Show that, when $x\to 0$,

1. $\text{Lt}\ \dfrac{1-\cos x}{3x^2}=\dfrac{1}{6}$. *(Pb., 1960)*

2. $\text{Lt}\ \dfrac{\sin ax}{\sin bx}=\dfrac{a}{b}$. **3.** $\text{Lt}\ \dfrac{1-\cos x}{x\log(1+x)}=\dfrac{1}{2}$.

4. $\text{Lt}\ \dfrac{a^x-1}{b^x-1}=\dfrac{\log a}{\log b}$. **5.** $\text{Lt}\ \dfrac{a\sin x-\sin ax}{x(\cos x-\cos ax)}=\dfrac{a}{3}$.

6. Show that $\underset{\theta\to 0}{\text{Lt}}\ \dfrac{e^{\theta}\sin\theta-\theta-\theta^2}{\theta^3}=\dfrac{1}{3}$.

7. Evaluate $\underset{x\to 0}{\text{Lt}}\ \dfrac{x-\sin x}{x^3}$. *(Panjab, 1948)*

8. $\underset{x\to 0}{\text{Lt}}\ \dfrac{x-\tan x}{x^3}$. *(Delhi, 1936)*

9. $\underset{x\to b}{\text{Lt}}\ \dfrac{x^b-b^x}{x^x-b^b}$. *(Panjab, 1953)*

10. $\underset{x\to 0}{\text{Lt}}\ \dfrac{\log(1+x^3)}{\sin^3 x}$. *(Panjab, 1952)*

11. $\underset{x\to 0}{\text{Lt}}\ \dfrac{\sin x\sin^{-1}x}{x^2}$. *(Agra, 1941)*

12. $\underset{x\to 1}{\text{Lt}}\ \dfrac{x^x-x}{1-x+\log x}$. *(Nagpur, 1940)*

13. $\underset{x\to 0}{\text{Lt}}\ \dfrac{e^{ax}-e^{-ax}}{\log(1+bx)}$. *(Delhi, 1952)*

14. $\underset{x \to 0}{\text{Lt}} \dfrac{e^x - \log (e+ex)}{x^2}$ (*Agra, 1949*)

15. $\underset{x \to 0}{\text{Lt}} \dfrac{x \cos x - \log (1+x)}{x^2}$. (*Delhi, 1950*)

16. $\underset{x \to 0}{\text{Lt}} \dfrac{xe^x - \log (1+x)}{x^2}$. (*Delhi, 1957 ; Panjab, '49*)

17. $\underset{x \to 0}{\text{Lt}} \dfrac{e^x - 2 \cos x + e^{-x}}{x \sin x}$. (*Panjab, 1946*)

18. $\underset{x \to 0}{\text{Lt}} \dfrac{\log (1-x^2)}{\log \cos x}$. (*Agra, 1950*)

10·3. The form $\dfrac{\infty}{\infty}$. The second rule for the form 0/0 is applicable in this case also under suitable modifications. We have the following rule.

If $f(x)$ and $F(x)$ both tend to infinity as x tends to a and $f'(x)$ and $F'(x)$ exist in the neighbourhood of a, are finite and do not vanish simultaneously, then

$$\underset{x \to a}{\text{Lt}} \frac{f(x)}{F(x)} = \underset{x \to a}{\text{Lt}} \frac{f'(x)}{F'(x)},$$

provided the latter limit exists, finite or not.

The proof of this rule also depends on Cauchy's form of the Mean Value theorem and we shall assume its truth.

The previous rule can be easily extended to the case when x tends to infinity, that is, we can show that

$$\underset{x \to \infty}{\text{Lt}} \frac{f(x)}{F(x)} = \underset{x \to \infty}{\text{Lt}} \frac{f'(x)}{F'(x)}.$$

Let $x = 1/y$, then

$$\underset{x \to \infty}{\text{Lt}} \frac{f(x)}{F(x)} = \underset{y \to 0}{\text{Lt}} \frac{f(1/y)}{F(1/y)} = \underset{y \to 0}{\text{Lt}} \frac{(-1/y^2) f'(1/y)}{(-1/y^2) F'(1/y)}$$

$$= \underset{y \to 0}{\text{Lt}} \frac{f'(1/y)}{F'(1/y)} = \underset{y \to \infty}{\text{Lt}} \frac{f'(x)}{F'(x)}.$$

Note 1. It can be shown easily that if $f(x) \to \infty$ as $x \to a$, then $f'(x)$ cannot remain finite as $x \to a$. This would appear to make the application of the above rule to the from ∞/∞ useless in the case $x \to a$. Nevertheless the rule is useful for the ratio of the derivatives may lend itself to simplifications which make it easier to calculate the limit while the same may not be true of the ratios of the functions.

Note 2. A factor which tends to a *non-zero finite limit* may be isolated while evaluating the limit of an indeterminate form.

Thus

$$\operatorname*{Lt}_{x\to 0}\frac{1-\cos x}{x^2\cos^3 x}=\operatorname*{Lt}_{x\to 0}\frac{1-\cos x}{x^2}\cdot\operatorname*{Lt}_{x\to 0}\frac{1}{\cos^3 x}$$

$$=\operatorname*{Lt}_{x\to 0}\frac{1-\cos x}{x^2}.$$

as $1/\cos^3 x$ tends to 1 as $x\to 0$. Here we have isolated the factor $1/\cos^3 x$ which tends to a non-zero finite limit. This device helps in simplifying calculations which, at times, tend to be tedious.

Ex. 1. *Evaluate* $\operatorname*{Lt}_{x\to\infty}\dfrac{\log x}{x}$.

This is of the form ∞/∞, therefore

$$\operatorname*{Lt}_{x\to\infty}\frac{\log x}{x}=\operatorname*{Lt}_{x\to\infty}\frac{1/x}{1}=\operatorname*{Lt}_{x\to\infty}\frac{1}{x}=0.$$

Ex. 2. *Evaluate the limit of* $\dfrac{\log(x-\frac{1}{2}\pi)}{\tan x}$ *as* $x\to\frac{1}{2}\pi$.

This is also of the form ∞/∞. Hence

$$\operatorname*{Lt}_{x\to\frac{1}{2}\pi}\frac{\log(x-\frac{1}{2}\pi)}{\tan x}=\operatorname*{Lt}_{x\to\frac{1}{2}\pi}\frac{1/(x-\frac{1}{2}\pi)}{\sec^2 x}=\operatorname*{Lt}_{x\to\frac{1}{2}\pi}\frac{\cos^2 x}{x-\frac{1}{2}\pi}.$$

The last fraction is now of the form 0/0, the transformation at the second step being necessary. If we had applied the rule for ∞/∞ a second time we would have been led to a more difficult indeterminate form. Applying now the rule for the form 0/0 to the last fraction, the required limit

$$=\operatorname*{Lt}_{x\to\frac{1}{2}\pi}\frac{-\sin 2x}{1}=0.$$

10·4. The calculation of the limiting values of other indeterminate forms can be reduced easily to those of the forms 0/0 or ∞/∞ by suitable transformations.

I. The form $0\times\infty$.

If Lt $f(x)=0$ while Lt $F(x)=\infty$, then the product $f(x)\times F(x)$ assumes the form $0\times\infty$. It may be transformed into the form 0/0 or ∞/∞ by one of the relations

$$f(x)\,F(x)=\frac{f(x)}{1/F(x)}=\frac{F(x)}{1/f(x)}.$$

Ex. 1. *Show that* $\operatorname*{Lt}_{x\to 0+} x\log x=0$.

This is of the form $0\times\infty$. We convert it into the form ∞/∞ by taking the factor x into the denominator, so that

$$\operatorname*{Lt}_{x\to 0+} x\log x=\operatorname*{Lt}_{x\to 0+}\frac{\log x}{1/x}=\operatorname*{Lt}_{x\to 0+}\frac{1/x}{-1/x^2}$$

$$=\operatorname*{Lt}_{x\to 0+}(-x)=0.$$

II The form $\infty-\infty$.

If $\text{Lt} f(x)=\infty$ and also $\text{Lt}\, F(x)=\infty$, then $f(x)-F(x)$ assumes the form $\infty-\infty$. It can be reduced to the form $0/0$ by the relation.

$$f(x)-F(x)=\left\{\frac{1}{F(x)}-\frac{1}{f(x)}\right\}\Big/\frac{1}{f(x)F(x)}.$$

Ex. 2. *Evaluate* Lt (sec x—tan x) *when* $x\to\frac{1}{2}\pi$,

(Panjab, Sept. 1954)

This is of the form $\infty-\infty$. We have, as $x\to\frac{1}{2}\pi$,

$$\text{Lt}\,(\sec x-\tan x)=\text{Lt}\frac{1-\sin x}{\cos x}, \text{ which is of the form } \frac{0}{0},$$

$$=\text{Lt}\ \frac{-\cos x}{-\sin x}=\frac{0}{1}=0.$$

III. The exponential forms 0^0, ∞^0 snd 1^∞

By taking logarithms all the three forms can be reduced to the form $0\times\infty$, which can further be reduced to $0/0$ or ∞/∞, whichever may be convenient. We illustrate by solved examples.

Ex. 3. *Prove that* $\underset{x\to+\infty}{\text{Lt}}\ (1+x)^{1/x}=1.$

This is of the form ∞^0. Let u be the required limiting value, then

$$\log u=\log \underset{x\to+\infty}{\text{Lt}}\ (1+x)^{1/x}=\underset{x\to+\infty}{\text{Lt}}\ \log\,(1+x)^{1/x}$$

$$=\underset{x\to+\infty}{\text{Lt}}\ \frac{1}{x}\log\,(1+x)=\underset{x\to+\infty}{\text{Lt}}\ \frac{\log\,(1+x)}{x}$$

(which is now of the form ∞/∞)

$$=\underset{x\to+\infty}{\text{Lt}}\ \frac{1/(1+x)}{1}=\underset{x\to+\infty}{\text{Lt}}\ \frac{1}{1+x}=0.$$

$$\therefore \qquad u=e^0=1.$$

Ex. 4. *Evaluate* $\underset{x\to0}{\text{Lt}}\ \left(\frac{\sin x}{x}\right)^{1/x^2}.$

This is of the form 1^∞. If u be the limiting value, then

$$\log u=\log\ \underset{x\to0}{\text{Lt}}\left(\frac{\sin x}{x}\right)^{1/x^2}=\underset{x\to0}{\text{Lt}}\ \log\left(\frac{\sin x}{x}\right)^{1/x^2}$$

$$=\underset{x\to0}{\text{Lt}}\ \frac{1}{x^2}\log\left(\frac{\sin x}{x}\right)=\underset{x\to0}{\text{Lt}}\ \frac{\log\left(\frac{\sin x}{x}\right)}{x^2}$$

(which is now of the form $0/0$)

$$= \underset{x\to 0}{\text{Lt}} \frac{\dfrac{\cos x}{\sin x} - \dfrac{1}{x}}{2x} = \underset{x\to 0}{\text{Lt}} \frac{x\cos x - \sin x}{2x^2 \sin x}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{-x\sin x}{4x\sin x + 2x^2\cos x} = \underset{x\to 0}{\text{Lt}} \frac{-\sin x}{4\sin x + 2x\cos x}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{-\cos x}{4\cos x + 2\cos x - 2x\sin x} = -\tfrac{1}{6}.$$

$$\therefore \quad u = e^{-\frac{1}{6}}.$$

It should be observed that in the above two examples we have assumed that $\log \{\text{Lt} f(x)\} = \text{Lt} \{\log f(x)\}$.

EXAMPLES XXXII

Evaluate the following limits :

Form ∞/∞.

1. $\underset{x\to\infty}{\text{Lt}} \dfrac{x^2+3x+5}{x^2-7x+6}$.

2. $\underset{x\to 0+}{\text{Lt}} \dfrac{\log \sin ax}{\log \sin bx}$, $a, b > 0$.

3. $\underset{x\to 0+}{\text{Lt}} \log_x \sin x$.

4. $\underset{x\to\infty}{\text{Lt}} \dfrac{\log x}{x^m}$, $m>0$.

5. $\underset{x\to\infty}{\text{Lt}} \dfrac{x^m}{e^x}$.

6. $\underset{x\to 0}{\text{Lt}} \dfrac{\log x^2}{\cot x^2}$. (*Panjab, 1947*)

Form $0\times\infty$.

7. $\underset{x\to\infty}{\text{Lt}}\ x\tan(1/x)$. (*Panjab, 1953*)

8. $\underset{x\to 0+}{\text{Lt}}\ x^m(\log x)^n$, m, n are positive integers. (*Panjab, 1944*)

9. $\underset{x\to 1}{\text{Lt}}\ (1-x)\tan \tfrac{1}{2}\pi x$. (*Agra, 1946*)

Form $\infty-\infty$.

10. $\underset{x\to 0}{\text{Lt}}\ (\operatorname{cosec} x - \cot x)$.

11. $\underset{x\to 0}{\text{Lt}} \left(\dfrac{1}{x} - \cot x\right)$. (*Panjab, 1957*)

12. $\underset{x\to 0}{\text{Lt}} \left\{\dfrac{1}{x(1+x)} - \dfrac{\log(1+x)}{x^2}\right\}$.

13. $\underset{x\to 1}{\text{Lt}} \left(\dfrac{x}{x-1} - \dfrac{1}{\log x}\right)$.

14. $\underset{x\to 0}{\text{Lt}} \dfrac{\cot x - (1/x)}{x}$. (*Punjab, 1956*)

15. $\underset{x\to 0}{\text{Lt}} \left(\dfrac{1}{x^2} - \dfrac{1}{\sin^2 x}\right)$. (*Agra, 1943*)

Form 0^0

16. $\underset{x\to 0}{\text{Lt}}\; x^x$.

17. $\underset{\theta\to\frac{1}{2}\pi-0}{\text{Lt}}\; (\cos\theta)^{\cos\theta}$. (*Pb., 1943*)

18. $\underset{x\to 0+}{\text{Lt}}\; (\sin x)^{\tan x}$. (*Panjab, 194*

Form ∞^0.

19. $\underset{x\to\frac{1}{2}\pi-0}{\text{Lt}}\; (\tan x)^{\sin 2x}$.

20. $\underset{x\to 0+}{\text{Lt}}\left(\dfrac{1}{x^m}\right)^{x^n}$, $m, n>0$.

21. $\underset{x\to 0+}{\text{Lt}}\; (-\log x)^x$.

22. $\underset{x\to 0+}{\text{Lt}}\; (\operatorname{cosec} x)^{1/\log x}$ (*Panjab, 1952*)

Form 1^∞

23. $\underset{x\to 0}{\text{Lt}}\; (1-x)^{1/x}$. (*Allahabad, 1942*)

24. $\underset{x\to 0}{\text{Lt}}\; (\cos x)^{1/x}$. (*Agra, 1949*)

25. $\underset{x\to\frac{1}{2}\pi}{\text{Lt}}\; (\sin x)^{\tan x}$. (*Patna, 1950*)

26. $\underset{x\to 0}{\text{Lt}}\; (\cos x)^{\cot^2 x}$. (*Agra, 1945*)

27. $\underset{x\to 0+}{\text{Lt}}\; (a^x+x)^{1/x}$. (*Patna, 1937*)

28. (*i*) $\underset{x\to 0+}{\text{Lt}}\left(\dfrac{\tan x}{x}\right)^{1/x}$ (*Allahabad, 1944*)

(*ii*) $\underset{x\to 0}{\text{Lt}}\left(\dfrac{\tan x}{x}\right)^{1/x^2}$. (*Sagar, 1949*)

(*iii*) $\underset{x\to 0+}{\text{Lt}}\left(\dfrac{\tan x}{x}\right)^{1/x^3}$. (*Agra, 1951*)

10·5. In applying the above rules it should be carefully seen that all the conditions laid down are satisfied. A careless application of these rules may sometimes lead to a false result. We give two simple examples.

(*i*) The fraction $\dfrac{x^2\cos(1/x)}{\sin x}$ is of the form $\dfrac{0}{0}$ when $x\to 0$. It can be written as

$$x.\frac{x}{\sin x}.\cos\left(\frac{1}{x}\right).$$

The first factor $\to 0$, the second $\to 1$ and the third remains bounded when $x\to 0$. Hence the given fraction has the limit 0 when $x\to 0$. But the ratio of the derivatives is

$$\frac{2x\cos(1/x)+\sin(1/x)}{\cos x},$$

which does not tend to any limit as $x\to 0$, for the term $\sin(1/x)$ in

the numerator does not tend to any limit. Hence the second rule of Art. 10·2 is not applicable.

However, the first rule of Art. 10·2 is applicable. For if $f(x)=x^2 \cos(1/x)$ when $x \neq 0$ and $f(0)=0$, then $f'(0)=0$; and if $F(x)=\sin x$, then $F'(0)=1$. Hence

$$\underset{x\to 0}{\mathrm{Lt}} \frac{f(x)}{F(x)}=\frac{f'(0)}{F'(0)}=\frac{0}{1}=0.$$

(*ii*) The fraction $(x-\sin x)/x$ is of the form ∞/∞ when $x\to\infty$, and

$$\mathrm{Lt} \frac{x-\sin x}{x}=\underset{x\to\infty}{\mathrm{Lt}}\left(1-\frac{\sin x}{x}\right)=1;$$

but the ratio of the derivatives is $1-\cos x$ which does not tend to any limit when $x\to\infty$.

10·6. Application of Taylor's Theorem. In many cases where the indeterminacy of a given expression disappears only after many applications of L'Hospital's rule, a careful application of Taylor's theorem may lead to the result much quicker. Thus if a fraction $\varphi(x)$ becomes indeterminate for $x=0$ and $\varphi(x)$ is composed of functions which can be expanded in ascending powers of x, then substituting the expansions of some or all of these functions in the given indeterminate form, and suppressing such powers of x which cancel, we make the indeterminacy disappear.

Ex. *Evaluate* $\underset{x\to 0}{\mathrm{Lt}} \dfrac{\sin x-x-\frac{1}{6}x^3}{x^5}$ (*Madras, 1946*)

Replacing $\sin x$ by its expansion

$$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots\ldots$$

in the numerator and cancelling out the common factor x^5, we get

$$\mathrm{Lt} \frac{\sin x-x+\frac{1}{6}x^3}{x^5}=\underset{x\to 0}{\mathrm{Lt}}\left(\frac{1}{5!}-\frac{x^2}{7!}+\ldots\ldots\right)=\frac{1}{120}.$$

EXAMPLES XXXIII

Prove that :

1. (*i*) $\underset{x\to 0}{\mathrm{Lt}} \dfrac{\cosh x-\cos x}{x^2}=1.$ (*ii*) $\underset{x\to 0}{\mathrm{Lt}} \dfrac{\sinh x-\sin x}{x^3}=\dfrac{1}{3}.$

(*iii*) $\underset{x\to 0}{\mathrm{Lt}} \dfrac{\tan x-\sin x}{\sin^3 x}=\dfrac{1}{2}.$ (*iv*) $\underset{x\to 0}{\mathrm{Lt}} \dfrac{3\tan x-3x-x^3}{x^5}=\dfrac{2}{5}.$

2. $\underset{x\to 0}{\mathrm{Lt}} \dfrac{a^x-1-x\log a}{x^2}=\dfrac{(\log a)^2}{2}.$

3. Evaluate $\mathrm{Lt} \dfrac{\sin^2 mx-\sin^2 nx}{1-\cos(m-n)x}$ when (*i*) $x\to 0$, (*ii*) $m\to n$.

4. Evaluate $\underset{h \to 0}{\text{Lt}} \dfrac{2f(x+h)-2f(x)-2hf'(x)-h^2 f''(x)}{h\{f(x+h)-f(x)-hf'(x)\}}$

5. Prove that (*i*) $\underset{h \to 0}{\text{Lt}} \dfrac{f(a+h)-f(a-h)}{2h}=f'(a)$,

and (*ii*) $\underset{h \to 0}{\text{Lt}} \dfrac{f(a+h)-2f(a)+f(a-h)}{h^2}=f''(a)$,

provided the right-hand sides exist. (*M.T.I.*, *1925*)

6. Find the limit as $x \to 1$ of $\dfrac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}$.

7. If Lt $(\sin 2x+a \sin x)/x^3$ be finite as $x \to 0$, then find the value of a and the limit. (*Panjab, 1940*)

8. If Lt $(\sinh 3x+a_1 \sinh 2x+x_2 \sinh x)/x^5$ as $x \to 0$ have a finite value, then find this value and the necessary values of a_1 and a_2.

9. Prove that $\underset{x \to n}{\text{Lt}} (x-n) \operatorname{cosec} \pi x=\dfrac{(-1)^n}{\pi}$

and $\underset{x \to n+\frac{1}{2}}{\text{Lt}} (x-n-\tfrac{1}{2}) \sec \pi x=\dfrac{(-1)^{n+1}}{\pi}$.

10. Find the limits as $x \to 0$ of

$$\frac{1}{x^3}\left(\operatorname{cosec} x-\frac{1}{x}-\frac{x}{6}\right), \quad \frac{1}{x^3}\left(\cot x-\frac{1}{x}+\frac{x}{3}\right).$$

11. Evaluate $\underset{x \to 0}{\text{Lt}} \left[\dfrac{1}{e^x-1}-\dfrac{1}{x}\right]$. X (*Panjab, 1947*)

12. Prove that (*i*) $\underset{x \to 0}{\text{Lt}} \dfrac{(1+x)^{1/x}-e}{x}=-\dfrac{e}{2}$. (*Pb., Sept. 1950*)

(*ii*) $\underset{x \to 0}{\text{Lt}} \dfrac{(1+x)^{1/x}-e+\frac{1}{2}ex}{x^2}=\dfrac{11e}{24}$. (*Raj., 1950*)

[**Hint.** Expand $(1+x)^{1/x}$ by using the expansions of $\log(1+x)$ and e^x.]

MISCELLANEOUS EXAMPLES III

1. (*a*) State Rolle's theorem. Give its geometric interpretation.

(*b*) State a sufficient set of conditions for

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a+\theta h),\ 0<\theta<1.$$

(*Kashmir, 1960*)

2. (*a*) State and prove the Mean value theorem of Lagrange.
(*b*) Deduce that if $f'(x)=0$ for all x, then $f(x)$ is constant.
(*Kashmir, 1960*)

3. Show that if $x>0$, and $0<\theta<1$, then

$$\log_{10}(x+1)=\frac{x\log_{10}e}{1+\theta x}.$$
(*Panjab Hons., 1959*)

4. If $f(h)=f(0)+hf'(0)+\frac{1}{2}h^2f''(\theta h)$, $0<\theta<1$, find θ when $h=1$ and $f(x)=(1-x)^{5/2}$.
(*Calcutta, 1945*)

5. Given $f(x)=x^{3/2}$, show that for this function the expansion of $f(x+h)$ fails when $x=0$, but that there exists a proper fraction θ such that

$$f(x+h)=f(x)+hf'(x)+\tfrac{1}{2}h^2f''(x+\theta h)$$

holds when $x=0$. Find θ.

6. Given $f(x+h)=f(x)+hf'(x)+\frac{1}{2}h^2f''(x+\theta h)$ and $f(x)=x^3$, find θ.

7. If $f''(x)$ exists in the open interval (a, b), show that

$$f(x+h)-f(x-h)-2f(x)=h^2f''(x+\theta h),$$

where θ is some number between -1 and $+1$.

8. If $\phi''(x)>0$ for every value of x, then

$$\phi[\tfrac{1}{2}(x_1+x_2)]\leqslant\tfrac{1}{2}[\phi(x_1)+\phi(x_2)]$$

for every pair of values of x_1 and x_2.
(*Bombay, 1937*)

9. Assuming $f''(x)$ continuous in (a, b), show that

$$f(c)-f(a)\frac{b-c}{b-a}-f(b)\frac{c-a}{b-a}=\tfrac{1}{2}(c-a)(c-b)f''(\xi)$$

where c and ξ both lie in (a, b).
(*Panjab, 1937*)

[**Hint.** Consider the function

$$\phi(x)=f(x)-\sum_{a,\,b,\,c}\frac{(x-b)(x-c)}{(a-b)(a-c)}f(a).$$

Now $\phi(a)=\phi(b)=\phi(c)=0$, etc.]

10. State carefully Taylor's theorem, and prove that Lagrange's remainder after n terms in the expansion of a^x is

$$\frac{(x\log a)^n a^{\theta x}}{n!},\ 0<\theta<1.$$
(*Panjab, 1951*)

11. If $y=e^{x\cos\alpha}\sin(x\sin\alpha)$, prove that

$$y_{n+2}-2y_{n+1}\cos\alpha+y_n=0.$$

Assuming that y can be expanded in a convergent series, prove that

$$y=\sum_{1}^{\infty} \frac{x^n}{n\,!} \sin n\,\alpha.$$

12. If $y=\dfrac{\log \{x+\sqrt{(1+x^2)}\}}{\sqrt{(1+x^2)}}$, prove that

$$(1+x^2)\frac{dy}{dx}+xy=1.$$

Assuming that y can be expanded in the form

$$a_0+a_1x+a_2x^2+\ldots\ldots+a_nx^n+\ldots\ldots,$$

show that for $n \geqslant 2$,

$$a_n=0 \text{ if } n \text{ is even},$$

and $$a_n=(-1)^{\frac{n-1}{2}} \frac{2.4.6\ldots(n-1)}{3.5.7\ldots n} \text{ if } n \text{ is odd.}$$ (*Panjab, 1937*)

13. State and prove Maclaurin's theorem with a remainder and show that the Maclaurin expansion of e^{-1/x^2} is not valid in any interval, however small. (*Panjab, 1955*)

14. Expand $\sin^{-1} \{2x/(1+x^2)\}$ in powers of x.

15. Prove that

$$f(mx)=f(x)+(m-1)xf'(x)+\frac{(m-1)^2\,x^2}{2\,!} f''(x)+\ldots\ldots$$

16. Apply Maclaurin's theorem to obtain the terms upto x^4 in the expansion of $\log(1+\sin^2 x)$ (*Agra, 1948*)

17. Prove that

$$\tan^{-1}(x+h)=\tan^{-1} x+h \sin z.\frac{\sin z}{1}-(h \sin z)^2 . \frac{\sin 2z}{2}$$

$$+(h \sin z)^3 . \frac{\sin 3z}{3}+\ldots\ldots$$

where $r^2=1+x^2$ and $\cot z=x$. (*Panjab Hons., 1959*)

18. Given that $f(a+h)=f(a)+hf'(a+\theta h)$, where $0<\theta \leqslant 1$, prove that the limiting value of θ when h is diminished indefinitely is $\frac{1}{2}$, provided $f''(a)\neq 0$. (*Panjab Hons., 1943*)

[Here $f(a+h)=f(a)+hf'(a+\theta h)$

$$=f(a)+h\{f'(a)+\theta hf''(a)+\frac{\theta^2 h^2}{2\,!} f''(a+\theta_1\,\theta h)\},$$

where $0<\theta_1<1$. Also by **Taylor's** theorem

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2\,!}f''(a)+\frac{h^3}{3\,!}f'''(a+\theta_2 h),$$

where $0<\theta_2<1$. Equating the two values of $f(a+h)$ and cancelling out the common terms, we get

$$\theta f''(a)+\frac{\theta^2 h}{2\,!}f'''(a+\theta\theta_1 h)=\frac{1}{2\,!}f''(a)+\frac{h}{3\,!}f'''(a+\theta_2 h).$$

Proceeding to the limits when $h\to 0$, we get $\theta\to\frac{1}{3}$.]

19. In the equation

$$f(a+h)=f(a)+hf'(a)+\ldots\ldots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{h^n}{n\,!}f^n(a+\theta h),$$

show that the limiting value of θ as $h\to 0$ is $1/(n+1)$, provided $f^{n+1}(x)$ is continuous in $(a, a+h)$ and $f^{n+1}(a)\neq 0$.

20. Show that

$$e^{\theta}=1+\sin\theta+\frac{1}{2\,!}\sin^2\theta+\frac{2}{3\,!}\sin^3\theta+\ldots\ldots$$

(Panjab Hons., 1956)

[**Hint.** Expand $e^{\sin^{-1}x}$ in powers of x and put $x=\sin\theta$.]

21. Find a and b in order that

$$\operatorname*{Lt}_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{x^3}$$

may be equal to 1. *(Panjab, 1959 ; Sagar, '50)*

22. Find whether the following limits exist, and where possible obtain their values :

$$(a)\ \lim_{n\to\infty}\frac{(n^2+1)\sin(\pi/n)}{n};\quad (b)\ \lim_{n\to\infty}\frac{\sin\frac12 n^2\pi}{1+\cos\frac12 n^2\pi};$$

$$(c)\ \lim_{x\to\pi/2}\frac{1-\sin x}{\log_e\sin x}.$$

(M.T.I., 1946)

23. Prove that

$$(i)\ \operatorname*{Lt}_{x\to 0}\frac{e^x-e^{\sin x}}{x-\sin x}=1.$$

(Allahabad, 1943)

$$(ii)\ \operatorname*{Lt}_{x\to 0}\frac{(a+x)^x-a^x}{x^2}=\frac{1}{a}.$$

$$(iii)\ \operatorname*{Lt}_{x\to\infty}x(a^{1/x}-1)=\log a.$$

(Panjab, 1945)

$$(iv)\ \operatorname*{Lt}_{x\to 0}\left[\frac{1}{x}-\frac{\log(1+x)}{x^2}\right]=\frac{1}{2}.$$

(Agra, 1950 ; Panjab, '51)

(v) $\underset{x\to 0}{\text{Lt}}\left(\dfrac{1}{x^2}-\cot^2 x\right)=\dfrac{2}{3}$. (*Panjab, 1942*)

(vi) $\underset{x\to 0}{\text{Lt}}\left(\dfrac{1}{x}\right)^{\tan x}=1$. (*Panjab, 1946*)

(vii) $\underset{x\to 0}{\text{Lt}}\left(\dfrac{\sinh x}{x}\right)^{1/x^2}=e^{1/6}$. (*Delhi, 1958, '49*)

(viii) $\underset{x\to 0}{\text{Lt}}\ \{\tfrac{1}{2}(a^x+b^x)\}^{1/x}=\tfrac{1}{2}\log(ab)$. (*Panjab, 1953*)

24. Evaluate :

(i) $\underset{x\to 0}{\text{Lt}}\ \dfrac{e^x-e^{x\cos x}}{x-\sin x}$ (*Panjab, 1956*)

(ii) $\underset{x\to \frac{1}{2}\pi}{\text{Lt}}\ (\sec x-\tan x)$. (*Panjab, 1954 S*)

(iii) $\underset{x\to 0}{\text{Lt}}\ \dfrac{\sinh^2 x}{x^2\cos x}$ (*Pb., 1941*) (iv) $\underset{x\to 1}{\text{Lt}}\ x^{1/(1-x)}$. (*Pb., 1960*)

(iv) $\underset{x\to 0}{\text{Lt}}\ \dfrac{\log(1+x)-\log(1-x)+4\sin x-6x}{x^3\tan^2 x}$.

(*Calcutta, 1952*)

CHAPTER XI

PARTIAL DIFFERENTIATION

11·1. Functions of several variables. If the values of a quantity u depend upon the values taken up by several other quantities $x, y, z, \ldots\ldots$etc., then u is said to be a function of the quantities $x, y, z, \ldots\ldots$, and we write $u=f(x, y, \ldots\ldots)$ or, to economise symbols, as $u=u(x, y, z, \ldots\ldots)$. If the quantities $x, y, z, \ldots\ldots$, etc., are not dependent on each other in any way, then they are called the independent variables and u is called the dependent variable. Unlike the case of functions of a single variable, the terms independent and dependent variables are no longer relative. In any question concerning functions of several variables, it is of primary importance to know what are the independent and what are the dependent variables.

The general theory of the functions of several variables is much more difficult than that of single variables and we shall be concerned in this book with only the definitions and the elementary results concerning them. We shall give all our definitions and results in terms of functions of two variables only. These are sufficiently illustrative of the general theory and extend themselves at once to the case of several variables.

Let $z=f(x, y)$ be a single-valued function of the two variables x, y which is defined for all pairs of values of x, y which lie within an area D of the x-y plane. To each point (x, y) of this area there corresponds a unique value of z given by the relation $z=f(x, y)$. If we represent all these values (x, y, z) by points in space, then the relation $z=f(x, y)$ is represented graphically by a surface. For functions of more than two variables such a graphical representation breaks down, but by analogy we may say that functional relations between more than three variables can be represented graphically by figures drawn in *hyper-space*.

11·2. Limits of functions of two variables. Let $f(x, y)$ be defined at all points in the neighbourhood of a point (a, b) with the possible exception of the point (a, b) itself. Then *the function $f(x, y)$ is said to tend to the limit l as $x \to a, y \to b$, if corresponding to every positive number ϵ, however small, we can find a positive number δ such that*

$$|f(x, y)-l| < \epsilon$$

whenever $\quad 0<|x-a|<\delta$ *and* $0<|y-b|<\delta$,

In symbols, we write

$$\operatorname*{Lt}_{\substack{x\to a\\ y\to b}} f(x, y)=l. \qquad \ldots(1)$$

As before we are not concerned with the value of the function $f(x, y)$ at the point $x=a, y=b$. For all we care, the function need not even be defined at this point.

Formally, the definition of a limit given above is the same as for functions of a single variable, but in practice it imposes much more stringent conditions on the function $f(x, y)$. The definition above means that whatever point (x, y), not (a, b) itself, is taken within the square whose centre is (a, b), a side is 2δ in length and the sides are parallel to the axes, then

$$|f(x,y)-l| < \varepsilon.$$

In other words, in whatever manner x and y tend to the values a and b, the function $f(x, y)$ must tend to the same limit l. Two particular ways of the approach of the point (x, y) to (a, b) are worthy of mention.

(*i*) We may first make $x \to a$ and obtain the limit of $f(x, y)$ and in the resulting limit make $y \to b$. We then get what we call a *repeated limit* or a *double limit*. If the limit thus obtained be l, then we write

$$\underset{y\to b}{\text{Lt}} \quad \underset{x\to a}{\text{Lt}} \; f(x,y)=l. \qquad \ldots(2)$$

(*ii*) We may reverse the order and first make $y \to b$ and in the resulting limit make $x \to a$. If the limit thus obtained be l', then we write

$$\underset{x\to a}{\text{Lt}} \quad \underset{y\to b}{\text{Lt}} \; f(x, y)=l'. \qquad \ldots(3)$$

In general $l=l'$, but it is not always so. For example,

$$\underset{x\to 0}{\text{Lt}} \; \underset{y\to 0}{\text{Lt}} \; \frac{x-y}{x+y} = \underset{x\to 0}{\text{Lt}} \; \frac{x}{x} = 1,$$

whereas

$$\underset{y\to 0}{\text{Lt}} \; \underset{x\to 0}{\text{Lt}} \; \frac{x-y}{x+y} = \underset{y\to 0}{\text{Lt}} \; \frac{-y}{y} = -1.$$

In contrast with the two repeated limits (2) and (3), the limit (1) is called the *simultaneous* limit.

11·8. Continuity. *A function $f(x, y)$ is said to be continuous in the two variables x, y, at the point (a, b) if* Lt $f(x, y)=f(a, b)$ *as* $x \to a$ *and* $y \to b$ *in any manner whatsoever.*

It follows, therefore, that the function $f(x, y)$ must be defined at every point in the neighbourhood of (a, b) including (a, b) itself and the simultaneous limit of $f(x, y)$ as $x \to a, y \to b$ must be equal to the value $f(a, b)$ of the function at the point (a, b).

Like the definition of a limit, this definition also imposes more stringent conditions than the corresponding definition for functions of a single variable.

It can be easily shown that if a function $f(x, y)$ is continuous in the two variables together at a point (a, b) then it is continuous as a function of x alone at $x=a$ and also continuous as a function of y alone at $y=b$. The converse of this is not true.

11.4. **Partial Derivatives.** Let $z=f(x, y)$ be a continuous function of the two independent variables x and y. If we keep y constant and vary x alone, then z becomes a continuous function of x alone. If this function possesses a derivative, this derivative is called the **partial derivative of z with respect to x.** This partial derivative is denoted by various symbols such as

$$\frac{\partial z}{\partial x},\ \frac{\partial f}{\partial x},\ f'_x(x, y),\ f_x(x, y),\ f_x,\ D_x f(x, y), \text{ etc.}$$

In symbols, give an increment δx to x alone, then the corresponding increment in the value of z is

$$f(x+\delta x, y)-f(x, y),$$

and the partial derivative of z with respect to x is

$$\frac{\partial z}{\partial x}=\underset{\delta x\to 0}{\text{Lt}}\ \frac{f(x+\delta x, y)-f(x, y)}{\delta x}.$$

Similarly, by keeping x constant and allowing y alone to vary, we can define the **partial derivative of z with respect to y.** This is denoted by the symbols

$$\frac{\partial z}{\partial y},\ \frac{\partial f}{\partial y}, f'_y(x, y),\ f_y(x, y),\ f_y,\ D_y f(x, y), \text{ etc.}$$

By definition,

$$\frac{\partial z}{\partial y}=\underset{\delta y\to 0}{\text{Lt}}\ \frac{f(x, y+\delta y)-f(x, y)}{\delta y}.$$

Ex. *Find the first partial derivatives of the following :*

(*i*) $z=x^2y^2-a^2(x^2-y^2)$. (*ii*) $z=\tan^{-1}\dfrac{x^2+y^2}{x-y}$. (*iii*) $z=x^y$.

(*i*) $\dfrac{\partial z}{\partial x}=2xy^2-2a^2x,\ \dfrac{\partial z}{\partial y}=2yx^2+2a^2y.$

(*ii*) $$\frac{\partial z}{\partial x}=\frac{1}{1+\left(\dfrac{x^2+y^2}{x-y}\right)^2}\cdot\frac{(x-y)2x-(x^2+y^2).1}{(x-y)^2}$$

$$=\frac{x^2-2xy-y^2}{(x^2+y^2)^2+(x-y)^2},$$

and $$\frac{\partial z}{\partial x}=\frac{x^2+2xy-y^2}{(x^2+y^2)^2+(x-y)^2} \text{ similarly;}$$

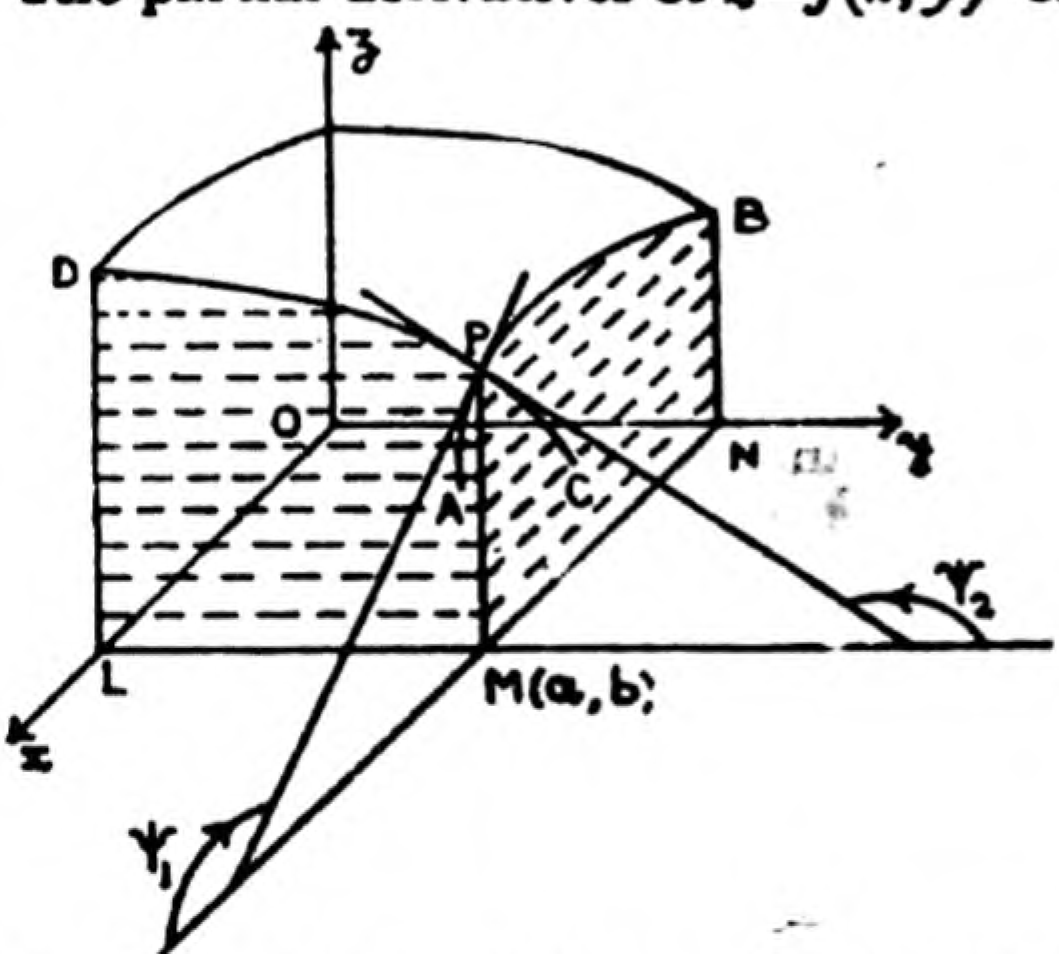
(*iii*) $\dfrac{\partial z}{\partial x}=yx^{y-1}$ and $\dfrac{\partial z}{\partial y}=x^y\log x.$

11·41. Partial Differentials. The partial differentials of z with respect to x and y respectively are denoted by the symbols $d_x z$ and $d_y z$ and are defined by the equations

$$d_x z = \frac{\partial z}{\partial x}\,\delta x \text{ and } d_y z = \frac{\partial z}{\partial x}\,\delta y.$$

It should be observed that the partial derivatives $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ are symbolic expressions and unlike $\frac{dy}{dx}$, they cannot be regarded as the ratios of differentials.

11·42. Geometrical significance of partial derivatives. The partial derivatives of $z=f(x, y)$ can also be given a geometrical interpretation. If we put $y=b$, a constant, then we get the section of the surface $z=f(x, y)$ with the plane $y=b$. This section APB is a plane curve given by the equations



$$y=b, \qquad z=f(x, b).$$

The slope of the tangent to this curve is given by the derivative of z with respect to x, keeping y constant at the value b. Hence *the partial derivative of z with respect to x gives the slope of the tangent drawn to the curve of intersection of the surface $z=f(x, y)$ with a plane parallel to the zOx plane.*

In the figure, the plane PMNB is $y=b$. It cuts the surface in the curve APB whose slope is $\tan\psi_1$, the value of $\partial z/\partial x$ at $P(a, b)$. Similarly, the plane $x=a$ cuts the surface in the curve CPD whose slope at P is $\tan\psi_2$, the value of $\partial z/\partial y$ at $P(a, b)$.

11·43. Partial Derivatives of higher orders. The first partial derivatives of $z=f(x, y)$ are themselves functions of x and y and can therefore be differentiated partially with respect to x or y. The four derivatives thus obtained, called the **second order partial derivatives** of z or $f(x, y)$, are

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right), \ \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right), \ \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right), \ \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right),$$

and are denoted as

$$\frac{\partial^2 z}{\partial x^2}, \ \frac{\partial^2 z}{\partial y \partial x}, \ \frac{\partial^2 z}{\partial x \partial y}, \ \frac{\partial^2 z}{\partial y^2}.$$

or as $f_{xx}(x, y),\ f_{yx}(x, y),\ f_{xy}(x, y),\ f_{yy}(x, y),$

or simply as $f_{xx},\quad f_{yx},\quad f_{xy},\quad f_{yy}.$

The distinction between $\partial^2 z/\partial y\partial x$ and $\partial^2 z/\partial x\partial y$ should be noted carefully. The first is obtained by first differentiating z partially with respect to x and then differentiating the result so obtained partially with respect to y, whereas the second is obtained by performing the two operations of differentiation in the reverse order. It was pointed out in § 11·2 that a reversal of the order of two limiting operations need not give identical results. The two derivatives, $\partial^2 z/\partial y\partial x$ and $\partial^2 z/\partial x\partial y$, may therefore be different. But in almost all the cases that we meet with in practice, we have

$$\frac{\partial^2 z}{\partial y\partial x}=\frac{\partial^2 z}{\partial x\partial y},$$

i.e., in general, the two differentiations *w.r.* to x and y may be performed in any order. There are, therefore, in general, only three distinct partial derivatives of the second order. We shall not give here the conditions under which the two mixed partial derivatives of the second order are equal.

We can similarly define partial derivatives of the third and higher orders. If we assume that we can perform the differentiations with respect to x and y in any order we please, we obtain the following four distinct derivatives of the third order :

$$\frac{\partial^3 z}{\partial x^3},\ \frac{\partial^3 z}{\partial x^2\partial y},\ \frac{\partial^3 z}{\partial x\partial y^2},\ \frac{\partial^3 z}{\partial y^3}.$$

Ex. 1. *Find all the second order partial derivatives of z, where* $z=\log\,(e^x+e^y)$.

Here $$\frac{\partial z}{\partial x}=\frac{e^x}{e^x+e^y},\qquad \frac{\partial z}{\partial y}=\frac{e^y}{e^x+e^y},$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{(e^x+e^y)\,e^x-e^x.e^x}{(e^x+e^y)^2}=\frac{e^x.e^y}{(e^x+e^y)^2},$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}=-\frac{e^x.e^y}{(e^x+e^y)^2},$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{(e^x+e^y)e^y-e^y.e^y}{(e^x+e^y)^2}=\frac{e^x.e^y}{(e^x+e^y)^2}.$$

Ex. 2. *Prove that* $\dfrac{\partial^2 u}{\partial x\partial y}=\dfrac{\partial^2 u}{\partial y\partial x}$ *for the following functions :*

(*i*) $u=ax^3+3bx^2y+3cxy^2+dy^3$, (*ii*) $u=e^{ax}\cos by$.

(*i*) $\dfrac{\partial u}{\partial x}=3ax^2+6bxy+3cy^2,\ \dfrac{\partial^2 u}{\partial y\partial x}=6bx+6cy,$

$\dfrac{\partial u}{\partial y}=3bx^2+6cxy+3dy^2,\ \dfrac{\partial^2 u}{\partial x\partial y}=6bx+6cy.$

Hence $\dfrac{\partial^2 u}{\partial y \partial x}=\dfrac{\partial^2 u}{\partial x \partial y}$.

(ii) $\dfrac{\partial u}{\partial x}=ae^{ax}\cos by,\ \dfrac{\partial^2 u}{\partial y \partial x}=-abe^{ax}\sin by,$

$\dfrac{\partial u}{\partial y}=-be^{ax}\sin by,\ \dfrac{\partial^2 u}{\partial x \partial y}=-abe^{ax}\sin by.$

Hence $\dfrac{\partial^2 u}{\partial y \partial x}=\dfrac{\partial^2 u}{\partial x \partial y}$.

EXAMPLES XXXIV

1. Find $\partial z/\partial x$ and $\partial z/\partial y$ in the following cases :

   (i) $z=x^3+y^3-3axy$. (ii) $z=\sin(e^{ax}+e^{by})$.

   (iii) $z=\sin^{-1}(x/y)$. (iv) $z=1/\sqrt{(x^2+y^2)}$.

2. If $z=\log(\tan x+\tan y)$, show that

$$\sin 2x\frac{\partial z}{\partial x}+\sin 2y\frac{\partial z}{\partial y}=2.$$

3. Verify that $\dfrac{\partial^2 u}{\partial x \partial y}=\dfrac{\partial^2 u}{\partial y \partial x}$ for the following cases :

   (i) $u=ax^2+2hxy+by^2$. (ii) $u=\log\{(x^2+y^2)/(xy)\}$. *(Andhra, 1937)*

   (ii) $u=\tan^{-1}(x/y)$. (iv) $u=e^{ax}\sin by$.

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4. If $u=x^3+y^3+3xyz$, show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=3u.$$ *(Delhi, 1952)*

5. If $u=e^{xyz}$, show that

$$\frac{\partial^3 u}{\partial x \partial y \partial z}=(1+3xyz+x^2y^2z^2)e^{xyz}.$$

(Lucknow, 1949 ; Panjab, '60)

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6. If $z=x^2\tan^{-1}(y/x)-y^2\tan^{-1}(x/y)$, prove that

$$\frac{\partial^2 z}{\partial x \partial y}=\frac{x^2-y^2}{x^2+y^2}.$$ *(Delhi, 1953 ; Aligarh, '50)*

7. If $u=x^3-3xy^2,\ v=3x^2y-y^3$, prove that

   (i) $\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}$.

   (ii) $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}=0.$

8. If $z=\frac{1}{2}\log(x^2+y^2)$ or $\tan^{-1}(y/x)$, prove that

   (i) $\left(\dfrac{\partial z}{\partial x}\right)^2+\left(\dfrac{\partial z}{\partial y}\right)^2=\dfrac{1}{x^2+y^2}$. (ii) $\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2}=0,$

9. If $u=\sin^{-1}(x/y)+\tan^{-1}(y/x)$, show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0. \qquad \textit{(Allahabad, 1947)}$$

10. If $u=\sin^{-1}\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$, show that

$$\frac{\partial u}{\partial x}=-\frac{y}{x}\frac{\partial u}{\partial y}. \qquad \textit{(Kashmir, 1954; Sagar, '48)}$$

11. If $v=\dfrac{A}{\sqrt{t}}\cdot e^{-x^2/(4a^2t)}$; prove that $\dfrac{\partial v}{\partial t}=a^2\dfrac{\partial^2 v}{\partial x^2}$.

12. The equation $\dfrac{\partial u}{\partial t}=\mu\dfrac{\partial^2 u}{\partial x^2}$ refers to the condition of heat along a bar without radiation; show that if

$$u=Ae^{-gx}\sin(nt-gx),$$

where A, g, n are positive constants, then $g=\sqrt{(n/2\mu)}$.

13. If $\theta=t^n e^{-r^2/4t}$, find what value of n will make

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\frac{\partial\theta}{\partial t}\ ? \qquad \textit{(Allahabad, 1934)}$$

14. If $V=(x^2+y^2+z^2)^{-\frac{1}{2}}$, show that

(i) $x\dfrac{\partial V}{\partial x}+y\dfrac{\partial V}{\partial y}+z\dfrac{\partial V}{\partial z}=-V,$

and (ii) $\dfrac{\partial^2 V}{\partial x^2}+\dfrac{\partial^2 V}{\partial y^2}+\dfrac{\partial^2 V}{\partial z^2}=0.$ *(Allahabad, 1943 Panjab, '51)*

15. If $u=e^x(x\cos y-y\sin y)$, prove that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0. \qquad \textit{(Pa jab, 1958)}$$

11·5. We now proceed to establish a formula which is of fundamental importance. Let $u=f(x, y)$ be a function of the two variables x and y and let the two partial derivatives f_x and f_y exist and be continuous for all the ranges of values of x and y under consideration. Let x and y vary simultaneously and receive the increments δx and δy respectively and let δu be the corresponding increment of u, then

$$\begin{aligned}\delta u&=f(x+\delta x, y+\delta y)-f(x, y)\\&=f(x+\delta x, y+\delta y)-f(x, y+\delta y)+f(x, y+\delta y)-f(x, y).\end{aligned}$$

Now δu has been written as the sum of two differences, in the first of which y remains constant at the value $y+\delta y$ and x changes from x to $x+\delta x$, and in the second x remains constant and y changes from y to $y+\delta y$. We can apply the Mean Value Theorem to each of these differences as the partial derivatives are supposed to be continuous. Hence we get

$$f(x+\delta x, y+\delta y)-f(x, y+\delta y)=\delta x\, f_x(x+\theta_1\delta x, y+\delta y)$$

and $$f(x, y+\delta y)-f(x, y)=\delta y\, f_y(x, y+\theta_2\delta y),$$
where θ_1 and θ_2 are positive proper fractions. Carrying these values into the value of δu, we get,
$$\delta u=\delta x\, f_x(x+\theta_1\delta x, y+\delta y)+\delta y\, f_y(x, y+\theta_2\delta y). \quad ...(1)$$

This formula may be called the **Mean Value Theorem** for functions of two variables.

Since $f_x(x, y)$ is supposed to be continuous for all values of x and y under consideration, we have
$$\text{Lt}\, f_x(x+\theta_1\delta x, y+\delta y)=f_x(x, y)$$
when δx and δy both tend to zero. Hence we may write
$$f_x(x+\theta_1\delta x, y+\delta y)=f_x(x, y)+\varepsilon_1,$$
where $\varepsilon_1\to 0$ when δx, δy both$\to 0$. We have similarly
$$f_y(x, y+\theta_2\delta y)=f_y(x, y)+\varepsilon_2,$$
where ε_2 also tends to zero with δx and δy. Substituting these values in (1), we get
$$\delta u=\delta x f_x(x, y)+\delta y f_y(x, y)+\varepsilon_1\delta x+\varepsilon_2\delta y, \quad ...(2)$$
or $$\delta u=\frac{\partial u}{\partial x}\delta x+\frac{\partial u}{\partial y}\delta y+\varepsilon_1\delta x+\varepsilon_2\delta y, \quad ...(3)$$
where ε_1 and $\varepsilon_2\to 0$ as δx and $\delta y\to 0$. This is the formula referred to at the beginning of the article. It expresses the increment δu resulting from simultaneous increments δx and δy in x and y respectively, as a sum of two terms one of which is linear in δx and δy and the other which tends to zero more rapidly than δx and δy when these tend to zero.

11·51. Total differential. Let $u=f(x, y)$ be a function of the two variables x and y. Let δx, δy be the increments of x, y and δu be the corresponding increment in u. Then by equation (2) of the last article
$$\delta u=\frac{\partial u}{\partial x}\delta x+\frac{\partial u}{\partial y}\delta y+\varepsilon_1\delta x+\varepsilon_2\delta y.$$

When δx and $\delta y\to 0$, δu also tends to zero and is, therefore, an infinitesimal. Since ε_1, $\varepsilon_2\to 0$ as δx, $\delta y\to 0$, and $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial u}{\partial y}$ are finite quantities, therefore the principal part of δu is the sum of the first two terms on the right-hand side. This is called **the total differential** of u with respect to x and y and is denoted by du. Hence
$$du=\frac{\partial u}{\partial x}\delta x+\frac{\partial u}{\partial y}\delta y. \quad ...(1)$$

If, in particular, we take $u=x$, then $\dfrac{\partial u}{\partial x}=1$, $\dfrac{\partial u}{\partial y}=0$ and so $du=\delta x$. Also $du=dx$, since $u=x$. Hence $dx=\delta x$. Similarly $dy=\delta y$. Hence (1) becomes
$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy. \quad ...(2)$$

Form (1) of the total differential presupposes that x and y are the independent variables, but the form (2) is not subject to such a restriction for it can be shown that the total differential of u is given by (2) even when x and y are not the independent variables.

Equation (2) may be written symbolically as

$$du=\left(\frac{\partial}{\partial x}\,dx+\frac{\partial}{\partial y}\,dy\right)u.$$

It may be observed that the total differential of u, *i e*, du is the sum of the two partial differentials

$$\frac{\partial u}{\partial x}\,dx \text{ and } \frac{\partial u}{\partial y}\,dy.$$

11 52. Successive total differentials. Let $u=f(x, y)$ be a function of the two independent variables x and y, then the first total differential of u is given by

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}\,dy. \qquad \ldots(1)$$

The total differential of du, *i.e.*, $d(du)$, is called the second total differential of u and is denoted by d^2u. It is calculated from du by substituting du for u in formula (1) and regarding dx and dy as constants. Thus

$$\begin{aligned} d^2u=d(du)&=d\left(\frac{\partial u}{\partial x}\,dx+\frac{\partial u}{\partial y}\,dy\right)\\ &=d\left(\frac{\partial u}{\partial x}\right)dx+d\left(\frac{\partial u}{\partial y}\right)dy\\ &=\left(\frac{\partial^2 u}{\partial x^2}dx+\frac{\partial^2 u}{\partial y\partial x}dy\right)dx+\left(\frac{\partial^2 u}{\partial x\partial y}\,dx+\frac{\partial^2 u}{\partial y^2}\right)dy\\ &=\frac{\partial^2 u}{\partial x^2}(dx)^2+2\,\frac{\partial^2 u}{\partial x\partial y}\,dxdy+\frac{\partial^2 u}{\partial y^2}(dy)^2, \qquad \ldots(2)\end{aligned}$$

on the assumption that $\dfrac{\partial^2 u}{\partial x\partial y}=\dfrac{\partial^2 u}{\partial y\partial x}$.

Formula (2) may be written symbolically as

$$d^2u=\left(\frac{\partial}{\partial x}\,dx+\frac{\partial}{\partial y}dy\right)^2u.$$

Proceeding similarly, we can calculate higher order total differentials of u, and we have in general

$$d^nu=\left(\frac{\partial}{\partial x}\,dx+\frac{\partial}{\partial y}\,dy\right)^nu.$$

It should be observed that the above calculation of d^2u has been made on the assumption that x and y are the independent variables and therefore their differentials dx and dy may be treated as constants. In case x and y are nct the independent variables, but are dependent themselves on some other variables, then their

differentials dx and dy cannot be regarded as constants and their second differentials d^2x and d^2y will enter into the expression for d^2u. In fact, we have

$$d^2u=d(du)=d\left(\frac{\partial u}{\partial x}dx\right)+d\left(\frac{\partial u}{\partial y}dy\right)$$

$$=d\left(\frac{\partial u}{\partial x}\right)dx+d\left(\frac{\partial u}{\partial y}\right)dy+\frac{\partial u}{\partial x}d^2x+\frac{\partial u}{\partial y}d^2y$$

$$=\frac{\partial^2 u}{\partial x^2}(dx)^2+2\frac{\partial^2 u}{\partial x\partial y}dxdy+\frac{\partial^2 u}{\partial y^2}(dy)^2+\frac{\partial u}{\partial x}d^2x+\frac{\partial u}{\partial y}d^2y.$$

The expressions for d^3u, d^4u, etc., will be much more complicated in this case.

Ex. *If $u=x^2y+xy^2$, find d^2u (i) when x and y are the independent variables, (ii) when x and y are functions of t.*

We have, in both the cases (*i*) and (*ii*),

$$du=(2xy+y^2)dx+(x^2+2xy)dy. \qquad \ldots(1)$$

(*i*) If x and y are the independent variables, then dx and dy are to be treated as constants for the second differentiation. Hence from (1),

$$\begin{aligned}d^2u&=d(2xy+y^2)dx+d(x^2+2xy)dy\\&=[2ydx+(2x+2y)dy]dx+[(2x+2y)dx+2xdy]dy\\&=2[y(dx)^2+2(x+y)dxdy+x(dy)^2].\end{aligned}$$

(*ii*) If x and y are functions of t, then dx and dy are no longer to be treated as constants. Then from (1)

$$\begin{aligned}d^2u&=d(2xy+y^2)\,dx+(2xy+y^2)\,d(dx)\\&\quad+d(x^2+2xy).dy+(x^2+2xy)d(dy)\\&=[2ydx+(2x+2y)dy]dx+(2xy+y^2)d^2x\\&\quad+[(2x+2y)dx+2xdy]dy+(x^2+2xy)d^2y.\\&=2[y(dx)^2+2(x+y)dxdy+x(dy)^2]\\&\quad+(2xy+y^2)d^2x+(x^2+2xy)d^2y.\end{aligned}$$

11·53. Differentiation of composite functions or funtions of functions.

I. Let $u=f(x, y)$ be a function of the two veriables x and y possessing continuous first partial derivatives and let x, y be themselves functions of a variable t possessing finite derivatives, then u is a composite function of the variable t possessing a derivative. Let t receive an increment δt and let δx, δy and δu be the corresponding increments of x, y and u respectively, then by formula (2) of Art. 11·5,

$$\delta u=\frac{\partial u}{\partial x}\delta x+\frac{\partial u}{\partial y}\delta y+\varepsilon_1\,\delta x+\varepsilon_2\,\delta y,$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $\delta x, \delta y \to 0$. Dividing this equation by δt, we get

$$\frac{\delta u}{\delta t}=\frac{\partial u}{\partial x}\,\frac{\delta x}{\delta t}+\frac{\partial u}{\partial y}\,\frac{\delta y}{\delta t}+\varepsilon_1\frac{\delta x}{\delta t}+\varepsilon_2\frac{\delta y}{\delta t}.$$

Now when $\delta t\to 0$, δx and δy both tend to zero, for x and y are both differentiable functions of t. Hence ε_1 and $\varepsilon_2 \to 0$ when $\delta t \to 0$. Also $\delta x/\delta t$ and $\delta y/\delta t \to dx/dt$ and dy/dt respectively. Hence proceeding to the limits when $\delta t \to 0$, we obtain

$$\frac{du}{dt}=\underset{\delta t\to 0}{\mathrm{Lt}}\ \frac{\delta u}{\delta t}=\frac{\partial u}{\partial x}\,\frac{dx}{dt}+\frac{\partial u}{\partial y}\,\frac{dy}{dt}. \tag{1}$$

It may be observed that the rule for the differentiation of a function of a function is applied twice—firstly; considering u as a function of x alone and x as a function of t; secondly, considering u as a function of y alone and y as a function of t, and then adding the two results so obtained.

If $u=f(x, y)$, $x=\varphi(t)$ and $y=\psi(t)$, the derivative of u with respect to t can also be obtained by first expressing u directly in terms of t in the form

$$u=f\{\varphi(t), \psi(t)\},$$

but this method is not always suitable, particularly in theoretical work.

A **special case** of importance occurs when we take $t=x$ itself. In this case

$$u=f(x, y),\ x=x,\ y=\psi(x),$$

i.e., u is a function of x, y, and y is a function of x so that u is a function of x. Then u has two possible derivatives with respect to x, one a partial derivative calculated from the relation $u=f(x, y)$ on the assumption that y remains a constant, and the second an ordinary derivative when we take $u=f(x, y)$ where y is supposed to vary with x by means of the relation $y=\psi(x)$. This second derivative which may be called the **total derivative** of u with respect to x, is obtained by using formula (1) above and is given by the equation

$$\frac{du}{dx}=\frac{\partial u}{\partial x}\,\frac{dx}{dx}+\frac{\partial u}{\partial y}\,\frac{dy}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\,\frac{dy}{dx}, \tag{2}$$

where $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}$ are obtained from $u=f(x, y)$ and $\dfrac{dy}{dx}$ from $y=\psi(x)$. It is this case which necessitated the introduction of separate notations for ordinary and partial derivatives.

Ex. 1. *Find the total derivative of u with respect to t when $u=\cosh(y/x)$ where $x=t^2$, $y=e^t$.*

We have $\dfrac{\partial u}{\partial x}=\sinh\left(\dfrac{y}{x}\right)\left(-\dfrac{y}{x^2}\right)$, $\dfrac{\partial u}{\partial y}=\sinh\left(\dfrac{y}{x}\right)\left(\dfrac{1}{x}\right)$,

$$\frac{dx}{dt}=2t, \qquad \frac{dy}{dt}=e^t,$$

Hence

$$\frac{du}{dt}=\frac{\partial u}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$=-\frac{y}{x^2}\sinh\left(\frac{y}{x}\right).2t+\frac{1}{x}\sinh\left(\frac{y}{x}\right).e^t$$

$$=\frac{1}{x^2}\sinh\left(\frac{y}{x}\right)(xe^t-2yt).$$

Ex. 2. *If* $z=\sqrt{(x^2+y^2)}$ *and* $x^3+y^3+3axy=5a^2$, *find the value of* $\dfrac{dz}{dx}$ *when* $x=a, y=a.$ *(Panjab, 1953)*

Here $\dfrac{\partial z}{dx}=\dfrac{x}{\sqrt{(x^2+y^2)}}$, $\dfrac{\partial z}{\partial y}=\dfrac{y}{\sqrt{(x^2+y^2)}}$, and differentiating the given relation between x and y, we get

$$3x^2+3y^2\frac{dy}{dx}+3a\left(y+x\frac{dy}{dx}\right)=0 \text{ whence } \frac{dy}{dx}=-\frac{x^2+ay}{y^2+ax}.$$

Hence by formula (2) of Art. 11·53,

$$\frac{dz}{dx}=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\frac{dy}{dx}=\frac{x}{\sqrt{(x^2+y^2)}}-\frac{y}{\sqrt{(x^2+y^2)}}\cdot\frac{x^2+ay}{y^2+ax}.$$

Putting $x=a$ $y=a$ the required value of

$$\frac{dz}{dx}=\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}=0.$$

II. Let $u=f(x, y)$, where x, y are themselves functions of the two variables r and s given by the equations

$$x=\varphi(r, s), \qquad y=\psi(r, s),$$

then u is a composite function of the two variables r and s.

To calculate the partial derivative of u with respect to r, give an increment δr to r keeping s fixed and let δx, δy, δu be the corresponding increment of x, y, u respectively. By (2) of Art 11·5,

$$\delta u=\frac{\partial u}{\partial x}\delta x+\frac{\partial u}{\partial y}\delta y+\epsilon_1\delta x+\epsilon_2\delta y.$$

Divide throughout by δr and proceed to the limits making $\delta r\to 0$, then by a reasoning similar to that of case I above, we get

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\,\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\,\frac{\partial y}{\partial r}. \tag{3}$$

Proceeding similarly, we obtain

$$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\,\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\,\frac{\partial y}{\partial s}. \tag{4}$$

The formulae (1) and (3), (4) can be easily extended to the case when u is a function of several variables $x, y, z, \ldots\ldots$, which are themselves functions of one or more independent variables.

Ex. 1. *If* $u=f(x^2+y^2)$, *prove that* $\dfrac{\partial u}{\partial x} : \dfrac{\partial u}{\partial y}=x : y.$

Let $z=x^2+y^2$, then $u=f(z)$, where $z=x^2+y^2$. Hence

$$\frac{\partial u}{\partial x}=\frac{du}{dz}\cdot\frac{\partial z}{\partial x}=f'(z).2x=2xf'(x^2+y^2),$$

and
$$\frac{\partial u}{\partial y}=\frac{du}{dz}\cdot\frac{\partial z}{\partial y}=f'(z).2y=2yf'(x^2+y^2).$$

Hence
$$\frac{\partial u}{\partial x} : \frac{\partial u}{\partial y}=x : y.$$

Ex. 2. *If* $z=e^{ax+by} f(ax-by)$, *prove that*

$$b\frac{\partial z}{\partial x}+a\frac{\partial z}{\partial y}=2abz.$$

Let $u=ax+by$, $v=ax-by$, then $z=e^u f(v)$.

$$\therefore \quad \frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}=ae^u f(v)+ae^u f'(v),$$

and
$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}=be^u f(v)-be^u f'(v),$$

and the given relation is easily verified.

11·54. Differentiation of Implicit Functions. If $f(x, y)$ be a function of the two variables x, y and y itself be a function of x, then by formula (2) of the previous article, the total derivative of $f(x, y)$ with respect to x is given by the equation

$$\frac{df}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}.$$

Now consider the implicit relation between x and y given by the equation

$$f(x, y)=0,$$

and suppose that this relation defines y as a differentiable function of x, then differentiating this equation totally with respect to x by the above formula, we get

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}=0. \qquad (1)$$

whence
$$\frac{dy}{dx}=-\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y}=-\frac{f_x}{f_y}, \qquad (2)$$

provided $f_y \neq 0$.

Differentiating equation (1) again totally with respect to x, we get

$$\frac{d}{dx}\left(\frac{\partial f}{\partial x}\right)+\frac{d}{dx}\left(\frac{\partial f}{\partial y}\frac{dy}{dx}\right)=0,$$

or $$\frac{d}{dx}\left(\frac{\partial f}{\partial x}\right)+\frac{d}{dx}\left(\frac{\partial f}{\partial y}\right)\frac{dy}{dx}+\frac{\partial f}{\partial y}\frac{d^2y}{dx^2}=0,$$

or $$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial x\partial y}\frac{dy}{dx}+\left(\frac{\partial^2 f}{\partial x\partial y}+\frac{\partial^2 f}{\partial y^2}\frac{dy}{dx}\right)\frac{dy}{dx}+\frac{\partial f}{\partial y}\frac{d^2y}{dx^2}=0,$$

i.e., $$\frac{\partial^2 f}{\partial x^2}+2\frac{\partial^2 f}{\partial x\partial y}\frac{dy}{dx}+\frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dx}\right)^2+\frac{\partial f}{\partial y}\frac{d^2y}{dx^2}=0, \tag{3}$$

whence, after substituting the value of dy/dx from (2),

$$\frac{d^2y}{dx^2}=-\frac{\left(\frac{\partial f}{\partial y}\right)^2\frac{\partial^2 f}{\partial x^2}-2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial x\partial y}+\left(\frac{\partial f}{\partial x}\right)^2\frac{\partial^2 f}{\partial y^2}}{\left(\frac{\partial f}{\partial y}\right)^3} \tag{4}$$

$$=-\frac{f_{xx}f_y^2-2f_{xy}f_xf_y+f_{yy}f_x^2}{f_y^3}. \tag{4}$$

By differentiating (3) further, we can obtain the higher derivatives of y with respect to x in terms of the partial derivatives of $f(x, y)$.

Ex. *If* $x^3+y^3-3axy=0$, *find* $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Here $f(x, y)=x^3+y^3-3axy$, and so

$$f_x=3x^2-3ay, \qquad f_y=3y^2-3ax,$$

$$f_{xx}=6x, \qquad f_{xy}=-3a, \qquad f_{yy}=6y.$$

$$\therefore \quad \frac{dy}{dx}=-\frac{f_x}{f_y}=-\frac{3x^2-3ay}{3y^2-3ax}=\frac{ay-x^2}{y^2-ax},$$

and $$\frac{d^2y}{dx^2}=-\frac{f_{xx}f_y^2-2f_xf_yf_{xy}+f_{yy}f_x^2}{f_y^3}$$

$$=-\frac{2a(x^2-ay)(y^2-ax)+2y(x^2-ay)^2+2x(y^2-ax)^2}{(y^2-ax)^3}$$

$$=-\frac{2xy(x^3+y^3-3axy+a^3)}{(y^2-ax)^3}$$

$$=\frac{2a^3xy}{(ax-y^2)^3}.$$

11 55. Cosinder the two equations

$$f(x, y, u, v)=0, \qquad \phi(x, y, u, v)=0.$$

Here we have four variables connected by two equations. We can consider these equations as solved for any two of the variables in terms of the remaining two. Thus we can think of any two of them as functions of the remaining two. Hence we may regard (*i*) u and v as functions of x and y, or (*ii*) u and y as functions of x and v, or (*iii*) v and y as functions of x and u, and *vice versa*.

In case (i), when we differentiate u partially *w.r.* to x, we treat y as a constant, whereas in case (ii), when we differentiate u partially *w.r.* to x, we treat v as a constant. The two values of $\frac{\partial u}{\partial x}$ thus obtained are not identical. To avoid confusion regarding the choice of the independent variables, we denote $\frac{\partial u}{\partial x}$ in case (i) by $\left(\frac{\partial u}{\partial x}\right)_y$ and in case (ii) by $\left(\frac{\partial u}{\partial x}\right)_v$ where the suffix y in case (i) and v in case (ii) indicates the choice of the second independent variable. When, however, there is no ambiguity regarding the choice of the second independent variable, the suffixes outside the brackets are dropped.

Ex. *If* $x=r\cos\theta$, $y=r\sin\theta$, *find the value of*

(i) $\left(\frac{\partial x}{\partial r}\right)_\theta$, $\left(\frac{\partial y}{\partial r}\right)_\theta$, $\left(\frac{\partial x}{\partial \theta}\right)_r$, $\left(\frac{\partial y}{\partial \theta}\right)_r$.

(ii) $\left(\frac{\partial r}{\partial x}\right)_y$, $\left(\frac{\partial r}{\partial x}\right)_x$, $\left(\frac{\partial \theta}{\partial x}\right)_y$, $\left(\frac{\partial \theta}{\partial y}\right)_x$.

(iii) $\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}$.

For (i), we have to express x and y as functions of r and θ. This is already the case with the given equations. Hence

$$\left(\frac{\partial x}{\partial r}\right)_\theta=\cos\theta, \qquad \left(\frac{\partial y}{\partial r}\right)_\theta=\sin\theta,$$

$$\left(\frac{\partial x}{\partial \theta}\right)_r=-r\sin\theta, \quad \left(\frac{\partial y}{\partial \theta}\right)_r=r\cos\theta.$$

For (ii), we have to regard r and θ as functions of the two independent variables x and y and have, therefore, to solve the given equations for r and θ in terms of x and y. Squaring and adding the given equations, we get

$$x^2+y^2=r^2 \qquad \text{whence} \qquad r=\sqrt{(x^2+y^2)}.$$

Also, dividing one equation by the other, we get

$$y/x=\tan\theta \qquad \text{whence} \qquad \theta=\tan^{-1}(y/x).$$

Hence $\left(\frac{\partial r}{\partial x}\right)_y=\frac{x}{\sqrt{(x^2+y^2)}}$, $\left(\frac{\partial r}{\partial y}\right)_x=\frac{y}{\sqrt{(x^2+y^2)}}$.

$$\left(\frac{\partial \theta}{\partial x}\right)_y=-\frac{y}{x^2+y^2}, \quad \left(\frac{\partial \theta}{\partial y}\right)_x=\frac{x}{x^2+y^2}.$$

For (iii), we have once again to express r and θ as function of x and y. Hence using the results of (ii), we have

$$\frac{\partial^2\theta}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial\theta}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right)=\frac{2xy}{(x^2+y^2)^2}.$$

$$\frac{\partial^2\theta}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial\theta}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)=\frac{-2xy}{(x^2+y^2)^2}.$$

Hence $$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=0.$$

Examples XXXV

1. If $u=\tan^{-1}\left(\frac{y}{x}\right)$, prove that $du=\frac{xdy-ydx}{x^2+y^2}$.

2. If $pv=k\theta$, k constant, show that

$$dp=-\frac{p}{v}\,dv+\frac{p}{\theta}\,d\theta.$$

3. If $x=r\cos\theta$, $y=r\sin\theta$, where r and θ are independent, show that

(i) $dx=\cos\theta\,dr-r\sin\theta\,d\theta$. (ii) $dy=\sin\theta\,dr+r\cos\theta\,d\theta$.

(iii) $x\,dy-y\,dx=r^2d\theta$. (iv) $x\,dx+y\,dy=rdr$.

(v) $d^2x=-2\sin\theta\,dr\,d\theta-\cos\theta\,d\theta^2$.

(vi) $d^2y=2\cos\theta\,dr\,d\theta-r\sin\theta\,d\theta^2$.

4. If $x=r\cos\theta$, $y=r\sin\theta$, show that

(i) $\left(\frac{\partial x}{\partial r}\right)_\theta=\left(\frac{\partial r}{\partial x}\right)_y$, (ii) $\left(\frac{\partial x}{r\partial\theta}\right)_r=\left(\frac{r\partial\theta}{\partial x}\right)_y$,

(*Panjab, 1931*)

(iii) $\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}=\frac{1}{r}\left\{\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right\}$. (*Panjab, 1953*)

(iv) $\frac{\partial^2 r}{\partial x^2}\cdot\frac{\partial^2 r}{\partial y^2}=\left(\frac{\partial^2 r}{\partial x\partial y}\right)^2$. (*M.T.I., 1945*)

and (v) $\frac{\partial^2\theta}{\partial x\partial y}=-\frac{\cos 2\theta}{r^2}$.

5. If $x=r\cos\theta$, $y=r\sin\theta$, and r, θ are themselves functions of a parameter t, then show that

(i) $x'=\cos\theta\,.\,r'-r\sin\theta\,.\,\theta'$. (ii) $y'=\sin\theta.r'+r\cos\theta.\theta'$, X

(iii) $x''=(r''-r\theta'^2)\cos\theta-(2r'\theta'+r\theta'')\sin\theta$,

(iv) $y''=(r''-r\theta'^2)\sin\theta+(2r'\theta'+r\theta'')\cos\theta$,

where dashes denote differentiations with respect to t.

6. Find the total derivative of u w.r. to t when

(i) $u=e^x\sin y$, where $x=\log t$, $y=t^2$.

(ii) $u=\tan^{-1}(y/x)$, where $x=\log t$, $y=e^t$.

7. Find the partial differential coefficients of x^2y with respect to x and y, and its total differential coefficient with respect to x when x and y are connected by the relation

$$x^2+xy+y^2=1.$$ (*Aligarh, 1946*)

8. If $u = e^{y/x}$, where $x = r\cos\theta,\ y = r\sin\theta$, find $\dfrac{\partial u}{\partial r}, \dfrac{\partial u}{\partial \theta}$.

9. If $z = f(x, y)$, where $x = e^{u} + e^{-v},\ y = e^{-u} - e^{v}$, prove that

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$ *(Delhi, 1955)*

10. If $u = x^2 + 4xy$, where $x = re^{s},\ y = re^{-s}$, find

$$\frac{\partial^2 u}{\partial r^2},\quad \frac{\partial^2 u}{\partial r\,\partial s},\quad \frac{\partial^2 u}{\partial s^2}.$$

11. Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ from the following implicit relations :

(*i*) $x^2 + y^2 = a^2$. (*ii*) $x^2/a^2 + y^2/b^2 = 1$.

(*iii*) $xy^2 + x^2 y = a^3$. (*iv*) $x^5 + y^5 - 5a^3xy = 0$

12. If $f(x, y, z) = 0$, show that $\left(\dfrac{\partial y}{\partial z}\right)_x \left(\dfrac{\partial z}{\partial x}\right)_y \left(\dfrac{\partial x}{\partial y}\right) = -1.$

[**Hint.** Considering z as a function of x and y and differentiating partially *w.r.t.* x treating y as a constant, we get

$$f_x + f_z\left(\frac{\partial z}{\partial x}\right)_y = 0 \quad \text{so that} \quad \left(\frac{\partial z}{\partial x}\right)_y = -\frac{f_x}{f_z} \text{ etc.}$$]

13. If x, y, z are connected by two equations of the form $f(x, y, z) = 0$, $\varphi(x, y, z) = 0$, then determine the expressions for $\dfrac{dy}{dx}$ and $\dfrac{dz}{dx}$ regarding y and z as functions of x.

14. Find $\dfrac{dy}{dx}$ and $\dfrac{dz}{dx}$ for the following :

(*i*) $x^3 + y^3 + z^3 = 3xyz$, $x + y + z = a$.

(*ii*) $x^2 + y^2 + z^2 = a^2$, $lx + my + nz = p$.

15. Find dB/dA, where A, B, C, the angles of a triangle, satisfy

$$\Sigma \sin B \sin C = h.$$ *(Andhra, 1936)*

16. If $u = f(y + ax)$, prove that

$$\frac{\partial u}{\partial x} = a\frac{\partial u}{\partial y} \text{ and } \frac{\partial^2 u}{\partial x^2} = a^2\frac{\partial^2 u}{\partial y^2}.$$

17. If $u = f(y + ax) + \varphi(y - ax)$, prove that

$$\frac{\partial^2 u}{\partial x^2} = a^2\frac{\partial^2 u}{\partial y^2}.$$ *(Agra, 1943)*

18. If $u = f(x^2 + y^2 + z^2)$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 4(x^2 + y^2 + z^2)\, f''(x^2 + y^2 + z^2) + 5f'(x^2 + y^2 + z^2).$$

11·6. Homogeneous functions. A function $f(x, y)$ of the two variables x and y is said to be a *homogeneous function of degree or order n if, for all values of t,*

$$f(tx, ty)=t^n f(x, y).$$

Thus if $u=ax^2+2hxy+by^2$, then u is a homogeneous function of degree 2, for

$$a(tx)^2+2h(tx)(ty)+b(ty)^2=t^2(ax^2+2hxy+by^2).$$

Again if $u=x \tan^{-1}(y/x)$, then u is a homogeneous function of degree 1, for

$$(tx) \tan^{-1}(ty/tx)=t.x \tan^{-1}(y/x).$$

The definition can be extended to a function of any number of variables.

Ex. 1. Prove that the following are homogeneous functions of the degrees indicated :

(*i*) $\dfrac{x^3+y^3}{x-y}$, degree 2. (*ii*) $\sqrt{(xy)} \sin^{-1}\dfrac{y}{x}$, degree 1.

(*iii*) $\dfrac{x^2+y^2}{x^2-y^2} \cos \dfrac{y}{x}$, degree 0. (*iv*) $\left(\dfrac{x}{x^3+y^3}\right)^{1/2}$, degree -1.

Ex. 2. If $f(x, y)$ and $\phi(x, y)$ are homogeneous of degree m and n respectively, prove that

(*i*) $f(x, y).\phi(x, y)$ is homogeneous of degree $m+n$.

(*ii*) $f(x, y)/\phi(x, y)$ is homogeneous of degree $m-n$.

11·61. Euler's Theorem on homogeneous functions. *If $f(x, y)$ be a homogeneous function of degree n, and possesses continuous partial derivatives, then*

$$x \frac{\partial f}{\partial x}+y \frac{\partial f}{\partial y}=nf(x, y).$$

Since $f(x, y)$ is a homogeneous function of degree n, therefore, for all values of t,

$$f(tx, ty)=t^n f(x, y).$$

Put $tx=u$ and $ty=v$, then

$$f(u, v)=t^n f(x, y). \tag{1}$$

Differentiating both sides with respect to t and noting that $\dfrac{du}{dt}=x$ and $\dfrac{dv}{dt}=y$, we get

$$\frac{\partial f}{\partial u} \frac{du}{dt}+\frac{\partial f}{\partial t} \frac{dv}{dt}=nt^{n-1} f(x, y).$$

or

$$x \frac{\partial f}{\partial u}+y \frac{\partial f}{\partial y}\equiv nt^{n-1} f(x, y). \tag{2}$$

But this relation is true for all values of t. Hence putting $t=1$, (2) becomes

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf(x,y).$$

since $u=x$ and $v=y$ when $t=1$. This proves the theorem.

The Theorem can be ~~easily extended~~ to homogeneous functions of any number of variables.

The converse of the above theorem is also true. *If $f(x,y)$ is a function such that*

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf(x,y)$$

for all values of x, y then $f(x,y)$ is a homogeneous function of degree n. The proof of the converse is beyond the scope of this book.

If we differentiate equation (2) again with respect to t, then noting that $\frac{\partial f}{\partial u}$, $\frac{\partial f}{\partial v}$ are themselves functions of u, v, we get

$$x\left(\frac{\partial^2 f}{\partial x^2}\frac{du}{dt}+\frac{\partial^2 f}{\partial v\partial u}\frac{dv}{dt}\right)+y\left(\frac{\partial^2 f}{\partial u\partial v}\frac{du}{dt}+\frac{\partial^2 f}{\partial v^2}\frac{du}{dt}\right)=n(n-1)t^{n-2}f(x,y)$$

or, since $\frac{du}{dt}=x$, $\frac{dv}{dt}=y$ and $\frac{\partial^2 f}{\partial v\partial u}=\frac{\partial^2 f}{\partial u\partial v}$,

$$x^2\frac{\partial^2 f}{\partial u^2}+2xy\frac{\partial^2 f}{\partial u\partial v}+y^2\frac{\partial^2 f}{\partial v^2}=n(n-1)t^{n-2}f(x,y). \qquad (4)$$

Putting $t=1$, (4) becomes

$$x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\partial y}+y^2\frac{\partial^2 f}{\partial y^2}=n(n-1)f(x,y). \qquad (5)$$

Differentiating (4) again with respect to t, we obtain other similar equations involving higher order derivatives.

Ex. 1. *If $u=\frac{xy}{x+y}$, verify, that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=u$,*

and
$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0.$$

Since u is a homogeneous function of degree 1, these results are immediate consequence of equations (3) and (5) of the last article. To verify these, we have to calculate all the partial derivatives upto the second order. Here

$$\frac{\partial u}{\partial x}=\frac{y^2}{(x+y)^2}, \qquad \frac{\partial u}{\partial y}=\frac{x^2}{(x+y)^2},$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{-2y^2}{(x+y)^3}, \quad \frac{\partial^2 u}{\partial x\partial y}=\frac{2xy}{(x+y)^3}, \quad \frac{\partial^2 u}{\partial y^2}=\frac{-2x^2}{(x+y)^3},$$

and the given equations are easily verified.

Ex. 2. *If* $u=\sin^{-1}\{(x^2+y^2)/(x+y)\}$ *prove that*

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u. \qquad \textit{(Delhi, 1958 ; Panjab, 1956)}$$

From the given relation, we have $\sin u=\dfrac{x^2+y^2}{x+y}$, Let $v(x,y)=\dfrac{x^2+y^2}{x+y}$

then $v(tx,ty)=\dfrac{t^2x^2+t^2y^2}{tx+ty}=t\dfrac{x^2+y^2}{x+y}=tv(x,y)$ which shows that v is a homogeneous function of degree 1. Hence by Euler's theorem

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=v$$

or $$x\frac{\partial}{\partial x}(\sin u)+y\frac{\partial}{\partial y}(\sin u)=\sin u,$$

i.e., $$x\frac{\partial u}{\partial x}\cos u+y\frac{\partial u}{\partial y}\cos u=\sin u,$$

or $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u.$$

EXAMPLES XXXVI

1. If $u=f(y/x)$, show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=0.$ *(Delhi, 1957)*

2. If $u=x^nf(y/x)$, show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$$

and $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u.$$

3. Show that if $z=x\phi(y/x)+\psi(y/x)$, then

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=0.$$

4. If $z=xy\,f(y/x)$, show that $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=2z.$

(Kashmir, 1957)

Show also that if z is a constant, then

$$y\left(y-x\frac{dy}{dx}\right)f'\left(\frac{y}{x}\right)=x\left(y+x\frac{dy}{dx}\right)f\left(\frac{y}{x}\right) \qquad \textit{(Panjab, 1934)}$$

5. If $u=\tan^{-1}\{(x^3+y^3)/(x-y)\}$, show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u. \qquad \textit{(Panjab, 1959)}$$

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6. If $u=\sin^{-1}\{\sqrt{(x^2+y^2)}/(x+y)\}$, prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0.$$

7. $u=f(H_n)$, where H_n is a homogeneous function of the nth degree; also suppose that we get from this relation $H_n=F(u)$, then show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{F(u)}{F'(u)}.$$

8. If $u=f(x, y, z)$ is a homogeneous function of x, y, z of degree n and possesses continuous partial derivatives, prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu.$$

[$\because f(x, y, z)$ is homogeneous of degree n, we have

$$f(tx, ty, tz)=t^n f(x, y, z).$$

Now proceed as in Art. 11·61.]

11·7. Applications to small errors. Suppose that the value of a quantity u is calculated from the observed values of two quantities x and y by means of the formula $u=f(x, y)$. Suppose further that there are small errors of amounts δx, δy respectively in the observed values of x and y, so that their true values are $x+\delta x$ and $y+\delta y$, then the true value of u is $f(x+\delta x,\ y+\delta y)$. If δu be the consequent small error in the value of u, then

$$\delta u=f(x+\delta x, y+\delta y)-f(x, y)$$
$$=\frac{\partial u}{\partial x}\delta x+\frac{\partial u}{\partial y}\delta y+\epsilon_1\delta x+\epsilon_2\delta y$$

by equation (2) of Art. 11·5, where ϵ_1, ϵ_2 both tend to zero as δx, δy tend to zero. If now δx and δy be very small, then the terms $\epsilon_1\,\delta x$, $\epsilon_2\,\delta y$ are smaller still and may be neglected. Hence *the error in the value of u is given approximately by the formula*

$$\delta u=\frac{\partial f}{\partial x}\delta x+\frac{\partial f}{\partial y}\delta y. \tag{1}$$

It should be observed that (1) is not an exact equation and therefore, gives the value of δu only approximately. Equation (1) can be obtained from the equation of the total differential by merely writing δu instead of du.

The quantity $\delta u/u$ which measures the error per unit in the value of u is called the *relative error* or the *proportional error* and is obtained most easily by taking logarithmic derivatives. The *percentage error* is equal to the relative error multiplied by one hundred.

In case u is a function of a single variable x, then equation (1) becomes

$$\delta u=\frac{\partial f}{dx}\,\delta x.$$

The generalisation of formula (1) when u is a function of more than two variables is almost evident.

Ex. 1. *The area S of a triangle is calculated from the lengths of the sides a, b, c; if a be diminished and b be increased by small amounts x, prove that the consequent change in the area is given by*

$$\frac{\delta S}{S}=\frac{2(a-b)x}{c^2-(a-b)^2}.$$

We know that the area S of a triangle is given in terms of its sides by means of the formula

$$S^2=s(s-a)(s-b)(s-c)$$

or
$$16S^2=(a+b+c)(b+c-a)(c+a-b)(a+b-c), \qquad (1)$$

since
$$2s=a+b+c.$$

Here only the sides a and b change and c remains constant. Let δa, δb represent the changes in a and b and let δS be the consequent chage in S. We are given that $\delta a=-x$ and $\delta b=x$. Hence $\delta a+\delta b=0$ and $\delta a-\delta b=-2x$. Keeping these facts in view and differentiating (1) totally, we get

$$2\frac{\delta S}{S}=\frac{\delta a+\delta b}{a+b+c}+\frac{\delta b-\delta a}{b+c-a}+\frac{\delta a-\delta b}{c+a-b}+\frac{\delta a+\delta b}{a+b-c}$$

$$=\frac{2x}{b+c-a}-\frac{2x}{c+a-b}=\frac{4x(a-b)}{c^2-(a-b)^2}.$$

Hence
$$\frac{\delta S}{S}=\frac{2x(a-b)}{c^2-(a-b)^2}.$$

Ex. 2. *In a triangle ABC, the sides b, c and the angle A are measured. If small errors e and φ are made in measuring the sides and angle respectively, show that the error in the calculated value of a is*

$$(\cos B+\cos C)\, e+b\varphi \sin C.$$

If $b=c=4$ inches, $A=\pi/3$, $e=0{\cdot}1$ inch, $\varphi=0{\cdot}01$ radian, find the error in a.

By the cosine formula, the value of a is given by

$$a^2=b^2+c^2-2bc\cos A. \qquad (1)$$

Differentiating totally, we get

$$2a\delta a=2b\delta b+2c\delta c-2(c\cos A\,\delta b+b\cos A\,\delta c-bc\sin A\,\delta A)$$

or
$$a\delta a=(b-c\cos A)\,\delta b+(c-b\cos A)\,\delta c+bc\sin A\,\delta A \qquad (2)$$

But $b=c\cos A+a\cos C$, $c=b\cos A+a\cos B$ and $c\sin A=a\sin C$, therefore

$$a\delta a=a\cos C.\delta b+a\cos B.\delta c+ab\sin C.\delta A$$

or
$$\delta a=\cos C.\delta b+\cos B.\delta c+b\sin C.\delta A.$$

But we are given that $\delta b=\delta c=e$ and $\delta A=\varphi$, hence the error δa in the value of a is given by

$$\delta a=(\cos B+\cos C)e+b\varphi\sin C.$$

For the numerical case, since $b=c=4$ in. and $A=\pi/3$, hence from (1),

$$a^2=4^2+4^2-2.4\,4\cos 60^\circ=32-32\times\tfrac{1}{2}=16$$

Hence $a=4$ in. Also $\delta b=\delta c=0{\cdot}1$ in., and $\delta A=0{\cdot}01$ radian. Therefore substituting in (2) above,

$$4\delta a=2(4-4\times\tfrac{1}{2})\times 0{\cdot}1+16\times\tfrac{1}{2}\sqrt{3}\times 0{\cdot}01$$

or
$$\delta a=0{\cdot}1+0{\cdot}02\times 1{\cdot}732=0{\cdot}1+0{\cdot}3462$$
$$=0{\cdot}1346 \text{ in. approximately.}$$

EXAMPLES XXXVII

1. The error in the area A of an ellipse due to small errors in the lehgths of the semi axes a, b is given by

$$\frac{\delta A}{A}=\frac{\delta a}{a}+\frac{\delta b}{b}.$$

2. The area of a triangle ABC is calculated by measuring b, c and A ; show that the relative error in area is given by

$$\frac{\delta\Delta}{\Delta}=\frac{\delta b}{b}+\frac{\delta c}{c}+\cot A\ \delta A.$$

3. Two sides of a triangle are measured and found to be 32·5 in., and 24·2 in., the included angle being 57° ; find the area of the triangle. If the true lengths of the sides are really 32·6 in. and 24·1 in., what is the percentage error in the area ?

4. In measuring two sides of a triangle which include an angle of 30°, one side is found to be 27 inches with a possible error of 0·1 inch., and the other 13 inches with a possible error of 0·05 inch. What is an approximate value for the largest possible error in the area of the triangle due to the errors in measuring the sides ?

(*Panjab, 1945*)

5. The area Δ of a triangle is found from measurements of the side a and the angles B, C. Prove that the error $\delta\Delta$ in the calculated value of the area due to small errors δa, δB, δC is given approximately by

$$\frac{\delta\Delta}{\Delta}=2\frac{\delta a}{a}+\frac{c}{a}\ \frac{\delta B}{\sin B}+\frac{b}{a}\ \frac{\delta C}{\sin C}.$$ (*M.T.I., 1945*)

6. The sides of an acute-angled triangle are measured. Prove that increment in A due to small increments in the sides a, b, c is given by the equation

$$bc\sin A\ \delta A=-a(\cos C\ \delta b+\cos B\ \delta c-\delta a).$$

Supposing that the limits of error in the length of any side are $\pm\mu$ per cent, where μ is small, prove that the limits of error in A are approximately

$$\pm 1{\cdot}15\frac{\mu a^2}{bc\sin A}\text{ degrees.}$$ (*M.T.I., 1927*)

7. The side a and the opposite angle A of a triangle ABC remain constant ; show that when the other sides and angles are slightly varied,

$$\frac{\delta b}{\cos B}+\frac{\delta c}{\cos C}=0.$$

8. If the sides and angles of a plane triangle ABC vary in such a way that its circum-radius remains constant, prove that

$$\frac{da}{\cos A}+\frac{db}{\cos B}+\frac{dc}{\cos C}=0,$$

where da, db, dc denote small increments in the sides a, b, c respectively. (*Agra, 1944*)

9. A triangle ABC is determined from observed values of b, c and A, but it is afterwards found that there are small errors δb, δc and δA in these quantities. Prove that the consequent error in R, the radius of the circle ABC, is given approximately by

$$\frac{\delta R}{R}=\frac{\cos C\,\delta b+\cos B\,\delta c}{a}+\frac{\cos B\cos C}{\sin A}\delta A.$$

10. If the density ρ of a body be calculated from its weights W, w in air and in water respectively, show that the relative error in ρ due to errors δW, δw in W and w is given by

$$\frac{\delta\rho}{\rho}=\frac{-w}{W-w}\,\frac{\delta W}{W}+\frac{\delta w}{W-w}.$$

11·8. Extreme values of implicit functions.

Consider y as a function of x given by the equation

$$f(x,y)=0. \tag{i}$$

Differentiating *w.r.* to x, we get

$$f_x+f_y\frac{dy}{dx}=0, \tag{ii}$$

whence
$$\frac{dy}{dx}=-\frac{f_x}{f_y}.$$

At an extremum, $\dfrac{dy}{dx}=0, \quad \therefore f_x=0.$ (*iii*)

Let (x_1, y_1), (x_2, y_2)......be the simultaneous solutions of (*i*) and (*iii*). Then the values of x which make y extremum are included in x_1, x_2, ..., it being assumed that f_y is not zero at any of these points. The corresponding extrema values of y are y_1, y_2.........

Differentiating (*ii*) again *w.r.* to x, we get

$$\frac{d^2y}{dx^2}=-\left[f_{xx}+2f_{xy}\frac{dy}{dx}+f_{yy}\left(\frac{dy}{dx}\right)^2\right]\Big/f_y.$$

At extreme points $\dfrac{dy}{dx}=0, \therefore \dfrac{d^2y}{dx^2}=-\dfrac{f_{xx}}{f_y}.$

y is a maximum or a minimum according as $\dfrac{d^2y}{dx^2}$ is negative or positive.

If $\dfrac{d^2y}{dx^2}$ is zero, we proceed to find $\dfrac{d^3y}{dx^3}$, etc., and proceed as indicated earlier.

Ex. *Find the maximum and minimum values of y when*

$$x^3+y^3-3axy=0.$$

Let $\qquad f(x, y)=x^3+y^3-3axy=0. \qquad (i)$

Here $\qquad f_x=3(x^2-ay).$

At an extremum, $f_x=0$, $\therefore$ $x^2-ay=0. \qquad (ii)$

Solving (i) and (ii), $x=0$, or $\sqrt[3]{2}a$

The corresponding values of y are 0 and $\sqrt[3]{4}a$ respectively.

$\therefore$ possible extreme points are $(0, 0)$ and $(\sqrt[3]{2}a, \sqrt[3]{4}a)$.

Again, $f_y=3(y^2-ax)$, which vanishes at $(0, 0)$. Hence at this point $\dfrac{dy}{dx}$ fails to be determined by the equation

$$f_x+f_y\frac{dy}{dx}=0.$$

Differentiating this equation again *w.r.* to x, we get

$$f_{xx}+2f_{xy}\frac{dy}{dx}+f_{yy}\left(\frac{dy}{dx}\right)^2+f_y\frac{d^2y}{dx^2}=0$$

which, on substitution for the various derivatives, reduces to

$$6x-6a\frac{dy}{dx}+6y\left(\frac{dy}{dx}\right)^2+3(y^2-ax)\frac{d^2y}{dx^2}=0. \qquad (iii)$$

At $(0, 0)$ this gives $\dfrac{dy}{dx}=0.$

Differentiating (iii) again *w.r.* to x, we have

$$6-6a\frac{d^2y}{dx^2}+6\left(\frac{dy}{dx}\right)^3+12y\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}+3(y^2-ax)\frac{d^3y}{dx^3}$$
$$+3\left(2y\frac{dy}{dx}-a\right)\cdot\frac{d^2y}{dx^2}=0.$$

At $(0, 0)$, this gives

$$6-9a\frac{d^2y}{dx^2}=0 \quad \text{or} \quad \frac{d^2y}{dx^2}=2/3a$$

which is positive. Hence there is a minimum at $(0, 0)$.

Now consider the other point $(\sqrt[3]{2}a, \sqrt[3]{4}a)$.

Here $\dfrac{d^2y}{dx^2}=-\dfrac{f_{xx}}{f_y}=-\dfrac{6x}{3(y^2-ax)}=-\dfrac{2}{a}$, which is negative.

Hence there is a maximum at $(\sqrt[3]{2}a, \sqrt[3]{4}a)$.

EXAMPLES XXXVIII

Find the maximum and minimum values of y given by the following equations :—

1. $y^3+x^3=3(x^2y+1)$.
2. $x^4-y^4-4xy+2=0$.
3. $x^2-4xy+y^2+8x-16=0$.
4. $x^3+y^3-9xy+6x+7y-6=0$.

5. $x^3+y^3=3ax^2$. 6. $x^2-2xy+2y^2-4x-3y=16$.

7. If $ay^2-y^2x^2+x^4=0$, show that the minimum value of y is $4a$.

8. If $x^2\sec^2\theta-4ax\tan\theta-4ah=0$, show that x is maximum when $\tan\theta=\sqrt{\{a/(a+h)\}}$ and the maximum value of

$$x=2\sqrt{\{a(a+h)\}}.$$

11·81. Conditional maxima and minima.

Let $u=f(x, y)$ be a function of two variables x and y which are connected by the relation

$$F(x, y)=0. \qquad (i)$$

From (i), y (or x) may be considered as a function of x (or y) so that u may ultimately be looked upon as a function of one variable, say x (or y).

In certain cases, it may be possible to solve (i) for one of the variables in terms of the other, say y in terms of x. If the solution be $y=\varphi(x)$, we have

$$u=f[x, \varphi(x)].$$

We may now proceed to find the extreme values of u by the methods indicated earlier.

At times, however, this method is cumbersome and impracticable. We then proceed as follows :

$$u=f(x, y) \qquad (ii)$$

where

$$F(x, y)=0. \qquad (iii)$$

$\because$ u is ultimately a function of x, say, for u to be maximum or minimum,

$$\frac{du}{dx}=0.$$

From (ii),

$$\frac{du}{dx}=f_x+f_y\frac{dy}{dx}, \qquad (iv)$$

and from (iii),

$$F_x+F_y\frac{dy}{dx}=0 \text{ whence } \frac{dy}{dx}=-\frac{F_x}{F_y} \qquad ...(v)$$

From (iv) and (v)

$$\frac{du}{dx}=\frac{f_xF_y-f_yF_x}{F_x}.$$

For extreme values,

$$f_xF_y-f_yF_x=0. \qquad ...(vi)$$

Equations (iii) and (vi) are now solved for x and y. The simultaneous solutions of these two equations give the pairs of values of x and y for which u can have an extreme value.

In order to find out whether a particular pair of values determines a maximum or a minimum, we find the higher derivatives of u *w.r.* to x and apply the results obtained earlier.

The method indicated above when u is a function of two variables which are themselves connected by a relation is perfectly general and may easily be extended to a function of n variables which are themselves connected by $(n-1)$ relations.

Ex. *Find the maximum and minimum values of*

$$u=\frac{a^2}{x^2}+\frac{b^2}{y^2} \qquad (i)$$

where $$x+y=c.$$

We consider y as a function of x as given by the second relation. Differentiating *w.r.* to x, we get

$$\frac{du}{dx}=-\frac{2a^2}{x^3}-\frac{2b^2}{y^3}\cdot\frac{dy}{dx} \qquad (iii)$$

and $$1+\frac{dy}{dx}=0, \text{ or } \frac{dy}{dx}=-1. \qquad (iv)$$

$$\therefore \quad \frac{du}{dx}=-\frac{2a^2}{x^3}+\frac{2b^2}{y^3}.$$

At an extremum, $\dfrac{du}{dx}=0.$

$$\because \quad -\frac{2a^2}{x^3}+\frac{2b^2}{y^3}=0, \text{ or } \frac{a^2}{x^3}=\frac{b^2}{y^3}. \qquad (v)$$

Solving (ii) and (v) for x and y, we get

$$\frac{x}{a^{2/3}}=\frac{y}{b^{2/3}}=\frac{x+y}{a^{2/3}+b^{2/3}}=\frac{c}{a^{2/3}+b^{2/3}}.$$

$$\therefore \quad x=\frac{ca^{2/3}}{a^{2/3}+b^{2/3}}, \qquad y=\frac{cb^{2/3}}{a^{2/3}+b^{2/3}}.$$

Again, $$\frac{d^2u}{dx^2}=\frac{6a^2}{x^4}-\frac{6b^2}{y^4}\cdot\frac{dy}{dx}=\frac{6a^2}{x^4}+\frac{6b^2}{y^4}$$

which is evidently positive.

Therefore u is minimum for the above values of x and y, and the minimum value

$$=\frac{a^2(a^{2/3}+b^{2/3})^2}{c^2a^{4/3}}+\frac{b^2(a^{2/3}+b^{2/3})^2}{c^2b^{4/3}}=\frac{(a^{2/3}+b^{2/3})^3}{c^2}.$$

EXAMPLES XXXIX

1. Find the maximum and minimum values of

(i) xy, $\qquad$ (ii) x^2+y^2,

where $(x/a)+(y/b)=1$.

2. Find the maximum and minimum values of

$$u=ax+by \text{ when } xy=c^2.$$

3. If $ax+by=1$, find the extreme values of

(i) x^2+y^2. $\qquad$ (ii) $(x-a)^2+(y-b)^2$.

4. Show that the maximum and minimum values of x^2+y^2 where $ax^2+2hxy+by^2=1$, are given by the roots of the equation

$$\left(a-\frac{1}{r^2}\right)\left(b-\frac{1}{r^2}\right)=a^2. \qquad \textit{(Panjab, Sept. 1950)}$$

5. Find the maximum and minimum values of

(i) $u=\sin^2\theta+\sin^2\varphi$, where $\theta+\varphi=\alpha$.

(ii) $u=3\sin^2 x+4\sin^2 y$, where $y-x=\frac{1}{2}\pi$.

11·9. Greatest and least values. From the definitions of maximum and minimum values it is clear that these values are only the *relatively* greatest and least values of the function in the immediate neighbourhood of the points concerned. They may not be the greatest or the smallest values of the function in a given interval. To find the *absolute* maximum and minimum values of a *continuous* function $f(x)$ in a given interval $[a, b]$, we must find all the points of maximum and minimum values in the iuterval (a, b) and also calculate $f(a)$ and $f(b)$. Then

(i) the greatest of the maxima values and $f(a)$ and $f(b)$ is the absolutely greatest value of the function in $[a, b]$.

(ii) the least of the minima values and $f(a)$ and $f(b)$ is the absolutely least value of the function in the range $[a, b]$, and

Ex. *Find the greatest and the least values of* $2x^3+3x^2-12x+4$ *in the range* $[-3, 3]$.

Here $\quad f(x)=2x^3+3x^2-12x+4$,

$\therefore \quad f'(x)=6(x^2+x-2)=6(x-1)(x+2)$.

$\therefore \quad f'(x)=0$ for $x=-2$ and 1.

Again $\quad f''(x)=12x+6$.

$\therefore\ f''(1)$ is positive and $f''(-2)$ is negative.

Hence $x=1$ gives a minimum and the minimum value $=-3$ and $x=2$ gives a maximum and the maximum value $=24$.

Again, $f(-3)=13, f(3)=49$.

Hence maximum value of the function $=24$.

Minimum value	$=-3$.
Greatest value	$=49$.
Least value	$=-3$.

EXAMPLES XL

Find the greatest and the least values of the function

1. x^3-18x^2+96x in the interval $[0, 9]$. *(Delhi, 1956 ; Panjab, 1948)*

2. x^3-12x^2+45x in the interval $0\leqslant x\leqslant 7$.

3. $x^3-9x^2+24x+1$ in the interval $0\leqslant x\leqslant 5$.

4. $x^2/(x^3+16)$ in the interval $-\infty<x<\infty$.

CHAPTER XII

ENVELOPES

12·1. Consider the equation $f(x, y, \alpha)=0$. The form and position of this curve depends upon α. The different curves obtained by giving different values to α are said to constitute a *family* and α is called the *parameter* of the family. Considering the totality of curves, α is variable but, for any particular curve, α is a constant.

Two curves whose parameters differ by an infinitesimal are said to be *consecutive* or *contiguous* members of the family. The limiting position of a point of intersection of two consecutive members of a family of curves is called an *ultimate point of intersection* of two such curves.

Def. *The locus of the ultimate points of intersection of two consecutive curves of a family is called the envelope of the family of curves.*

Ex. Let the equation of a family of straight lines be

$$x \cos \alpha+y \sin \alpha=p,$$

where α is the parameter. Consider two consecutive members of the family :

$$x \cos \alpha+y \sin \alpha=p, \qquad (i)$$

and
$$x \cos (\alpha+\delta\alpha)+y \sin (\alpha+\delta\alpha)=p. \qquad (ii)$$

Solving these simultaneously and simplifying, we get, for the point of intersection,

$$x=\frac{p \cos (\alpha+\frac{1}{2} \delta\alpha)}{\cos \frac{1}{2} \delta\alpha}, y=\frac{p \sin (\alpha+\frac{1}{2} \delta\alpha)}{\cos \frac{1}{2} \delta\alpha}.$$

When $\delta\alpha \to 0$, we have for the ultimate point of intersection

$$x=p \cos \alpha, \qquad y=p \sin \alpha.$$

Eliminating α between these equations, we get

$$x^2+y^2=p^2$$

as the equation of the envelope.

Note. It may be verified that the equation of the tangent to the envelopes at $(p \cos \alpha, p \sin \alpha)$ is the line $x \cos \alpha+y \sin \alpha=p$. As this tangency relation holds for all values of α, it follows that every member of the given family of straight lines is a tangent to the envelope.

12·2. Equation of the envelope. *To find the equation of the envelope of the family of curves* $f(x, y, \alpha)=0$.

Let
$$f(x, y, \alpha)=0, \qquad (i)$$

and
$$f(x, y, \alpha+\delta\alpha)=0, \qquad (ii)$$

where $\delta\alpha$ is an infinitesimal, be the equations of two consecutive members of the family of curves.

The co-ordinates x, y of a point of intersection of these two curves satisfy both the equations (i) and (ii) and therefore also the equation

$$f(x, y, \alpha+\delta\alpha)-f(x, y, \alpha)=0,$$

or
$$\frac{f(x, y, \alpha+\delta\alpha)-f(x, y, \alpha)}{\delta x}=0.$$

As $\delta\alpha\to 0$, we observe that the co-ordinates of the ultimate point of intersection satisfy the equation

$$\underset{\delta\alpha\to 0}{\text{Lt}} \frac{f(x, y, \alpha+\delta x)-f(x, y, \alpha)}{\delta\alpha}=0, \text{ i.e., } \frac{\partial f}{\partial\alpha}=0. \qquad (iii)$$

The equation of the envelope is thus obtained by eliminating α between the equations $f(x, y, \alpha)=0$ and $\frac{\partial f}{\partial x}=0$.

Cor. The two equations representing the envelope may be solved simultaneously for x and y in terms of α. Let the solution be

$$x=\phi(\alpha), \; y=\psi(\alpha). \qquad \text{(A)}$$

The equations (A) may be taken as the parametric equations of the envelope.

Ex. 1. *Find the envelope of the family of lines*

$$x \cos \alpha+y \sin \alpha=p, \qquad (1)$$

α being the parameter.

Differentiating (1) partially *w.r.* to α, we get

$$-x \sin \alpha+y \cos \alpha=0. \qquad (2)$$

To eliminate α between (1) and (2), square these equations and add. Then

$$(x \cos \alpha+y \sin \alpha)^2+(-x \sin \alpha+y \cos \alpha)^2=p^2$$

or
$$x^2(\cos^2 \alpha+\sin^2 \alpha)+y^2 (\cos^2 \alpha+\sin^2 \alpha)=p^2$$

or
$$x^2+y^2=p^2. \qquad \text{(cf. Ex. § 12·1.}$$

Ex. 2. *Find the envelope of the family of circles*

$$(x-\alpha)^2+y^2=r^2, \qquad (i)$$

α being the parameter.

Differentiating partially *w.r.* to α, we get

$$-2(x-\alpha)=0, \text{ or } \alpha=x. \qquad (ii)$$

Substituting for α in (i), the equation of the envelope is

$$y^2=r^2,$$

which represents the pair of parallel straight lines $y=\pm r$.

12·21. Every family of curves need not necessarily have an envelope. The following example demonstrates this fact.

Ex. *Find the envelope of the circles* $(x-\alpha)^2+y^2=\alpha^2$.

On simplification, the equation to the family of circles may be written as

$$f(x, y, \alpha) \equiv x^2 + y^2 - 2\alpha x = 0. \qquad (i)$$

Differentiating partially *w.r.* to α, we get

$$-2x = 0. \qquad (ii)$$

It is not possible to eliminate α between equations (*i*) and (*ii*).

Hence the given family of circles do not have an envelope.

Note. The eliminant of the equations

$$f(x, y, \alpha) = 0 \text{ and } \frac{\partial f}{\partial \alpha} = 0$$

is the condition that the first of these two equations may have a pair of equal roots. Hence

(*i*) A family of curves in which the parameter appears linearly will not have an envelope.

(*ii*) In order that a family of rational algebraic curves may have an envelope, the parameter must appear at least in the second degree.

Ex. *Find the envelope of the family of curves.*

$$A\lambda^2 + B\lambda + C = 0, \qquad (1)$$

where A, B, C are functions of x and y, and λ *is the parameter.*

Differentiating (*i*) partially *w.r.* to λ, we get

$$2A\lambda + B = 0. \qquad (2)$$

From (2), $\lambda = -B/2A$. Substituting in (1) for λ, we get

$$A\left(\frac{B^2}{4A^2}\right) + B\left(-\frac{B}{2A}\right) + C = 0, \text{ or } B^2 - 4AC = 0, \qquad (3)$$

which is the required equation of the envelope.

It may be observed that (3) is precisely the condition that equation (1) in λ may have a pair of equal roots. (*Cf. Note above*)

12 3. Tangency property. *In general, the envelope of a family of curves touches each member of the family.*

Let the family of curves be

$$f(x, y, \alpha) = 0. \qquad (1)$$

Consider any particular member of this family for which $\alpha = c$, a constant. Then the equation of this particular curve is

$$f(x, y, c) = 0. \qquad (2)$$

Let $P(x, y)$ be a point common to this curve and the envelope. The slope of the curve at P is given by

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \text{ whence } \frac{dy}{dx} = -\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y}, \qquad (3)$$

where $\alpha = c$.

Now the envelope of the family of curves is given by

$$f(x, y, \alpha)=0 \text{ and } \frac{\partial f}{\partial \alpha}=0.$$

Solving these two equations for x and y in terms of α, we get parametric equations of the envelope as

$$x=\varphi(\alpha), \qquad y=\psi(\alpha).$$

Hence the slope of the envelope at P is given by

$$\frac{dy}{dx}=\frac{dy}{d\alpha}\Big/\frac{dx}{d\alpha}. \tag{4}$$

Differentiating equation (1) totally regarding x, y as functions of α, we get

$$\frac{\partial f}{\partial x}\frac{dx}{d\alpha}+\frac{\partial f}{\partial y}\frac{dy}{d\alpha}+\frac{\partial f}{\partial \alpha}=0\ ;$$

but $\dfrac{\partial f}{\partial \alpha}=0$ for the envelope, hence we get

$$\frac{\partial f}{\partial x}\frac{dx}{d\alpha}+\frac{\partial f}{\partial y}\frac{dy}{d\alpha}=0 \text{ or } \frac{dy}{d\alpha}\Big/\frac{dx}{d\alpha}=-\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y}. \qquad \text{...(5)}$$

Comparing (4) and (5), the slope of the envelope at $P(x, y)$ is equal to $-\dfrac{\partial f}{\partial x}\Big/\dfrac{\partial f}{\partial y}$ which by (3) is also the slope of the curve $f(x, y, c)=0$ at P. Hence the envelope touches the curve (2) at $P(x, y)$. By giving different values to α, it follows that the envelope touches all the members of the family.

Note. The above argument fails if $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ both vanish at the point $P(x, y)$. In this case $\dfrac{dy}{dx}$ is indeterminate and we cannot assert, without further examination, that the envelope touches the curve.

EXAMPLES XLI

Find the envelopes of the following families of curves

1. $y=mx+(a/m)$, m being the parameter.
2. $y=mx+\sqrt{(a^2m^2+b^2)}$, " " "
3. $y=mx+cm^n$, " " "
4. $x \cos m\alpha+y \sin m\alpha=a (\cos n\alpha)^{m/n}$, α being the parameter.
5. $\dfrac{a^2 \cos\theta}{x}-\dfrac{a^2 \sin\theta}{y}=\dfrac{c^2}{a}$, θ being the parameter.
6. $f(x, y) \cos\theta+\varphi(x, y) \sin\theta=\psi(x, y)$, " " "

7. $(x-\alpha)^2+y^2=4a$, α being the parameter. (*Panjab, 1951*)

8. $\dfrac{x^2}{a^2}+\dfrac{y^2}{(k-a)^2}=1$, a being the parameter.

9. $\dfrac{x^2}{a^2}+\dfrac{y^2}{k^2-a^2}=1$, where a is the parameter.

(*Allahabad, 1946*)

10. $y=x\tan\alpha-\dfrac{gx^2}{2U^2\cos^2\alpha}$, where α is the parameter.

(*Panjab, 1956*)

12·4. Envelope of a family of curves involving two parameters connected by a relation. Let the equation of a family of curves

$$f(x, y, \alpha, \beta)=0 \tag{1}$$

contain two parameters α and β connected by the relation

$$\phi(\alpha, \beta)=0. \tag{2}$$

To find the envelope, we may eliminate one of the two parameters between (1) and (2). The resulting equation will now contain only one parameter and the envelope may be found by the usual method.

At times, however, elimination of one of the parameters is not convenient. In such a case, we may consider one of the parameters, say β, to be a function of α given by the relation (2). Then differentiating (1) and (2) *w.r.* to α, we get

$$\frac{\partial f}{\partial \alpha}+\frac{\partial f}{\partial \beta}\,\frac{d\beta}{d\alpha}=0, \tag{3}$$

$$\frac{\partial \phi}{\partial \alpha}+\frac{\partial \phi}{\partial \beta}\,\frac{d\beta}{d\alpha}=0. \tag{4}$$

Eliminating α, β, $\dfrac{d\beta}{d\alpha}$ between the four equations (1)—(4), we obtain the equation of the envelope.

The process of elimination is sometimes simplified as follows. From (3) and (4),

$$\frac{\partial f}{\partial \alpha}\Big/\frac{\partial \phi}{\partial \alpha}=\frac{\partial f}{\partial \beta}\Big/\frac{\partial \phi}{\partial \beta}=\lambda, \text{ say.}$$

$$\therefore \qquad \frac{\partial f}{\partial \alpha}=\lambda\frac{\partial \phi}{\partial \alpha}, \text{ and } \frac{\partial f}{\partial \beta}=\lambda\frac{\partial \phi}{\partial \beta}. \tag{5}$$

Now eliminate α, β, λ between equations (1), (2), (5). We get the equation of the envelope. This is called the method of *intermediate multipliers*.

Ex. 1. *Find the envelope of a system of concentric and coaxal ellipses of constant area.* (*Panjab, 1946*)

The equation of any such ellipse is of the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \tag{1}$$

The area of this ellipse is πab. Since this is constant, πab is constant, *i.e.*, ab is constant. Hence

$$ab=c^2, \text{ a constant.} \tag{2}$$

Differentiating (1) and (2) *w. r.* to a regarding b as a function of a, we get

$$-\frac{2x^2}{a^3}-\frac{2y^2}{b^3}\frac{db}{da}=0, \tag{3}$$

and
$$b+a\frac{db}{da}=0 \tag{4}$$

rom (3) and (4), on equating the value of $\dfrac{db}{da}$,

$$\frac{x^2}{a^2}=\frac{y^2}{b^2}. \tag{5}$$

From (1) and (5), $x^2=\frac{1}{2}a^2$ and $y^2=\frac{1}{2}b^2$, whence

$$a=\sqrt{2}x, \qquad b=\sqrt{2}y.$$

Substituting these in (2), the envelope is

$$2xy=c^2,$$

which is a rectangular hyperbola having the axes for its asymptotes.

Ex. 2. *Find the envelope of the lines* $\dfrac{x}{a}+\dfrac{y}{b}=1$, (1)

where a and b are connected by the relation $a^n+b^n=c^n$. (2)

(*Panjab,* **1954**)

Differentiating (1) and (2) *w. r.* to a regarding b as a function of a, we get

$$-\frac{x}{a^2}-\frac{y}{b^2}\frac{db}{da}=0, \tag{3}$$

ano
$$na^{n-1}+nb^{n-1}\frac{db}{db}=0. \tag{4}$$

Equating the value of $\dfrac{db}{da}$, we get

$$\frac{x}{a^{n+1}}=\frac{y}{b^{n+1}} \text{ or } \frac{x/a}{a^n}=\frac{y/b}{b^n}=\frac{x/a+y/b}{a^n+b^n}=\frac{1}{c^n}$$

by (1) and (2). Hence

$$a^{n+1}=c^n x \text{ and } b^{n+1}=c^n y,$$

or
$$a=(c^n x)^{\frac{1}{n+1}} \text{ and } b=(c^n y)^{\frac{1}{n+1}}.$$

Substituting in (2) for a, b the envelope is

$$(c^n x)^{\frac{n}{n+1}} + (c^n y)^{\frac{n}{n+1}} = c^n,$$

or
$$x^{\frac{n}{n+1}} + y^{\frac{n}{n+1}} = c^{\frac{n}{n+1}}.$$

12 2. Further examples. We add a few more examples.

Ex. 1. *Find the envelope of the circles drawn on the semi-diameters of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *as diameters.*

(Agra, 1948 ; Panjab, '58)

The origin O is the centre of the ellipse. Let OP be any radius vector drawn from O to any point P of the ellipse and let the co-ordinates of P be $(a \cos \theta, b \sin \theta)$. Then the equation of the circle on OP as diameter is

$$x(x-a \cos \theta)+y(y-b \sin \theta)=0,$$

or
$$x^2+y^2=ax \cos \theta+by \sin \theta. \tag{1}$$

We have to find the envelope of the circle (1), θ being the parameter. Differentiating (1) *w. r.* to θ, we get

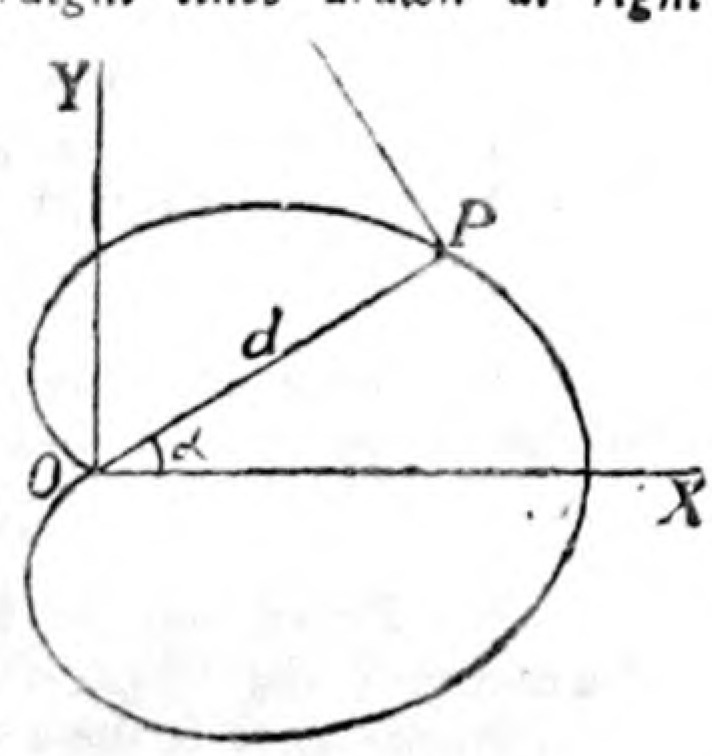
$$0=-ax \sin \theta+by \sin \theta. \tag{2}$$

Squaring (1) and (2) and adding, the equation of the envelope is

$$(x^2+y^2)^2=a^2x^2+b^2y^2.$$

Ex. 2. *Find the envelope of the straight lines drawn at right angles to the radii vectores of the cardioid* $r=a(1+\cos \theta)$ *through their extremities.*

(Aligarh, 1930)



Let $P(d, \alpha)$ be any point on the cardioid, therefore,

$$d=a(1+\cos \alpha). \tag{1}$$

Taking the initial line as the x-axis and the pole as origin, the equation of the line through P perpendicular to OP is

$$x \cos \alpha+y \sin \alpha=d=a(1+\cos \alpha)$$

or
$$(x-a) \cos \alpha+y \sin \alpha=a \tag{2}$$

Differentiating *w. r.* to α, we get

$$-(x-a) \sin \alpha+y \cos \alpha=0. \tag{3}$$

Squaring (2) and (3) and adding, the equation of the envelope is

$$(x-a)^2+y^2=a^2.$$

Ex. 3. *Find the envelope of the circles described on the radii vectores of the curve $r=a(1+\cos\theta)$ as diameters.*

Let $P\,(d,\alpha)$ be any point on the curve, then

$$d=a(1+\cos\alpha).$$

The equation of the circle on the radius vector OP as diameter is

$$r=d\cos(\theta-\alpha)$$

or
$$r=a\,(1+\cos\alpha)\cos(\theta-\alpha) \tag{1}$$

Differentiating (1) $w.r.$ to α, we get

$$0=a[(1+\cos\alpha)\sin(\theta-\alpha)-\sin\alpha\cos(\theta-\alpha)],$$

or
$$\sin(\theta-\alpha)+\sin(\theta-\alpha)\cos\alpha-\sin\alpha\cos(\theta-\alpha)=0,$$

or
$$\sin(\theta-\alpha)-\sin(2\alpha-\theta)=0.$$

or
$$\sin(\theta-\alpha)=\sin(2\alpha-\theta),$$

$\therefore$
$$\theta-\alpha=2\alpha-\theta \quad \text{or} \quad \alpha=\tfrac{2}{3}\theta.$$

Substituting this value of α in (1), the equation of the envelope is

$$r=a(1+\cos\tfrac{2}{3}\theta)\cos\tfrac{1}{3}\theta \quad \text{or} \quad r=2a\cos^3\tfrac{1}{3}\theta.$$

EXAMPLES XLII

1. Find the envelope of the straight line $x/a+y/b=1$, where

   (*i*) $a+b=c$. (*ii*) $a^2+b^2=c^2$.

   (*iii*) $a^n+b^n=c^n$. (*Panjab, 1954 S*) (*iv*) $ab=c^2$.

   (*v*) $a^m b^n=c^{m+n}$, c being a constant. (*Panjab, 1949*)

2. Prove that the envelope of the ellipses, having the axes of co-ordinates as principal axes and the sum of their axes constant and equal to $2c$ is the astroid

$$x^{2/3}+y^{2/3}=c^{2/3}.$$ (*Panjab, 1941*)

3. Show that the envelope of a circle whose centre lies on the parabola $y^2=4ax$ and which passes through its vertex is

$$2ay^2+x(x^2+y^2)=0.$$ (*Agra, 1947*)

4. Prove that the envelope of the circles which pass through the centre of the ellipse $x^2/a^2+y^2/b^2=1$ and have their centres upon its circumference is the curve

$$(x^2+y^2)^2=4(a^2x^2+b^2y^2).$$ (*Bombay, 1947*)

5. Show that the envelope of the family of parabolas

$$\sqrt{(x/a)}+\sqrt{(y/b)}=1$$

under the condition $ab=c^2$, is a hyperbola. (*Panjab, Sept. 1950*)

6. From any point on the ellipse $x^2/a^2+y^2/b^2=1$, perpendiculars are drawn to the axes, and the feet of these perpendiculars

are joined. Show that the straight line thus formed always touches the curve.

$$(x/a)^{2/3}+(y/b)^{2/3}=1. \quad \textit{(Panjab, 1957 ; Lucknow, 1945)}$$

7. Through a variable point $P(at^2, 2at)$ of the parabola $y^2=4ax$ a line is drawn perpendicular to SP, where S is the focus. Show that the envelope of this line is the curve

$$27ay^2=x(x-9a)^2. \qquad \textit{(M.T.I., '46)}$$

8. Show that the envelope of the family of circles whose diameters are double ordinates of the parabola $y^2=4ax$ is the parabola $y^2=4a(x+a)$. *(Nagpur, 1942)*

9. Find the envelope of the straight lines drawn through the extremities of, and prependicular to the radii vectores of the following curves :

(*i*) $r\cos(\theta-\alpha)=p$. (*ii*) $r=ae^{m\theta}$.

(*iii*) $r^n=a^n\cos n\theta$. *(Lucknow, 1950)*

10. Find the envelope of the circles described on the radii vectores of the following curves as diameters :

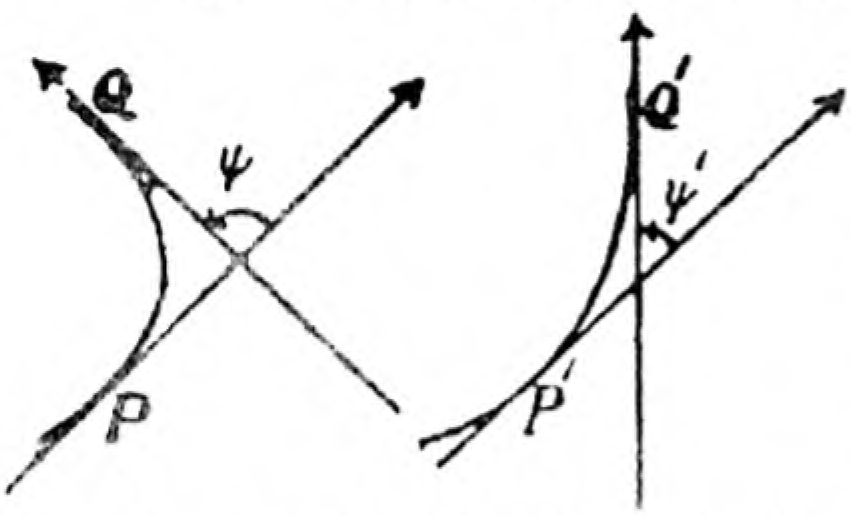
(*i*) $(l/r)=1+e\cos\theta$. (*ii*) $r\sin^2\theta=4a\cos\theta$.

(*iii*) $r^n=a^n\cos n\theta$. *(Banaras, 1949)*

CHAPTER XIII

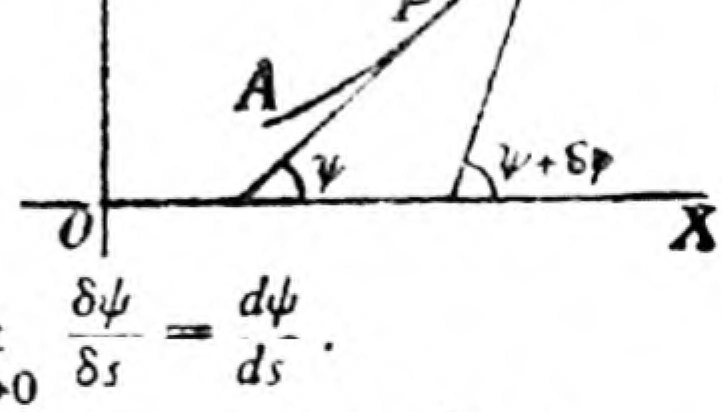
CURVATURE

13·1. Curvature. Consider the arcs PQ, $P'Q'$ of two different curves and let the length of each arc be s. As is clear from the diagram, in traversing the arc PQ, the direction of motion along the curve turns through a greater angle than it does in moving along $P'Q'$, *i.e.*, the average rate at which the tangent deflects is greater along PQ than along $P'Q'$. In common language, we say the curve PQ is sharper than $P'Q'$. In mathematical language, we say the average curvature of the curve PQ is greater than that of $P'Q'$. If the angles turned through be ψ and ψ' respectively, the average rate of bending in the first case is ψ/s and in the second case ψ'/s.



The measure of the rate of change of direction along a curve is called its **curvature** and is obtained by comparing the angle turned through with the length of arc corresponding to which this change of direction has occurred. Evidently if the angle is measured in radians and the arc in inches, curvature will be measured in radians per inch. It is the object of this chapter to measure this rate of change of direction per unit length of the arc.

13·11 Curvature at a point. Let P be any point on the curve and Q a neighbouring point. Let the arcual distance of P from a fixed point A on the curve be s and let the arc PQ be δs. Let $\delta\psi$ be the angle between the tangents at P and Q so that $\delta\psi$ measures the deflection of the tangent as the arc PQ is traversed.



Average curvature of the arc PQ

$$=\frac{\delta\psi}{\delta s}.$$

$$\therefore \text{ curvature of the curve at } P = \underset{\delta\psi\to 0}{\text{Lt}} \frac{\delta\psi}{\delta s} = \frac{d\psi}{ds}.$$

If k(= Greek 'kappa') denotes the curvature at P, we have

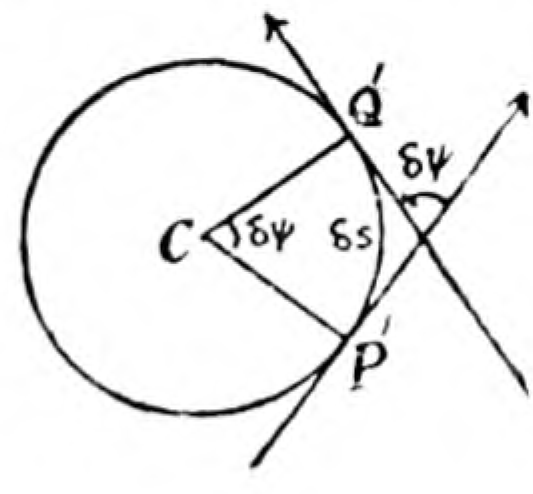
$$k=\frac{d\psi}{ds}.$$

Note 1. The curvature of the curve at P depends only on the position of P on the curve and is independent of any system of co-ordinates.

Note 2. The deflection of the tangent can be measured with respect to any fixed tangent to the curve or with regard to any fixed line in the plane of the curve.

Note 3. $\delta\psi$ is called the *angle of contingence.*

13·12. Curvature of a circle. Let us consider a circle of radius r. Imagine a point P to move along the circle. As P moves along an arc $P'Q'$ of length δs, let the tangent turn through an angle $\delta\psi$. Then the average curvature (*i.e.*, average rate of bending) along $P'Q' = \dfrac{\delta\psi}{\delta s}$ (measured in units of angle per unit length of the arc). To find the curvature of the circle at P', we take the limit of this ratio as $Q' \to P'$.



$\therefore$ curvature of the circle at P'

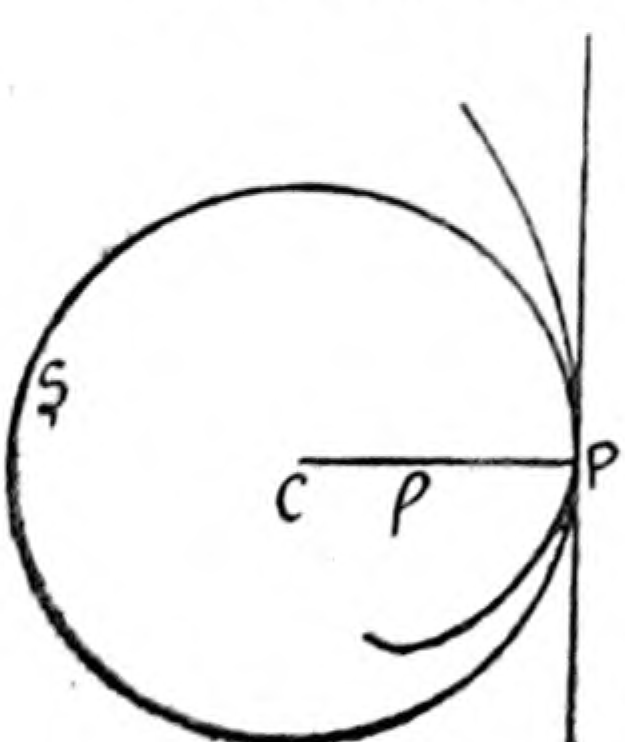
$$= \underset{\delta s \to 0}{\mathrm{Lt}} \frac{\delta\psi}{\delta s} = \underset{\delta\psi \to 0}{\mathrm{Lt}} \frac{\delta\psi}{r\delta\psi} = \frac{1}{r}.$$

Thus the curvature at any point of a circle is constant and equals the reciprocal of its radius. Consequently, the greater the circle, the smaller is its curvature. In covering equal arcual distances l along two different circles, the tangent deflects by a greater amount in the case of the smaller of the two circles.

13·2. Radius of curvature. The **radius of curvature** at any point P on a curve is defined as the radius of the circle S which touches the curve at P and has the same curvature as the given curve at this point. Now for a circle, the curvature at a point is constant and equals the reciprocal of the radius. Hence if ρ denotes the radius of curvature of the curve at P, then

$$\frac{d\psi}{ds} = \frac{1}{\rho} \quad \text{or} \quad \rho = \frac{ds}{d\psi} \quad \text{and} \quad \rho = \frac{1}{k}$$

The circle S is called the **circle of curvature** at P and its centre is called the **centre of curvature** at the point.



Since the curve and the circle S have the same curvature at P, they must bend away from the (common) tangent at P in the same direction. Hence the centre of curvature at P must lie on the normal at P on the concave side of the curve, *i.e.*, on the positive direction of the normal at P.

At a general point P of the curve, the curvature is either an increasing or a decreasing function as P moves along the curve. Since the curvature of a circle is constant and

at P the curve and the circle of curvature S have the same curvature, it follows that on one side of P the curve has greater curvature than the circle S and on the other side of P, the curve has smaller curvature than the circle. Hence, the curve will, in general, cross the circle of curvature.

As a point P where the curvature is a maximum, the circle of curvature lies within the curve and at a point P where the curvature is a minimum, the curve lies within the circle of curvature. Examples are the circles of curvature of an ellipse at the ends of the major and minor axes respectively.

13·3. The expression $\dfrac{d\psi}{ds}$ for curvature or $\dfrac{ds}{d\psi}$ for the radius of curvature are suitable only when the equation of the curve is given in the intrinsic form. We now proceed to find expressions for ρ when the equation of the curve is given in other forms.

13·31. Cartesian coordinates—Explicit Form. *When the equation of the curve is given explicitly in the form $y=f(x)$.*

Here $$\tan\psi=\frac{dy}{dx}=y_1, \therefore \psi=\tan^{-1} y_1.$$

Differentiating $w.r.$ to x, we get

$$\frac{d\psi}{dx}=\frac{1}{1+y_1^2}\cdot y_2$$

$$\therefore \quad \frac{d\psi}{ds}=\frac{d\psi}{dx}\cdot\frac{dx}{ds}=\frac{y_2}{1+y_1^2}\cdot\frac{1}{\sqrt{(1+y_1^2)}}$$

$$=\frac{y_2}{(1+y_1^2)^{3/2}}.$$

$$\therefore \quad \boldsymbol{\rho=\frac{ds}{d\psi}=\frac{(1+y_1^2)^{3/2}}{y_2}} \tag{I}$$

It has been assumed that s increases as x increases so that $\dfrac{dx}{ds}$ is positive. The sign of ρ, therefore, depends upon that of y_2. ρ is positive or negative according as y_2 is positive or negative, *i.e.*, according as the curve is concave or convex in the positive y-direction.

Cor. At a point of inflexion, (i) the curvature is zero and changes sign, and (ii) the radius of curvature is infinite.

13·32. Cartesian co-ordinates—Implicit Form. *When the equation of the curve is given implicitly in the form $f(x, y)=0$.*

Here $\dfrac{dy}{dx}=-\dfrac{f_x}{f_y}$ and $\dfrac{d^2y}{dx^2}=-\dfrac{f_{xx}f_y^2-2f_{xy}f_xf_y+f_{yy}f_x^2}{f_y^3}$

Hence substituting for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (I) above, and simplifying we get, in magnitude,

$$\rho = \frac{(\mathbf{f}_x^2+\mathbf{f}_y^2)^{3/2}}{\mathbf{f}_{xx}\,\mathbf{f}_y^2-2\mathbf{f}_{xy}\,\mathbf{f}_x\mathbf{f}_y+\mathbf{f}_{yy}\mathbf{f}_x^2}. \tag{II}$$

18·33. Cartesian co-ordinates. Parametric form. (*i*) *When the equations of the curve are given in terms of a parameter t in the form $x=f(t)$, $y=g(t)$.*

Let dashes denote differentiations *w.r.* to t, then

$$\frac{dy}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}=\frac{y'}{x'},$$

and
$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dt}\left(\frac{y'}{x'}\right)\frac{dt}{dx}$$

$$=\frac{x'y''-y'x''}{x'^2}\cdot\frac{1}{x'}=\frac{x'y''-y'x''}{x'^3}.$$

Hence substituting in (1) above and simplifying, we get

$$\rho=\frac{(\mathbf{x}'^2+\mathbf{y}'^2)^{3/2}}{\mathbf{x}'\mathbf{y}''-\mathbf{y}'\mathbf{x}''}. \tag{III}$$

(*ii*) *When the length of arc s measured from a fixed point on the curve is taken as the parameter and the equations of the curve are taken as $x=f(s)$, $y=g(s)$.*

We have

$$\frac{dx}{ds}=\cos\psi \text{ and } \frac{dy}{ds}=\sin\psi. \tag{1}$$

Differentiating these equations *w.r.* to s, we get

$$\frac{d^2x}{ds^2}=-\sin\psi\,.\,\frac{d\psi}{ds}=-\frac{\sin\psi}{\rho}, \tag{2}$$

and
$$\frac{d^2y}{ds^2}=\cos\psi\,.\,\frac{d\psi}{ds}=\frac{\cos\psi}{\rho}. \tag{3}$$

Hence from (1), (2) and (3).

$$\rho=-\frac{\sin\psi}{\dfrac{\mathbf{d^2x}}{\mathbf{ds^2}}}=-\frac{\dfrac{\mathbf{dy}}{\mathbf{ds}}}{\dfrac{\mathbf{d^2x}}{\mathbf{ds^2}}} \text{ and also } \rho=\frac{\cos\psi}{\dfrac{\mathbf{d^2y}}{\mathbf{ds^2}}}=\frac{\dfrac{\mathbf{dx}}{\mathbf{ds}}}{\dfrac{\mathbf{d^2y}}{\mathbf{ds^2}}}. \tag{IV}$$

Again, squaring and adding (2) and (3), we get

$$\frac{1}{\rho^2}=\left(\frac{\mathbf{d^2x}}{\mathbf{ds^2}}\right)^2+\left(\frac{\mathbf{d^2y}}{\mathbf{ds^2}}\right)^2. \tag{V}$$

Ex. 1. *Prove that if ρ be the radius of curvature at any point P on the parabola $y^2=4ax$ and S be its focus, then ρ^2 varies as $(SP)^3$.*

(*Panjab*, **1952**)

The co-ordinates of the focus S are $(a, 0)$. If $P(x, y)$ be any point on the parabola, then

$$SP=\sqrt{\{(x-a)^2+y^2\}}=\sqrt{(x^2-2ax+a^2+y^2)}$$

$$=\sqrt{(x^2+2ax+a^2)}=x+a. \qquad [\because y^2=4ax]$$

Also from the equation of the parabola,

$$y=2a^{1/2}x^{1/2} \text{ and } \therefore\ y_1=a^{1/2}x^{-1/2},\ y_2=-\tfrac{1}{2}a^{1/2}x^{-3/2}$$

Hence by formula (I),

$$\rho=\frac{(1+y_1^2)^{3/2}}{y_2}=\frac{(1+ax^{-1})^{3/2}}{-\frac{1}{2}a^{1/2}x^{-3/2}}=-\frac{2}{\sqrt{a}}(x+a)^{3/2},$$

and

$$\therefore\ \rho^2=\frac{4}{a}(x+a)^3=\frac{4}{a}SP^3.$$

Hence ρ^2 varies as SP^3.

Ex. 2. *Find the radius of curvature at any point of the curve*

$$x=a(\theta-\sin\theta),\quad y=a(1-\cos\theta).$$

Here

$$\frac{dx}{d\theta}=a(1-\cos\theta),\ \frac{dy}{d\theta}=a\sin\theta,$$

$$\frac{d^2x}{d\theta^2}=a\sin\theta,\qquad \frac{d^2y}{d\theta^2}=a\cos\theta.$$

$\therefore$ by formula (III) above,

$$\rho=\frac{\{a^2(1-\cos\theta)^2+a^2\sin^2\theta\}^{3/2}}{a(1-\cos\theta)\,.\,a\cos\theta-a\sin\theta\,.\,a\sin\theta}$$

$$=\frac{a\{2(1-\cos\theta)\}^{3/2}}{-(1-\cos\theta)}=-2\sqrt{2}a(1-\cos\theta)^{1/2}$$

$$=4a\sin\tfrac{1}{2}\theta \text{ in magnitude.}$$

Ex. 3. *Find the points of maximum and minimum curvature on the curve $y=\log\sin x$.*

We shall first find the curvature at any point (x, y) and then the values of x which make curvature numerically greatest or least. Differentiating the equation of the curve we have

$$y_1=\cot x,\quad y_2=-\operatorname{cosec}^2 x.$$

$$\therefore\quad \text{curvature}=\frac{y_2}{(1+y_1^2)^{3/2}}=\frac{-\operatorname{cosec}^2 x}{(1+\cot^2 x)^{3/2}}=-\sin x.$$

We have now to find the numerically greatest and least values of $(-\sin x)$.

The numerically greatest values are attained at

$$x=\pm\pi/2,\ \pm 3\pi/2,\ \ldots\ldots\ldots$$

where curvature $=1$, and the numerically least at

$$x=0,\ \pm\pi,\ \pm 2\pi,\ \ldots\ldots\ldots$$

where curvature $=0$.

EXAMPLES XLIII

Find the radius of curvature at any point of :

1. The catenary $s=c\tan\psi$. 2. The cycloid $s=4a\sin\psi$.

3. The tractrix $s=c\log\sec\psi$.

4. The parabola $s=a\log(\tan\psi+\sec\psi)+a\tan\psi\sec\psi$. *(Calcutta)*

5. $y=c\log\sec(x/c)$. *(Calcutta, 1955)* 6. $xy=c^2$. *(Kashmir, 1956)*

7. $x=a\cos t, y=b\sin t$. *(Allahabad, 1950)*

8. The catenary $x=c\log\{s+\sqrt{(c^2+s^2)}\}, y=\sqrt{(c^2+s^2)}$.

Find the radius of curvature of the curve :

9. $x^3+y^3=2$ at $(1, 1)$. 10. $(x^2+y^2)^2=a^2(y^2-x^2)$ at $(0, a)$.

11. Find the curvature at any point of the catenary $y=c\cosh(x/c)$ and show that it varies inversely as the square of the ordinate.

12. In the cycloid $x=a(\theta+\sin\theta), y=a(1-\cos\theta)$, prove that the radius of curvature is $4a\cos\frac{1}{2}\theta$. *(Delhi, 1959 ; Agra, '49)*

13. Prove that for the ellipse $x^2/a^2+y^2/b^2=1$, $\rho=a^2b^2/p^3$, p being the perpendilular from the centre upon the tangent at (x, y). *(Panjab, 1958 ; Nagpur, '49)*

14. (*i*) If CP, CD be a pair of conjugate semi-diameters of an ellipse with semi-axes of lengths a and b, prove that the radius of curvature at $P=CD^3/ab$. *(Panjab, 1961 S ; Calcutta, '47)*

(*ii*) If ρ_1, ρ_2 be the radii of curvature at the extremities of two conjugate diameters of an ellipse, prove that

$$(\rho_1^{2/3}+\rho_2^{2/3})(ab)^{2/3}=a^2+b^2. \quad \textit{(Panjab, 1947)}$$

[**Hint.** The extremities of a pair of conjugate diameters are $(a\cos\theta, b\sin\theta)$ and $(-a\sin\theta, b\cos\theta)$.]

Impt 15. Show that the radius of curvature at the point

$$(a\cos^3\theta, a\sin^3\theta)$$

on the curve $x^{2/3}+y^{2/3}=a^{2/3}$ is $3a\sin\theta\cos\theta$. *(Panjab, 1959 ; Utkal, '50)*

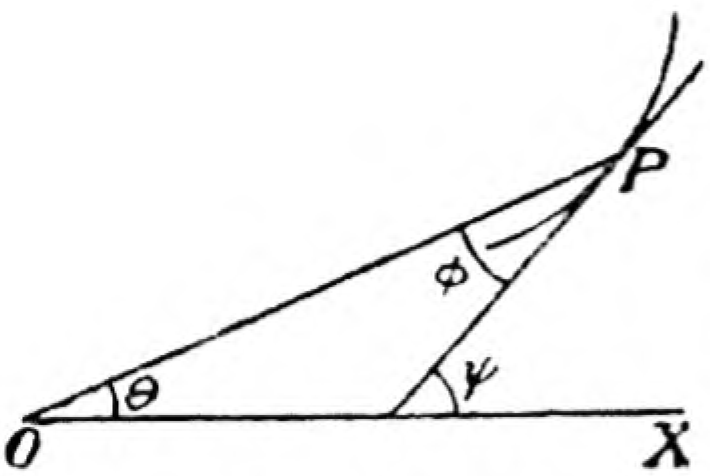
16. Find ρ for (*i*) $e^y=\cos x$, (*ii*) $y=e^x$ at the points where the curve crosses the y-axis.

17. Find the points on the parabola $x=at^2, y=2at$ at which the radius of curvature is equal to the latus rectum.

18. Find the points of maximum and minimum curvature on the curves : (*i*) $y=\sin x$, (*ii*) $y=e^x$.

19. Show that the radius of curvature of the envelope of the line $x\cos\alpha+y\sin\alpha=f(\alpha)$ is $f(\alpha)+f''(\alpha)$. *(Panjab, 1960)*

20. Prove that at a point of inflexion, the circle of curvature degenerates into a straight line.



13·34. Polar form. (*i*) *To find the radius of curvature of the curve* $r=f(\theta)$.

Let ψ be the angle which the tangent at any point $P(r, \theta)$ makes with the x-axis. Then from the figure,

$$\psi=\theta+\phi.$$

and $\therefore$ $\dfrac{1}{\rho}=\dfrac{d\psi}{ds}=\dfrac{d}{ds}(\theta+\phi)=\dfrac{d}{d\theta}(\theta+\phi)\dfrac{d\theta}{ds}$

$$=\left(1+\frac{d\phi}{d\theta}\right)\frac{d\theta}{ds}.$$

Now $\tan\phi=r\dfrac{d\theta}{dr}=\dfrac{r}{r_1}$ and $\therefore$ $\phi=\tan^{-1}\left(\dfrac{r}{r_1}\right)$, where r_1 stands for $\dfrac{dr}{d\theta}$. Let r_2 stand for $\dfrac{d^2r}{d\theta^2}$, then

$$\frac{d\phi}{d\theta}=\frac{1}{1+(r/r_1)^2}\cdot\frac{r_1^2-rr_2}{r_1^2}=\frac{r_1^2-rr_2}{r^2+r_1^2}.$$

Also $\qquad \dfrac{ds}{d\theta}=\sqrt{(r^2+r_1^2)}$ and so $\dfrac{d\theta}{ds}=\dfrac{1}{\sqrt{(r^2+r_1^2)}}$.

Hence $\qquad \dfrac{1}{\rho}=\left(1+\dfrac{d\phi}{d\theta}\right)\dfrac{d\theta}{ds}=\left(1+\dfrac{r_1^2-rr_2}{r^2+r_1^2}\right)\dfrac{1}{\sqrt{(r^2+r_1^2)}}$

$$=\frac{r^2+2r_1^2-rr_2}{(r^2+r_1^2)^{3/2}}.$$

$$\therefore \qquad \rho=\frac{(\mathbf{r}^2+\mathbf{r}_1^2)^{3/2}}{\mathbf{r}^2+2\mathbf{r}_1^2-\mathbf{r}\mathbf{r}_2}. \qquad \textbf{(VI)}$$

(*ii*) *To find the radius of curvature of the curve* $u=f(\theta)$, *where* $u=1/r$.

Since $\qquad r=1/u$, therefore

$$r_1=\frac{dr}{d\theta}=-\frac{1}{u^2}\frac{du}{d\theta}=-\frac{u_1}{u^2},$$

and $$r_2=\frac{d^2r}{d\theta^2}=-\frac{u^2u_2-u_1\,2uu_1}{u^4}=\frac{2u_1^2-uu_2}{u^3}.$$

Hence substituting for r, r_1, r_2, in (VI), we get, after simplification,

$$\rho=\frac{(\mathbf{u}^2+\mathbf{u}_1^2)^{3/2}}{\mathbf{u}^3(\mathbf{u}+\mathbf{u}_2)}. \qquad \textbf{(VII)}$$

Cor. At a point of inflexion, curvature vanishes and changes sign. Hence

$$u+u_2, \quad i.e., \quad u+\frac{d^2u}{d\theta^2}$$

vanishes and changes sign at a point of inflexion.

13·35 Pedal form. *To find the radius of curvature of the curve where* $r=f(p)$.

$\tan\phi = r\,\frac{d\theta}{dr} = r\,\frac{d\theta/ds}{dr/ds}$

We have $p=r\sin\phi$, therefore

$$\frac{dp}{dr}=\sin\phi+r\cos\phi\frac{d\phi}{dr}=r\frac{d\theta}{ds}+r\frac{dr}{ds}\;\frac{d\phi}{dr}$$

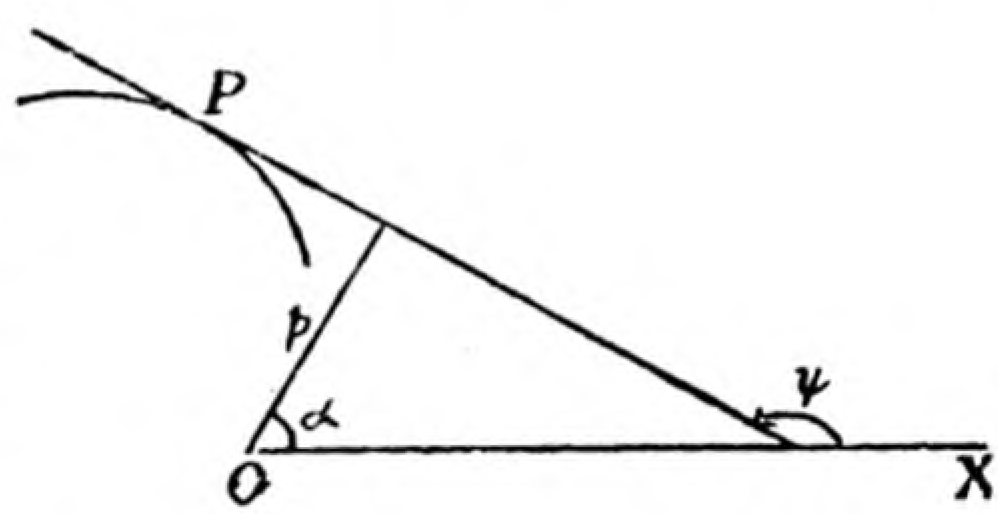
$$=r\left(\frac{d\theta}{ds}+\frac{d\phi}{ds}\right)=r\frac{d}{ds}(\theta+\phi)$$

$$=r\frac{d\psi}{ds}=\frac{r}{\rho}. \qquad (\because\ \theta+\phi=\psi)$$

$$\therefore \qquad \rho=\mathbf{r}\frac{\mathbf{dr}}{\mathbf{dp}}. \qquad \text{(VIII)}$$

13·36. Polar tangential form. *To find the radius of curvature of the curve* $p=f(\psi)$.

Let ψ be the angle which the tagent at any point P(x, y) on the curve makes with the x-axis. Let p be the length of the perpendicular from the origin on the tangent and let α be the angle which this perpendicular makes with the x-axis.



Then $\qquad \alpha=\psi-\frac{1}{2}\pi.$

Also $\qquad p=x\cos\alpha+y\sin\alpha$

$$=x\cos(\psi-\tfrac{1}{2}\pi)+y\sin(\psi-\tfrac{1}{2}\pi)$$

$$=x\sin\psi-y\cos\psi,$$

$\frac{dy}{dx}=\tan\psi$, $\frac{\sin\psi}{\cos\psi}=\frac{dy/ds}{dx/ds}$

$$\therefore \qquad \frac{dp}{d\psi}=x\cos\psi+\sin\psi\,.\,\frac{dx}{d\psi}+y\sin\psi-\cos\psi\,.\frac{dy}{d\psi}.$$

But $\frac{dx}{d\psi}=\frac{dx}{ds}\cdot\frac{ds}{d\psi}=\rho\cos\psi$, and $\frac{dy}{d\psi}=\frac{dy}{ds}\cdot\frac{ds}{d\psi}=\rho\sin\psi$.

Hence $\frac{dp}{d\psi}=x\cos\psi+y\sin\psi+\sin\psi\,.\,\rho\cos\psi-\cos\psi\,.\,\rho\sin\psi$

$$=x\cos\psi+y\sin\psi.$$

Differentiating again *w.r.* to ψ,

$$\frac{d^2p}{d\psi^2}=-x\sin\psi+\cos\psi\,.\frac{dx}{d\psi}+y\cos\psi+\sin\psi\,.\frac{dy}{d\psi}$$

$$=-(x\sin\psi-y\cos\psi)+\rho(\cos^2\psi+\sin^2\psi)$$

$$=-p+\rho.$$

$$\therefore \qquad \rho=\mathbf{p}+\frac{\mathbf{d^2p}}{\mathbf{d\psi^2}}. \qquad \text{(IX)}$$

not to done

Ex. 1. *Find the radius of curvature at any point of the parabola*

$$\frac{2a}{r}=1-\cos\theta.$$

Writing u for $1/r$, we get

$$2au=1-\cos\theta.$$

Differentiating *w.r.* to θ we have

$$2au_1=\sin\theta, \qquad 2au_2=\cos\theta.$$

Hence substituting for u, u_1, u_2 in (VII), we get

$$\rho=\frac{\left\{\dfrac{(1-\cos\theta)^2}{4a^2}+\dfrac{\sin^2\theta}{4a^2}\right\}^{3/2}}{\dfrac{(1-\cos\theta)^3}{8a^3}\left\{\dfrac{1-\cos\theta}{2a}+\dfrac{\cos\theta}{2a}\right\}}=\frac{4\sqrt{2a}}{(1-\cos\theta)^{3/2}}$$

$$=\frac{4\sqrt{2a}}{(2\sin^2\frac{1}{2}\theta)^{3/2}}=2a\operatorname{cosec}^3\tfrac{1}{2}\theta.$$

Ex. 2. *Find ρ for the ellipse $p^2=a^2\cos^2\psi+b^2\sin^2\psi$.*

Differentiating the equation twice *w.r.* to ψ, we get

$$p\frac{dp}{d\psi}=-(a^2-b^2)\sin\psi\cos\psi,$$

and

$$p\frac{d^2p}{d\psi^2}+\left(\frac{dp}{d\psi}\right)^2=(a^2-b^2)(\sin^2\psi-\cos^2\psi).$$

$$\therefore\quad p\frac{d^2p}{d\psi^2}=(a^2-b^2)(\sin^2\psi-\cos^2\psi)-\frac{(a^2-b^2)^2\sin^2\psi\cos^2\psi}{p^2}$$

$$=(a^2-b^2)(\sin^2\psi-\cos^2\psi)-\frac{(a^2-b^2)^2\sin^2\psi\cos^2\psi}{a^2\cos^2\psi+b^2\sin^2\psi}$$

$$=\frac{a^2b^2-(a^2\cos^2\psi+b^2\sin^2\psi)^2}{a^2\cos^2\psi+b^2\sin^2\psi}=\frac{a^2b^2}{p^2}-p^2.$$

$$\therefore\quad \frac{d^2p}{d\psi^2}=\frac{a^2b^2}{p^3}-p.$$

Hence $\rho=p+\dfrac{d^2p}{d\psi^2}=\dfrac{a^2b^2}{p^3}.$

Cor. 1. ρ is minimum when p has its maximam value a at an end of the major axis and ρ is maximum when p has its minimum value b at an end of the minor axis.

Hence $\rho_{min}=\dfrac{a^2b^2}{a^3}=\dfrac{b^2}{a}$ and $\rho_{max}=\dfrac{a^2b^2}{b^3}=\dfrac{a^2}{b}$.

Cor. 2. The curvature of the ellipse is maximum at the ends of the major axis and minimum at the ends of the minor axis.

EXAMPLES XLIV

Find the radius of curvature at any point of the following curves :

1. $r=a(1-\cos\theta)$. *(Allahabad, 1942)*

2. $r=a\cos m\theta$. *(Panjab, 1938)*

3. $r^3=a^2\sin 2\theta$. 4. $\theta=\sqrt{(r^2-a^2)}/a-\cos^{-1}(a/r)$.

5. (*i*) The cardioid $2ap^2=r^3$. (*ii*) The lemniscate $pa^2=r^3$.

6. (*i*) The ellipse $r^2=a^2+b^2-(a^2b^2/p^2)$.

(*ii*) The hyperbola $r^2=a^2-b^2+(a^2b^2/p^2)$.

7. (*i*) The hyperbola $p^2=a^2\cos^2\psi-b^2\sin^2\psi$.

(*ii*) The parabola $p\sin\psi=a$.

(*iii*) The circle $p=a(1+\sin\psi)$.

8. Show that the radius of curvature at any point on the cardioid $r=a(1-\cos\theta)$ is $\frac{2}{3}\sqrt{(2ar)}$. *(Lucknow, 1948)*

9. Show that in the rectangular hyperbola.

$$r^2\cos 2\theta=a^2,\ \rho=r^3/a^2.$$ *(Agra, 1950)*

10. Show that for the curve $r^n=a^n\cos n\theta$, the radius of curvature is $a^n r^{1-n}/(n+1)$. *(Patna, 1950)*

11. Show that for the curve $a^n p=r^{n+1}$, ρ varies inversely as the $(n-1)$th power of the radius vector. *(Delhi, 1958)*

12. In the conic $(l/r)=1+e\cos\theta$, show that

$$\rho=\frac{l(1+2e\cos\theta+e^2)^{3/2}}{(1+e\cos\theta)^3}.$$

13. If ϕ is the angle which the radius vector of the curve $r=f(\theta)$ makes with the tangent, prove that

$$\rho=\frac{r\operatorname{cosec}\phi}{1+(d\phi/d\theta)},$$

where ρ is the radius of curvature.

Apply this result to show that the radius of curvature of the circle $r=a\cos\theta$ is $\frac{1}{2}a$. *(Panjab, 1943)*

14. Show that at any point of the equiangular spiral $r=ae^{\theta\cot\alpha}$, $\rho=r\operatorname{cosec}\alpha$ and that it subtends a right angle at the pole. *(Delhi, 1950)*

15. If O be the pole and C the centre of curvature at any point P of the curve $r=\sqrt{2p}$, show that the triangle OCP is isosceles.

13·37. Radius of curvature at the origin. When a curve passes through the origin, the value of ρ at the origin may be obtained from formula (I) after calculating the values of the derivatives y_1 and y_2 at the origin.

In the case of the curve $f(x, y)=0$, if the origin is a double point on the curve, then $f_x=0, f_y=0$ at the origin and therefore formula (II) fails to give the value of ρ at the origin. The following method is useful in such cases.

Let $$y=px+\tfrac{1}{2}qx^2+\tfrac{1}{6}rx^3+\ldots\ldots \qquad (i)$$

be the Maclaurin expansion of y in ascending powers of x for any one branch of the curve through the origin, $p, q, r, \ldots\ldots$ denoting the values of $y_1, y_2, y_3 \ldots\ldots$ at the origin. The values $p, q, \ldots\ldots$ can be calculated by substituting the value of y from (i) in the equation of the curve and equating to zero the coefficients of various powers of x in the identity thus obtained. Then the radius of curvature at the origin is given by

$$\rho=\frac{(1+p^2)^{3/2}}{q}.$$

It has been assumed in (i) that the y-axis is not a tangent to the branch under consideration. If it be so, we can use an expansion of the form

$$x=p_1 y+\tfrac{1}{2}q_1 y^2+\tfrac{1}{6}r_1 y^3+\ldots\ldots$$

where $p_1, q_1, \ldots\ldots\ldots$, etc., denote the values of $\dfrac{dx}{dy}, \dfrac{d^2x}{dy^2}, \ldots\ldots\ldots$ etc., at the origin. The radius of curvature of the corresponding branch is then given by

$$\rho=\frac{(1+p_1^2)^{3/2}}{q_1}.$$

Cor. 1. Newton's Formulae. If the x-axis is a tangent to the curve at the origin, then $p=0$ and, therefore, $\rho=\dfrac{1}{q}$.

Also, from (i), $y=\tfrac{1}{2}qx^2+\tfrac{1}{6}rx^3+\ldots\ldots$, and so

$$\underset{x\to 0}{\mathrm{Lt}}\ \frac{2y}{x^2}=q.$$

Hence at the origin

$$\rho=\frac{1}{q}=\underset{x\to 0}{\mathrm{Lt}}\ \frac{x^2}{2y}. \qquad \text{(X)}$$

It may similarly be shown that if the y-axis is a tangent to the curve at the origin, then

$$\rho=\underset{x\to 0}{\mathrm{Lt}}\ \frac{y^2}{2x}. \qquad \text{(XI)}$$

The formulae (X) and (XI) are called **Newton's formulae.**

Cor. 2. In polar co-ordinates, if the initial line is a tangent to the curve at the origin, then from (X),

$$\rho=\underset{\theta\to 0}{\mathrm{Lt}}\ \frac{r^2\cos^2\theta}{2r\sin\theta}=\underset{\theta\to 0}{\mathrm{Lt}}\ \frac{r}{2\theta}\cdot\frac{\theta}{\sin\theta}\cdot\cos^2\theta=\underset{\theta\to 0}{\mathrm{Lt}}\ \frac{r}{2\theta}.$$

Ex. 1. *Show that the radii of curvature of the curve*

$$y^2(a-x)=x^2(a+x)$$

at the origin are $\pm a\sqrt{2}$.

The tangents at the origin are $y^2=x^2$, *i.e.*, $y=\pm x$, showing that the origin is a node on the curve. Sustituting

$$y=px+\tfrac{1}{2}qx^2+\ldots\ldots$$

in the equation of the curve, we get

$$(a-x)(px+\tfrac{1}{2}qx^2+\ldots\ldots)^2=x^2(a+x).$$

Equating coefficients of x^2 and x^3, we get

$$ap^2=a \text{ and } apq-p^2=1.$$

From the first $p^2=1$ giving $p=\pm 1$.

When $p=1$, $q=2/a$ and so $\rho=\dfrac{(1+1)^{3/2}}{2/a}=a\sqrt{2}$,

and when $p=-1$, $q=-2/a$ and so $\rho=\dfrac{(1+1)^{3/2}}{-2/a}=-a\sqrt{2}$.

Ex. 2. *Find the radius of curvature at the origin for the curve*

$$2x^3+4x^2y+xy^2+5y^3+x^2-2xy+y^2-4x=0.$$

The tangent at the origin is $x=0$, *i.e.*, the y-axis, and therefore, using formula (XI), $\rho=\underset{x\to 0}{\mathrm{Lt}}\,(y^2/2x)$.

Dividing the equation of the curve by $2x$, we get

$$x^2+2xy+\tfrac{1}{2}y^2+5y\left(\frac{y^2}{2x}\right)+\tfrac{1}{2}x-y+\left(\frac{y^2}{2x}\right)-2=0.$$

Making $x\to 0$, $y\to 0$, when $y^2/2x\to\rho$, we get

$$\rho-2=0 \text{ whence } \rho=2.$$

Ex. 3. *Apply Newton's formula to find the radius of curvature at the origin for the cycloid*

$$x=a(\theta+\sin\theta),\ y=a(1-\cos\theta).$$

The origin corresponds to $\theta=0$. Also

$$\frac{dy}{dx}=\frac{a\sin\theta}{a(1+\cos\theta)}=\tan\tfrac{1}{2}\theta.$$

Hence $dy/dx=0$ at the origin and, therefore, the tangent at the origin is the x-axis, and so

$$\rho=\underset{x\to 0}{\mathrm{Lt}}\frac{x^2}{2y}=\underset{\theta\to 0}{\mathrm{Lt}}\frac{a^2(\theta+\sin\theta)^2}{2a(1-\cos\theta)} \qquad \left(\frac{0}{0}\right)$$

$$=\underset{\theta\to 0}{\mathrm{Lt}}\frac{2a(\theta+\sin\theta)(1+\cos\theta)}{2\sin\theta} \qquad \left(\frac{0}{0}\right)$$

$$=\underset{\theta\to 0}{\mathrm{Lt}}\frac{a(1+\cos\theta)^2-a(\theta+\sin\theta)\sin\theta}{\cos\theta}$$

$$=4a.$$

Ex. 4. *Find the radius of curvature of the curve $r=a \sin n\theta$ at the origin.*

Since $\phi=0$ when $\theta=0$, the initial line is a tangent to the curve at the origin, hence by Newton's method, the radius of curvature at the origin.

$$= \underset{\theta\to 0}{\text{Lt}} \frac{r}{2\theta} = \underset{\theta\to 0}{\text{Lt}} \frac{a \sin n\theta}{2\theta} = \underset{\theta\to 0}{\text{Lt}} \frac{an}{2} \cdot \frac{\sin n\theta}{n\theta} = \frac{an}{2}.$$

EXAMPLES XLV

Find the radius or radii of curvature of each of the following curves at the origin :—

1. $x^3+y^3-2x^2+6y=0$. 2. $y-x=x^2+2xy+y^2$. (*Calcutta, 1948*)

3. $y^3=x(x+y)$. 4. $y^2-2xy-3x^2+x^3+x^2y^2=0$.

5. Apply Newton's method to find the radius of curvature at the origin for the curve

$$x=t-\tfrac{1}{3}t^3,\ y=t^2.$$

6. Find the radius of curvature at the origin of the two branches of the curve given by the equations

$$x=1-t^2,\ y=t-t^3.$$ (*Panjab, 1936*)

7. Show that the radius of curvature at the origin of the curve

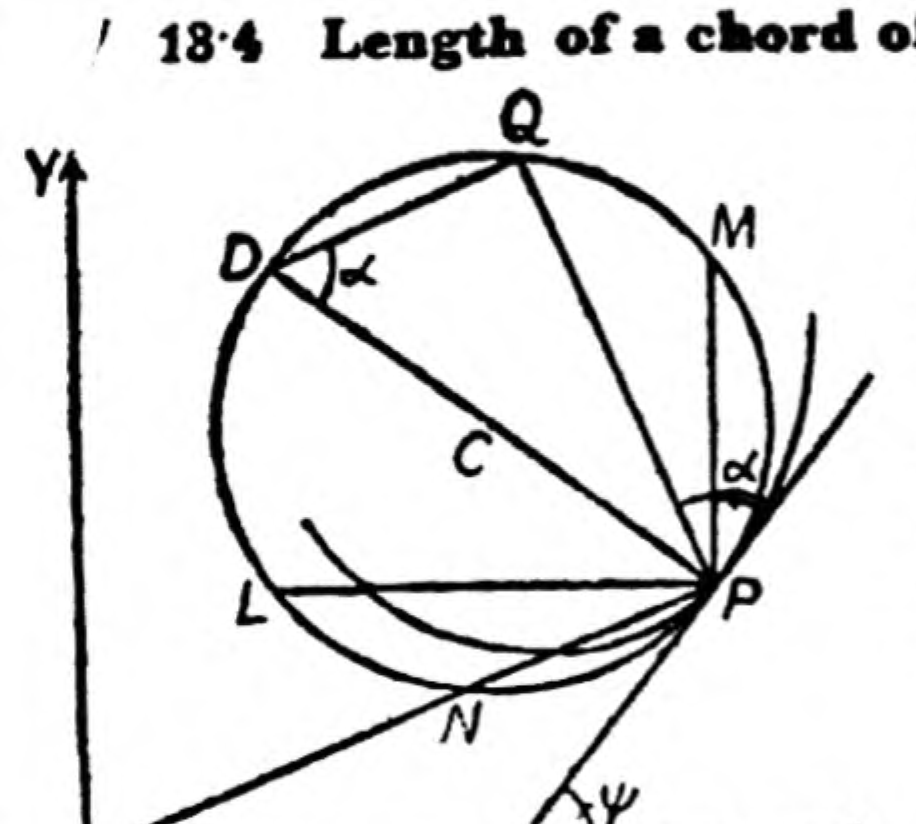
$$(ax+by)+(a_1x^2+2h_1xy+b_1y^2)+\ldots\ldots=0$$

is

$$\frac{(a^2+b^2)^{3/2}}{2(a'b^2-2h'ab+b'a^2)}.$$

8. Prove that the radius of curvature at the origin is $\dfrac{1}{2}\dfrac{dr}{d\theta}$, and apply the result to the curve $r=a(\theta+\sin\theta)$.

18·4 Length of a chord of curvature. Consider the circle of curvature at a given point P on the curve. Let PQ be any chord of this circle through the point P. Then PQ is called a **chord of curvature at** P.



Let PQ make an angle α with the tangent at P and let D be the other end of the diameter through P. Join QD. Then

$$\angle PDQ=\alpha \text{ and } \angle PQD=90^\circ.$$

Hence from the right-angled triangle PQD,

$$PQ=PD \sin\alpha=2\rho \sin\alpha,$$

where ρ is the radius of curvature at P.

We now calculate this length in some particular cases.

(*a*) **Cartesian co-ordinates.** (*i*) If C_x denote the length of the chord of curvature PL parallel to the x-axis, then $\alpha=\psi$, and so

$$C_x=2\rho\sin\psi=2\frac{(1+y_1^2)^{3/2}}{y_2}\cdot\frac{y_1}{(1+y_1^2)^{1/2}}=\frac{2y_1(1+y_1^2)}{y_2}.$$

(*ii*) If C_y denote the length of the chord of curvature PM parallel to the y-axis, then $\alpha=\frac{1}{2}\pi-\psi$, and so

$$C_y=2\rho\cos\psi=2\frac{(1+y_1^2)^{3/2}}{y_2}\quad\frac{1}{(1+y_1^2)^{1/2}}=\frac{2(1+y_1^2)}{y_2}.$$

(*b*) **Polar Co-ordinates.** If C_0 be the chord of curvature PN through the origin, then $\alpha=\phi$, the angle between the tangent and the radius vector, and so

$$C_0=2\rho\sin\phi=2\frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}\cdot\frac{r}{(r^2+r_1^2)^{1/2}}$$

$$=\frac{2r(r^2+r_1^2)}{r^2+2r_1^2-rr_2}.$$

(*c*) **Pedal equations.** If the pedal equation of a curve is given, then

$$C_0=2\rho\sin\phi=2r\frac{dr}{dp}\cdot\frac{p}{r}=2p\frac{dr}{dp}.$$

EXAMPLES XLVI

1970 1. Prove that the chord of curvature parallel to the axis of y for the curve $y=a\log\sec(x/a)$ is of constant length.

(*Lucknow, 1949 ; Panjab, 1960*)

2. If C_x, C_y be the chords of curvature parallel to the axes at any point on the curve $y=ae^{x/a}$, prove that

$$\frac{1}{C_x^2}+\frac{1}{C_y^2}=\frac{1}{2aC_x}.$$

(*Panjab, 1940 ; Agra, 1943*)

3. Prove that in the catenary $y=c\cosh(x/c)$,

$$C_x=C_y\sinh(x/c).$$

4. Show that the chord of curvature through the focus of a parabola is four times the focal distance of the point and the chord of curvature parallel to the axis has the same length (*Banaras, 1943*)

5. Show that the chord of curvature through the pole of the equiangular spiral $r=ae^{n\theta}$ is $2r$. (*Panjab, 1939 ; Patna, 1949*)

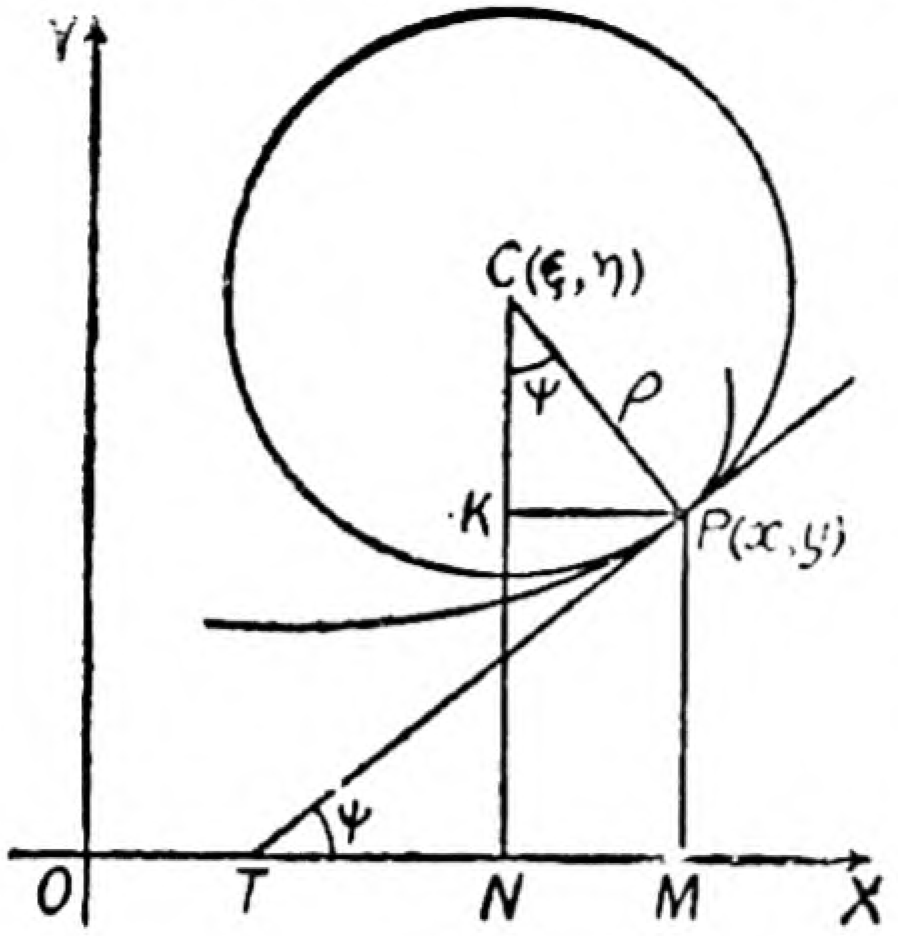
6. Show that the chord of curvature of the cardioid $r=a(1+\cos\theta)$ through the pole is $4r/3$. (*Sagar, 1949*)

7. If C_0 and C_p denote the chords of curvature of the cardioid $r=a(1+\cos\theta)$ along and perpendicular to the radius vector through any point, show that

$$3(C_0^2+C_y^2)=8aC_0.$$

8. Show that the chord of curvature through the pole of the curve $r^m=a^m\cos m\theta$ is $2r/(m+1)$.

18·5 Centre of curvature. Let $C(\xi, \eta)$ be the centre of curvature at any point $P(x, y)$ on a curve. We proceed to obtain the values of ξ and η. Draw PM and CN perpendiculars to OX and PK perpendicular to CN. Let PT be the tangent at P. Then



$$\angle PCK = \angle PTX = \psi.$$

From the figure,

$$\xi = ON = OM - NM = OM - KP$$
$$= x - \rho \sin \psi, \qquad \ldots(1)$$

and
$$\eta = NC = NK + KC$$
$$= MP + KC$$
$$= y + \rho \cos \psi. \qquad \ldots(2)$$

But

$$\cos \psi = \frac{1}{\sqrt{(1+y_1^2)}}, \ \sin \psi = \frac{y_1}{\sqrt{(1+y_1^2)}} \ \text{ and } \rho = \frac{(1+y_1^2)^{3/2}}{y_2},$$

where y_1 and y_2 stand for the values of $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at $P(x, y)$.

Hence substituting in (1) and (2), we get

$$\xi = x - \frac{y_1(1+y_1^2)}{y_2}, \ \eta = y + \frac{(1+y_1^2)}{y_2}. \qquad (3)$$

18·6 Evolutes and involutes. As the point $P(x, y)$ moves on the curve, the centre of curvature C will move and describe a second curve. We define :

The locus of the centres of curvature of a curve is called its **evolute.** *The curve itself is called an* **involute** *of its evolute.*

18·61 Evolute as envelope of normals. We now show that *the evolute of a curve may also be regarded as the envelope of its normals.*

The equation of the normal to the curve $y=f(x)$ at the point (x, y) is

$$(Y-y)y_1 + (X-x) = 0, \qquad (1)$$

where y_1 stands for the value of $\dfrac{dy}{dx}$ at the point (x, y) and X, Y are the current co-ordinates. Since y, y_1 are functions of x, x may be taken as the parameter. To find the envelope, we differentiate (1) *w.r.* to the parameter x and get

$$(Y-y)y_2 - y_1^2 - 1 = 0, \qquad (2)$$

where y_2 stands for $\dfrac{d^2y}{dx^2}$. Solving (1), (2) simultaneously, the co-

ordinates of the limiting position of the point of intersection of two neighbouring normals are given by

$$X=x-\frac{y_1(1+y_1^2)}{y_2},\ Y=y+\frac{1+y_1^2}{y_2}. \tag{3}$$

The envelope of the normals is the locus of the point (X, Y) given by equations (3) Since this point has the same co-ordinates as the centre of curvature (ξ, η) at (x, y), it follows that the envelope of the normals is the evolute of the curve and that the normal at (x, y) touches the evolute at the corresponding centre of curvature.

The evolute of a curve may, therefore, also be defined as the envelope of the normals to the curve.

13·62. The result of the preceding article may be proved in a different way as follows by proving the

Theorem. *The tangent to the evolute at any point on it is the normal to the curve at the corresponding point on the curve.*

The co-ordinates (ξ, η) of the centre of curvature C at the point $P(\xi, \eta)$ of the curve are

$$\xi=x-\rho\sin\psi, \qquad \eta=y+\rho\cos\psi. \tag{1}$$

Since y, ψ and ρ are functions of x, these may be taken as the parametric equations of the evolute, x being the parameter. From (1)

$$\begin{aligned}\frac{d\xi}{dx}&=1-\rho\cos\psi\frac{d\psi}{dx}-\sin\psi\frac{d\rho}{dx}\\&=1-\frac{ds}{d\psi}\cdot\frac{dx}{ds}\cdot\frac{d\psi}{dx}-\sin\psi\frac{d\rho}{dx}\\&=1-1-\sin\psi\frac{d\rho}{dx}=-\sin\psi\frac{d\rho}{dx},\end{aligned} \tag{2}$$

and

$$\begin{aligned}\frac{d\eta}{dx}&=\frac{dy}{dx}-\rho\sin\psi\frac{d\psi}{dx}+\cos\psi\frac{d\rho}{dx}\\&=\frac{dy}{dx}-\frac{ds}{d\psi}\cdot\frac{dy}{ds}\cdot\frac{d\psi}{dx}\\&=\frac{dy}{dx}-\frac{dy}{dx}+\cos\psi\frac{d\rho}{dx}=\cos\psi\frac{d\rho}{dx}.\end{aligned} \tag{3}$$

Hence the slope of the tangent to the evolute at the point $C(\xi, \eta)$ is

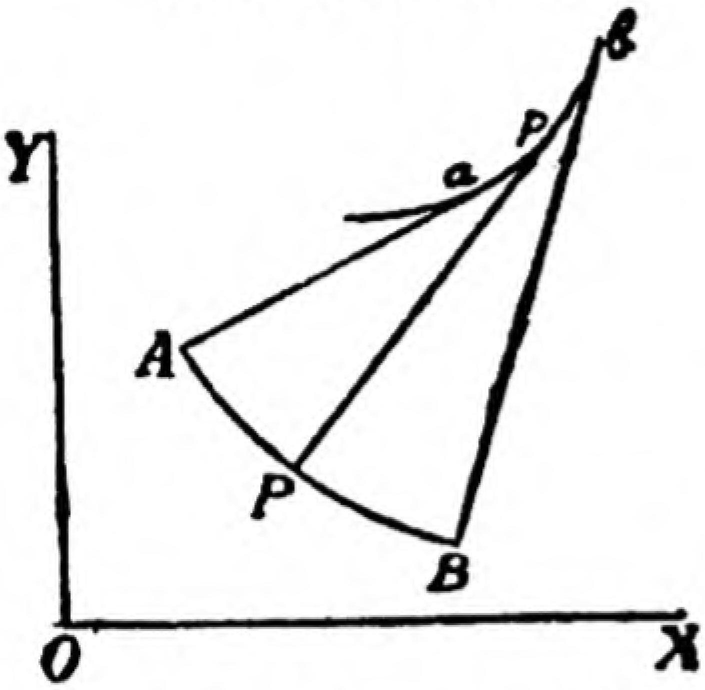
$$=\frac{d\eta}{d\xi}=\frac{d\eta}{dx}\Big/\frac{d\xi}{dx}=-\cot\psi \text{ by (2) and (3)}$$

$$=\text{slope of the normal at } P(x, y) \text{ to the curve.}$$

Since the normal at $P(x, y)$ passes through $C(\xi, \eta)$, the tangent at C to the evolute is the normal at P to the curve. This proves the theorem.

18·7 Length of arc of the evolute. *The arc of the evolute, corresponding to an arc of the original curve for which ρ constantly increases or decreases, is equal to the difference of the radii of curvature touching its extremities.*

Let AB be an arc of the curve and ab the corresponding arc of the evolute, so that $Aa=\rho_A$ and $Bb=\rho_B$ are the radii of curvature of the curve at A and B respectively. We prove that



$$\text{arc } ab=\rho_B-\rho_A,$$

provided $\dfrac{d\rho}{dx}$ maintains its sign along the arc AB.

Let $P(x, y)$ be any point on the arc AB so that arc $AP=s$. Let $p(\xi, \eta)$ be the point on the evolute corresponding to the point P on the curve, so that $p(\xi, \eta)$ is the centre of curvature of the curve at the point P. Let arc $ap=\sigma$, then σ is evidently a function of s. As P moves to a neighbouring point Q on the curve, let δs be the increment in s and $\delta\sigma$ the corresponding increment in σ.

From Arts. 13·5 and 13·62,

$$\xi=x-\rho \sin\psi, \quad \eta=y+\rho\cos\psi,$$

$$\frac{d\xi}{dx}=-\sin\psi\frac{d\rho}{dx}, \quad \frac{d\eta}{dx}=\cos\psi\frac{d\rho}{dx},$$

whence, squaring and adding,

$$\left(\frac{d\sigma}{dx}\right)^2=\left(\frac{d\xi}{dx}\right)^2+\left(\frac{d\eta}{dx}\right)^2=\left(\frac{d\rho}{dx}\right)^2$$

and $$\therefore \quad d\sigma=\pm d\rho.$$

Since $d\rho$ maintains its sign along the arc AB, so does $d\sigma$ along ab. Assuming that ρ increases, we have

$$d\sigma=d\rho \quad \text{and} \quad \sigma=\rho+C,$$

where C is the constant of integration. But $\sigma=0$ when $\rho=\rho_A$; hence $C=-\rho_A$ and so

$$\sigma=\rho-\rho_A.$$

In particular, arc $ab=\rho_B-\rho_A$.

Cor. *The radius of curvature of the evolute is* $\dfrac{d\rho}{d\psi}$.

Let ρ' be the radius of curvature of the evolute at p. The tangent at p to the evolute being the normal to the curve at P makes an angle ψ with the y-axis. Hence

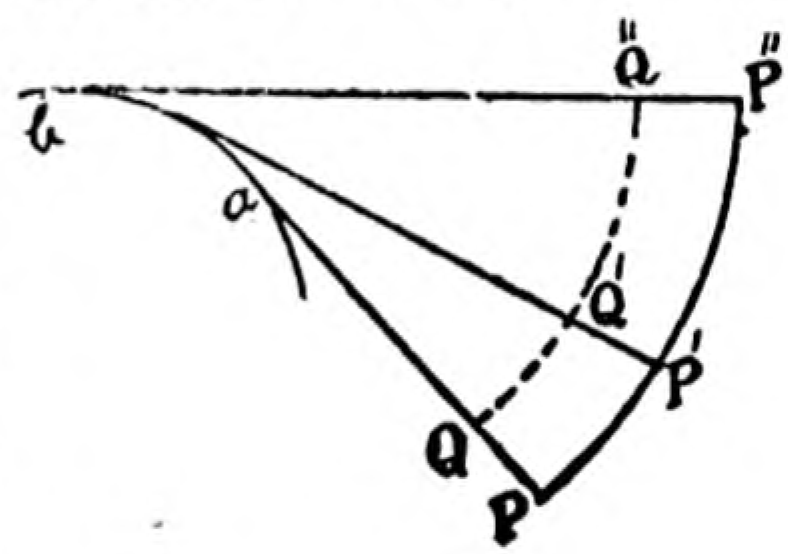
$$\rho'=\frac{d\sigma}{d\psi}=\frac{d\rho}{d\psi}.$$

Since $\rho=\frac{ds}{d\psi}$, therefore we have also,

$$\rho'=\frac{d^2s}{d\psi^2} \text{ and also } \rho'=\frac{d\rho}{ds} \cdot \frac{ds}{d\psi}=\rho\frac{d\rho}{ds}.$$

13 71 The following property of the evolute may be noted. Suppose a thin flexible string is wound round the evolute and leaves the evolute at a tightly stretched along the tangent to the evolute at a. If now, the string is gradually unwounded, all the time being held tight along the tangent, the point A and for that matter any point on the stretched part of the string will describe a curve to which the line of the string, *i e.*, the tangent line to the evolute is always normal. Thus the locus of every point on the string is an involute of the curve about which the string is wrapped. It follows that a *given curve can have but one evolute but an infinite number of involutes.*

Note. If P and Q be any two points on the string which is being unwound, the distance PQ along the common normals between the locus of P and that of Q remains constant. Two such curves are called parallel curves. We thus observe that the involutes of a given evolute are a set of parallel curves.



18·8 Evolute of the Parabola $y^2=4ax$.

First method : *Evolute as the locus of the centre of curvature.* Differentiating the equation of the curve, we get

$$2y\frac{dy}{dx}=4a, \quad \therefore y_1=\frac{dy}{dx}=\frac{2a}{y}.$$

Differentiating again,

$$y_2=\frac{d^2y}{dx^2}=-\frac{2a}{y^2}\frac{dy}{dx}=-\frac{4a^2}{y^3}.$$

Hence
$$\xi=x-\frac{y_1(1+y_1^2)}{y_2}=x+\frac{y^2+4a^2}{2a}$$

$$=x+\frac{4ax+4a^2}{2a}=3x+2a. \qquad [\because y^2=4ax$$

and
$$\eta=y+\frac{1+y_1^2}{y_2}=-\frac{y^3}{4a^2}=-\frac{2x^{3/2}}{a^{1/2}}.$$

Hence the centre of curvature at the point (x, y) is

$$\left(3x+2a, -\frac{2x^{3/2}}{a^{1/2}}\right). \tag{1}$$

To find the equation of the evolute, we have to eliminate x between the equations

$$\xi = 3x + 2a, \quad \eta = -\frac{2x^{3/2}}{a^{1/2}}.$$

From the first, $x = \frac{1}{3}(\xi - 2a)$. Squaring the second and substituting for x, we get

$$a\eta^2 = 4x^3 = \tfrac{4}{27}(\xi - 2a)^3.$$

Changing ξ, η into x, y, the equation of the evolute is

$$27ay^2 = 4(x - 2a)^3. \tag{2}$$

Second method. *Evolute as the envelope of normals.* Any normal to the parabola is

$$y + tx = 2at + at^3. \tag{3}$$

Differentiating *w.r.* to t, we get

$$x = 2a + 3at^2, \tag{4}$$

and $\therefore$ from (3),

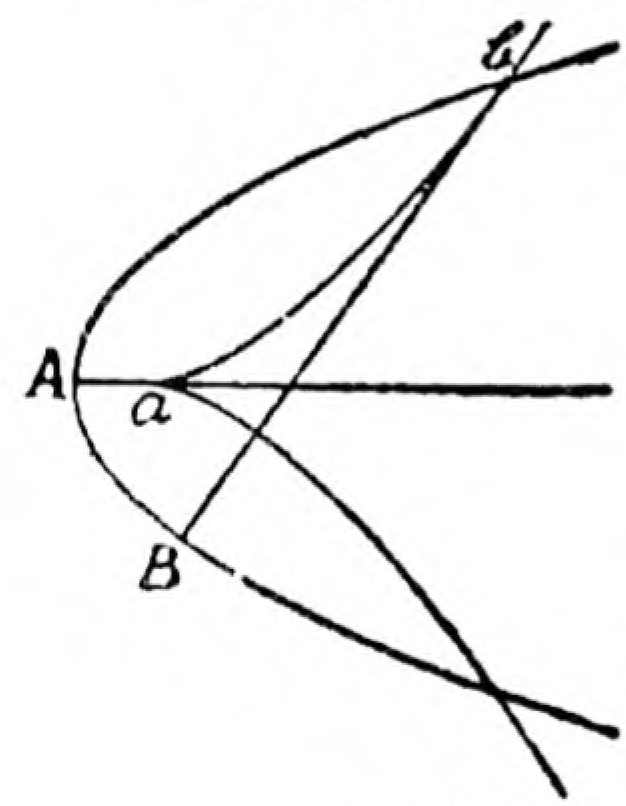
$$y = -2at^3. \tag{5}$$

Eliminating t between (4) and (5), the envelope of the normals, *i.e.*, the evolute of the parabola is, as before,

$$27ay^2 = 4(x - 2a)^3. \tag{2}$$

Cor. *Length of arc of the evolute inside the parabola.*

The evolute of the parabola is a semi-cubical parabola with a cusp* at the point $(2a, 0)$. The cusp on the evolute corresponds to the vertex A of the parabola where the curvature is maximum. Like the parabola the evolute is also symmetrical about the x-axis.



The abscissae of the points of intersection of the evolute and the parabola are the roots of

$$27a\,4ax = 4(x - 2a)^3$$

or

$$x^3 - 6ax^2 - 15a^2x - 8a^3 = 0$$

the only admissible root of which is $x = 8a$. When

$$x = 8a, \; y^2 = 4a \times 8a = 32a^2 \text{ and so } y = \pm 4\sqrt{2}a.$$

Hence the points of intersection of the evolute and the parabola are $(8a, \pm 4\sqrt{2}a)$.

Let $b(\xi, \eta)$ be the point $(8a, 4\sqrt{2}a)$ on the evolute and $B(x, y)$ the corresponding point on the parabola. Since b is the centre of curvature of the parabola at B, from (*i*) above

$$\xi = 3x + 2a.$$

*A point on a curve is said to be a cusp if (*i*) two branches of the curve pass through the point and (*ii*) the tangents to the two branches are real and coincident.

Here $\xi=8a$ and so $3x=6a$ or $x=2a$. Hence the abscissa of B is $2a$.

Next, the radius of curvature of the parabola at any point (x, y) is

$$\rho=\frac{(1+y_1^2)^{3/2}}{y_2}=-\frac{(y^2+4a^2)^{3/2}}{4a^2}$$

$$=-\frac{(4ax+4a^2)^{3/2}}{4a^2}=\frac{2(x+a)^{3/2}}{a^{1/2}} \text{ numerically.}$$

At A, $x=0$, $\therefore \rho_A=\frac{2a^{3/2}}{a^{1/2}}=2a.$

At B, $x=2a$, $\therefore \rho_B=\frac{2(2a+a)^{3/2}}{a^{1/2}}=6\sqrt{3}a.$

The total arc of the evolute inside the parabola

$=$twice the arc ab of the evolute

$=2(\rho_B-\rho_A)=2(6\sqrt{3}a-2a)=4(3\sqrt{3}-1)a.$

13·81 Evolute of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

First method. *Evolute as locus of centre of curvature.*

The parametric equations of the ellipse are

$$x=a\cos\theta, y=b\sin\theta.$$

If dashes denote differentiations *w.r.* to θ, then

$$x'=-a\sin\theta, y'=b\cos\theta.$$
$$x''=-a\cos\theta, y''=-b\sin\theta.$$

$$\therefore \qquad y_1=\frac{dy}{dx}=\frac{y'}{x'}=-\frac{b\cos\theta}{a\sin\theta},$$

and
$$y_2=\frac{d^2y}{dx^2}=\frac{x'y''-x''y'}{x'^3}=\frac{ab(\sin^2\theta+\cos^2\theta)}{-a^3\sin^3\theta}$$

$$=-\frac{b}{a^2\sin^3\theta}.$$

Hence the co-ordinates of the centre of curvature are

$$\xi=x-\frac{y_1(1+y_1^2)}{y_2}=a\cos\theta-\frac{\cos\theta\,(a^2\sin^2\theta+b^2\cos^2\theta)}{a}$$

$$=\frac{\cos\theta}{a}\left[a^2(1-\sin^2\theta)-b^2\cos^2\theta\right]$$

$$=\frac{a^2-b^2}{a}\cos^3\theta, \tag{1}$$

and
$$\eta=y+\frac{1+y_1^2}{y_2}=b\sin\theta-\frac{\sin\theta(a^2\sin^2\theta+b^2\cos^2\theta)}{b}$$

$$=\frac{\sin\theta}{b}\left[\,b^2(1-\cos^2\theta)-a^2\sin^2\theta\,\right]$$

$$=-\frac{a^2-b^2}{b}\sin^3\theta. \tag{2}$$

The equation of the evolute is found by eliminating θ between (1) and (2) ; we get, on eliminating θ,

$$(a\xi)^{2/3}+(b\eta)^{3/2}=(a^2-b^2)^{2/3}.$$

Changing ξ, η into x, y, the evolute is

$$(ax)^{2/3}+(by)^{2/3}=(a^2-b^2)^{2/3}. \tag{3}$$

Second method. *Evolute as the envelope of normals.*

The normal at $(a\cos\theta,\ b\sin\theta)$ to the ellipse is

$$ax\sec\theta-by\operatorname{cosec}\theta=a^2-b^2. \tag{4}$$

Differentiating *w.r.* to θ, we get

$$ax\sec\theta\tan\theta+by\operatorname{cosec}\theta\cot\theta=0. \tag{5}$$

Solving (4) and (5), we get

$$x=\frac{a^2-b^2}{a}\cos^3\theta,\quad y=-\frac{a^2-b^2}{b}\sin^3\theta \tag{6}$$

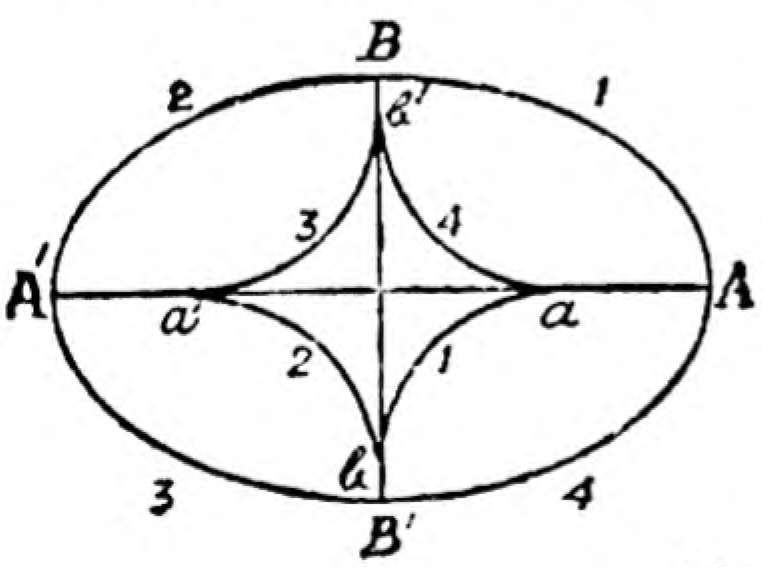
Eliminating θ between equations (6), the envelope of the normals, *i.e.*, the evolute is, as before,

$$(ax)^{2/3}+(by)^{2/3}=(a^2-b^2)^{2/3}.$$

Equations (6) may be regarded as the parametric equations of the ellipse, θ being the parameter.

Cor. Length of the evolute.

The evolute is symmetrical about both axes and has four cusps, two on each axis, which are the centres of curvature corresponding to the vertices of the ellipse which are points of maximum and minimum curvature on the ellipse.



The radius of curvature at any point P in the ellipse is

$$\rho=\frac{(x'^2+y'^2)^{3/2}}{x'y''-x''y'}$$

$$=\frac{(a^2\sin^2\theta+b^2\cos^2\theta)^{3/2}}{ab}.$$

At the vertex A, $\theta=0$ and at B, $\theta=\frac{1}{2}\pi$. Hence

$$\rho_A=\frac{b^2}{a}\ \text{and}\ \rho_B=\frac{a^2}{b}.$$

The length of the evolute corresponding to the arc AB of the ellipse

$$=\rho_B-\rho_A=\frac{a^2}{b}-\frac{b^2}{a}.$$

Hence the total length of the evaluate is $4\left(\frac{a^2}{b}-\frac{b^2}{a}\right)$.

[In the diagram the corresponding arcs of the ellipse and the evolute have been numbered alike.]

18 32 Evolute of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

First method. *Evolute as locus of the centre of curvature.*

The parametric equations of the hyperbola are

$$x=a\cosh\theta, \qquad y=b\sinh\theta.$$

If dashes denote differentiations *w.r.* to θ, then

$$x'=a\sinh\theta, \qquad y'=b\cosh\theta,$$

$$x''=a\cosh\theta, \qquad y''=b\sinh\theta.$$

$$\therefore\quad y_1=\frac{dy}{dx}=\frac{y'}{x'}=\frac{b\cosh\theta}{a\sinh\theta},$$

$$y_2=\frac{d^2y}{dx^2}=\frac{x'y''-x''y'}{x'^3}=\frac{ab(\sinh^2\theta-\cosh^2\theta)}{a^3\sinh^3\theta}$$

$$=-\frac{b}{a^2\sinh^3\theta}.$$

Hence the co-ordinates of the centre of curvature are

$$\xi=x-\frac{y_1(1+y_1^2)}{y_2}$$

$$=a\cosh\theta+\cosh\theta\ \frac{a^2\sinh^2\theta+b^2\cosh^2\theta}{a}$$

$$=\frac{a^2+b^2}{a}\cosh^3\theta.$$

$$\eta=y+\frac{1+y_1^2}{y_2}=b\sinh\theta-\sinh\theta\ \frac{a^2\sinh^2\theta+b^2\cosh^2\theta}{b}$$

$$=-\frac{a^2+b^2}{b}\sinh^3\theta. \tag{2}$$

Eliminating θ between (1) and (2), we get

$$(a\xi)^{2/3}-(b\eta)^{2/3}=(a^2+b^2)^{2/3}.$$

Changing ξ, η into x, y, the evolute is

$$(ax)^{2/3}-(by)^{2/3}=(a^2+b^2)^{2/3}. \tag{3}$$

Second method. *Evolute as envelope of the normals.*

The normal to the hyperbola at $(a\cosh\theta, b\sinh\theta)$ is

$$ax\operatorname{sech}\theta+by\operatorname{cosech}\theta=a^2+b^2. \tag{4}$$

Differentiating *w.r.* to θ, we get

$$-ax\operatorname{sech}\theta\tanh\theta-by\operatorname{cosech}\theta\coth\theta=0. \tag{5}$$

Solving (4) and (5) for x and y, we have

$$x=\frac{a^2+b^2}{a}\cosh^3\theta,\quad y=-\frac{a^2+b^2}{b}\sinh^3\theta. \tag{6}$$

Eliminating θ between the equations (6), the envelope of the normals, *i.e.*, the evolute is, as before,

$$(ax)^{2/3}-(by)^{2/3}=(a^2+b^2)^{2/3}. \tag{3}$$

Equations (6) give the co-ordinates of the centre of curvature at any point 'θ' on the hyperbola and may. therefore, be regarded as the parametric equations of the evolute, θ being the parameter.

Ex. 5. *Find the evolute of the rectangular hyperbola* $xy=c^2$.

Let $P(cp, c/p)$ be any point on the hyperbola. From the equation of the hyperbola

$$\frac{dy}{dx}=-\frac{c^2}{x^2}.$$

Hence the slope of the normal at P

$$=-\frac{dx}{dy}=\frac{x^2}{c^2}=p^2,$$

and therefore the equation of the normal at P is

$$y-\frac{c}{p}=p^2(x-cp). \tag{1}$$

The evolute is the envelope of the normal (1), p being the parameter. Differentiating (1) *w.r.* to p,

$$\frac{c}{p^2}=2px-3cp^2 \quad \text{whence} \quad 2x=3cp+\frac{c}{p^3}. \tag{2}$$

Substituting for x in (1),

$$2y=\frac{3c}{p}+cp^3. \tag{3}$$

From (2) and (3),

$$2(x+y)=c\left(p+\frac{1}{p}\right)^3 \quad \text{and} \quad 2(x-y)=-c\left(p-\frac{1}{p}\right)^3.$$

Hence eliminating p, the equation of the evolute is

$$2^{2/3}\{(x+y)^{2/3}-(x-y)^{2/3}\}=4c^{2/3}$$

or
$$(x+y)^{2/3}-(x-y)^{2/3}=(4c)^{2/3}.$$

EXAMPLES XLVII

(ξ, η) denete the co-ordinates of the centre of curvature at any point $P(x, y)$ on the curve.

1. For the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ whose parametric representation is $x=a \sec\theta$, $y=b\tan\theta$, show that

$$\xi=\frac{a^2+b^2}{a}\sec^3\theta,\quad \eta=-\frac{a^2+b^2}{b}\tan^3\theta.$$

Deduce that the equation of the evolute is

$$(ax)^{2/3}-(by)^{2/3}=(a^2+b^2)^{2/3}.$$

Show that this is also the envelope of the normal at any point 'θ' on the curve.

2. Find the envelope of the normal at any point on the $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1.$ *(Panjab, 1952)*

3. For the rectangular hyperbola $xy=a^2$, prove that

$$2\xi=3x+(y^2/x),\ 2\eta=3y+(x^2/y).$$

4. For the parabola $\sqrt{x}+\sqrt{y}=\sqrt{a}$, the centre of curvature at $(a/4, a/4)$ is $(3a/4, 3a/4)$.

5. For the catenary $y=c\cosh(x/c)$, show that

$$c\xi=cx-y\sqrt{(y^2-c^2)},\ \eta=2y.$$

6. If C is the centre of curvature corresponding to any point P on the curve $y=a\cosh(x/a)$ and G is the intersection of the normal at P and the x-axis, show that $PC=PG$. *(Agra, 1942)*

7. Prove that the co-ordinates of the centre of curvature at any point (x, y) on the curve $y=f(x)$ can be expressed in the form

$$x-\frac{dy}{d\psi}\quad\text{and}\quad y+\frac{dx}{d\psi}. \qquad \textit{(Panjab, Sept. 1949)}$$

8. For the curve $x=a\cos^3\theta$, $y=a\sin^3\theta$, show that

$$\xi=a\cos^3\theta+3a\sin^2\theta\cos\theta,\ \eta=a\sin^3\theta+3a\cos^2\theta\sin\theta.$$

Deduce that the evolute is

$$(x+y)^{2/3}+(x-y)^{2/3}=2a^{2/3}.$$

9. Find the equation of the normal to the tractrix

$$x=c\cos t+c\log\tan\tfrac{1}{2}t,\ y=c\sin t$$

at any point and hence show that the evolute of the tractrix is the catenary $y=c\cosh(x/c)$.

10 Prove that the centre of curvature at any point of the equiangular spiral $r=ae^{m\theta}$ is the extremity of the polar subnormal. Hence show that its evolute is another equiangular spiral.

MISCELLANEOUS EXAMPLES IV

1. If $f(x, y)=0$, show that $\dfrac{dy}{dx}=-\dfrac{f_x}{f_y}$.

Obtain dy/dz in terms of y and z where

$$a \sin x+b \sin y=c=a \cos x+b \cos z.$$ *(Calcutta, 1954)*

2. Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ where $f=\tan^{-1}\left(\dfrac{x^2+y^2}{x+y}\right)$.

(Kashmir, 1955)

3. If $u=f(y-z, z-x, x-y)$, show that

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0.$$ *(Rajputana, 1950)*

4. If $u=f(x, y)$, where $x=r \cos \theta, y=r \sin \theta$, prove that

(*i*) $$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2.$$

(*ii*) $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

5. If $x=f(u, v)$, $y=\phi(u, v)$; find $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ in terms of the partial derivatives of x and y *w.r.* to u and v.

6. If u and v are functions of x and y defined by the equations

$$x=u+e^{-v} \sin u, y=v+e^{-v} \cos u,$$

prove that $$\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}.$$ *(Panjab)*

7. If $\tan u=\dfrac{\cos x}{\sinh y}$ and $\tanh v=\dfrac{\sin x}{\cosh y}$, show that

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$ *(Agra, 1951)*

8. If $u=f(ax^2+2hxy+by^2)$ and $v=\varphi(ax^2+2hxy+by^2)$, prove that

$$\frac{\partial}{\partial y}\left(u \frac{\partial v}{\partial x}\right)=\frac{\partial}{\partial x}\left(u \frac{\partial v}{\partial y}\right).$$ *(M.T.)*

9. If $u=f(r)$ and $x=r \cos \theta, y=r \sin \theta$, prove that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''(r)+\frac{1}{r}f'(r).$$ *(Andhra, 1950)*

10. (*i*) If $F(x, y, z)=0$ find $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$.

(*ii*) If $f(x, y)=0$, $\phi(y, z)=0$, show that

$$\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dx}=\frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}.$$ *(Lucknow, 1940)*

11. Prove Euler's theorem that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu,$$

where u is a homogeneous function of x and y of degree n.

Deduce that $x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=(n-1)\frac{\partial u}{\partial x}$.

If $u=\sin^{-1}[(x^2+y^2)/(x+y)]$, show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u. \qquad \textit{(Panjab, 1948)}$$

12. If $u=H_n e^{ax+by}$ where H is homogeneous of the nth degree in x and y, then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=(ax+by+n)u.$$

13. Find the maximum and minimum values of

$a \sec x+b \operatorname{cosec} x$, $0<a<b$. *(Panjab, 1959)*

14. Show that the height of a closed cylinder of given volume and least surface is equal to its diameter. *(Delhi, 1952)*

15. The amount of fuel consumed per hour by a certain steamer varies as the cube of its speed. When the speed is 15 m.p.h., the fuel consumed is $4\frac{1}{4}$ tons of coal per hour at Rs. 4 per ton. The other expenses total Rs. 100 per hour. Find the most economical speed and the cost of a voyage of 1980 miles. *(Panjab, 1946)*

14. A lane runs at a right angle out of a road 18 ft. wide. How many feet wide is the lane if it is just possible to carry a pole 45 ft. long from the road into the lane, keeping it horizontal? *(M.T.I., 1934)*

17. (*a*) Prove that the minimum radius vector of the curve $(a^2/x^2)+(b^2/y^2)=1$ is of length $(a+b)$. *(Delhi, 1957 ; Panjab, 1949)*

(*b*) Find the maxima and minima of the radii vectors of the curve

$$\frac{c^4}{r^2}=\frac{a^2}{\sin^2\theta}+\frac{b^2}{\cos^2\theta}. \qquad \textit{(Delhi, 1949)}$$

18. Prove that a conical tent of a given capacity will require the least amount of canvas when the height is $\sqrt{2}$ times the radius of the base. *(Delhi, 1959 ; Allahabad, 1944)*

19. The greatest value of $x^m y^n$, where x and y are positive and $x+y=k$, is

$$m^m n^n k^{m+n}/(m+n)^{m+n}.$$

20. Tangents are drawn to the ellipse

$$(x^2/a^2)+(y^2/b^2)=1$$

and the circle $x^2+y^2=a^2$ at points where a common ordinate cuts them. Show that if θ be the greatest inclination of those tangents then

$$\tan\theta=\frac{a-b}{2\sqrt{(ab)}}. \qquad \textit{(Agra, 1944)}$$

21. Given the family of parabolas represented by the equation

$$ay=ax\tan\theta-x^2\sec^2\theta$$

for different values of θ, show that

(i) the member of the family, for which $\theta=\frac{1}{4}\pi$, cuts off the greatest intercept from the x-axis. Find the length of this maximum intercept ;

(ii) the envelope of this family of parabolas will itself be another parabola having its focus at the origin. (*Agra, 1942*)

22. A straight line of given length slides with its extremities on two fixed straight lines at right angles. Find the envelope of the circle drawn on the sliding line as diameter.

(*Panjab, 1947 ; Agra, '50*)

23. Find the envelope of a family of parabolas of given latus rectum and parallel axes when the locus of their foci is a given straight line $y=px+q$. (*Panjab, 1937*)

24. Find the evolute of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and show that it is the envelope of the family of ellipses given by

$$a^2x^2\sec^4\gamma+b^2y^2\operatorname{cosec}^4\gamma=(a^2-b^2)^2,$$

γ being the variable parameter. (*Panjab, 1944*)

25. Find the envelope of the ellipse

$$x=a\sin(\theta-\alpha),\ y=b\cos\theta,$$

where α is the parameter. (*Panjab, 1950 S*)

26. For the catenary $y=c\cosh(x/c)$, find the radius of curvature at any point and deduce the intrinsic equation.

(*Calcutta, 1956*)

27. Prove that in the curve $r^2=a^2\sin 2\theta$, the tangent turns three times as fast as the radius vector and that the curvature varies as the radius vector. (*Delhi, 1959*)

28. Prove that for the cardioid $r=a(1+\cos\theta)$, ρ^2/r is constant. (*Panjab, 1954*)

29. If ρ_1, ρ_2 be the radii of curvature at the extremities of any chord of the cardioid $r=a(1+\cos\theta)$, which passes through the pole, then

$$\rho_1^2+\rho_2^2=16a^2/9.$$

30. Find the point of the curve $y=e^x$ at which the curvature is a maximum and show that the tangent at the point forms with the axes of coordinates a triangle whose sides are in the ratio of $1:\sqrt{2}:\sqrt{3}$

(*Annamalai, 1940*)

ANSWERS

Examples XVI,

1. $x<-2$ and $x>6$. 2. $x<1$ and $x>2$; $1<x<2$.

Examples XVII, Pages 97—98

1. (*i*) $2\pi r$. (*ii*) $2\pi(h+2r)$ (*iii*) $\pi(2r^2+h^2)/(r^2+h^2)$. (*iv*) $4\pi r^2$.

2. $-\dfrac{X}{LD^2}, -\dfrac{X}{DL^2}, -\dfrac{X}{2DL\sigma}, \dfrac{X}{2DLT}$, where $X=\left(\dfrac{gT}{\pi\sigma}\right)^{\frac{1}{2}}$.

4. $\pi/3$. 5. $\sqrt{3}$ sq in./sec. 6. $3/8\pi$ cm./min.
10. $\sqrt{\{\mu(a^2-s^2)\}}, -\mu s$; a, 0 resp. 11. $2\frac{2}{3}$ ft./sec.
12 $ax/(a-b)$, $bx/(a-b)$ miles/hour.
13. $1\frac{1}{10}$ ft. sec. Decreasing. 14. $6\frac{2}{3}$ ft. sec.

Examples XVIII,

1. 99·995. 2. 6·994.
3. 4·004. 4. 8·005.

Examples XIX, Pages 105—106

1. (*i*) $Yy=2a(X+x)$, $2a(Y-y)+y(X-x)=0$.
(*ii*) $a^2Y-3x^2X+2x^3=0$, $3x^2(Y-y)+a^2(X-x)=0$.
(*iii*) $(x^2-ay)X+(y^2-ax)Y=axy$,
$(Y-y)(x^2-ay)-(X-x)(y^2-ax)=0$.
(*iv*) $ty=x+at^2$, $y+tx=2at+at^3$.
(*v*) $bx-ay\sin\theta=ab\cos\theta$, $ax\sin\theta+by=(a^2+b^2)\tan\theta$
(*vi*) $x\sin\frac{1}{2}t-y\cos\frac{1}{2}t=at\sin\frac{1}{2}t$,
$x\cos\frac{1}{2}t+y\sin\frac{1}{2}t=at\cos\frac{1}{2}t+2a\sin\frac{1}{2}t$.

2. (*i*) $4X+Y=4a$. (*ii*) $3X-2Y=a$. 4. $(2\pm1/\sqrt{3}, \pm2/3\sqrt{3})$.
6. $y=2x+3$. 7. $x+3y=\pm9$. 14. $n^nA^n=a^{n-1}B(nA^2+B^2)^{n-1}$.

Examples XX,

1. $90°$, $37° 57'$. 2. $45°$. 3. $\tan^{-1}\frac{12}{41}$.
5. 0, $\tan^{-1} 49$. 8. $90°$, $\tan^{-1}\{3/(2^{2/3}+2^{4/3})\}$.

Examples XXII,

1. $\frac{1}{3}\pi$. 2. $\frac{3}{4}\pi$. 3. $\frac{5}{6}\pi$. 4. $\tan^{-1}\theta$. 5. $m\theta-\frac{1}{2}\pi$.
6. $\frac{1}{3}\pi$. 7. $\frac{2}{3}\pi$. 8. $\frac{1}{2}\pi$. 9. $\tan^{-1}\frac{4}{3}$; $\frac{1}{4}\pi$ at the pole.
13. $-27a/5$, $-3a/5$, $9\sqrt{(10)}a/5$, $3\sqrt{(10)}a/5$.
14. $3a/4$, a, $\sqrt{(21)}\,a/4$, $a\sqrt{7}/2$.

Examples XXIII,

1. $p^2=ar$. 2. $r^2=a^2-3p^2$. 3. $r^2=a^2-b^2+a^2b^2/p^2$.
4. $(b^2/p^2)=(2a/r)-1$. 5. $2ap^2=r^3$.
6. $a^mp=r^{m+1}$. 7. $p=r\sin a$. 8. $(r^2+a^2)p^2=r^4$.
9. $a^2p=r^3$. 10. $a^2(e^2-1)/p^2=(2a/r)+1$.

Examples XXIV,

1. (i) $\sqrt{\left(\frac{a+x}{x}\right)}$. (ii) $\frac{y}{c}$. (iii) $\left(\frac{a}{x}\right)^{\frac{1}{3}}$. (iv) $\frac{a(8a-3x)^{1/2}}{(2a-x)^{3/2}}$.

2. (i) $a\sqrt{(1-e^2\cos^2 t)}$. (ii) $3a \sin t \cos t$. (iii) $2a \cos \frac{1}{2}t$.

5. (i) $2a \cos \frac{1}{2}\theta$. (ii) $a \operatorname{cosec} \alpha\, e^{\theta \cot \alpha}$ (iii) $a\sqrt{(\sec 2\theta)}$.

Examples XXV,

1. Max. at $x=-1$; min. at $x=0$.
2. Max. at $x=1$; min. at $x=7$.
3. Max. at $x=-2, 1$; min. at $x=-1, 2$.
4. Max. at $x=1$; min. at $x=6$.
5. Max at $x=-1$; min. at $x=\frac{7}{9}$; neither at $x=3$.
6. Max. at $x=2$; min. at $x=2\frac{6}{11}$; neither at $x=3$.
7. Min. at $x=\frac{1}{2}$. 8. Max at $x=a$; min. at $x=-a$.
10. Max. at $x=n\pi+\frac{1}{6}\pi$; min. at $x=n\pi-\frac{1}{6}\pi$.
11. If $\alpha=\tan^{-1}(\frac{1}{2}\sqrt{3})$, then $x=\alpha, \frac{1}{2}\pi, \pi-\alpha, \pi+\alpha, \frac{3}{2}\pi, \ldots\ldots$ give max. and min. alternately.
12. If $\alpha=\pi/(n+1)$, then there is max. at $x=\alpha, 3\alpha, \ldots\ldots$ and min. at $x=2\alpha, 4\alpha, \ldots\ldots$, provided $x\neq$ a multiple of π in either series.
13. If $a>b$, max. value$=a$, min. value$=b$ and *vice versa* if $a<b$.
14. If $a>b$, max. value$=(a+b)^2$.
17. Max. value$=4/e$; min. value$=0$.

Examples XXVI,

1. (i) 12, 12. (ii) 8, 16. (iii) 6, 18. 3. 220 yds., 440 yds.
5. $4\sqrt{3}$ sq. in 13. $(4\pi/27)h^3 \tan^2 \alpha$. 4. $a-b$.

Miscellaneous Examples II,

6. 120π sq. ft /sec.
7. Max.$=-(a^{2/3}+b^{2/3})^{3/2}$, min.$=(a^{2/3}+b^{2/3})^{3/2}$.
8. $9(5^{2/3}-2^{1/3})^{3/2}$ ft. 9. 32/9.
10. Increasing in $0<x<\frac{1}{4}\pi$ and $\frac{2}{3}\pi<x<\frac{3}{4}\pi$; decreasing in $\frac{1}{4}\pi<x<\frac{2}{3}\pi$ and $\frac{3}{4}\pi<x<\pi$. The greatest value is attained at $x=\frac{1}{4}\pi$.

Examples XXVII,

3. Not applicable ; not derivable at $x=1$.
4. (i) $c=4$. (ii) $c=\frac{1}{2}\pi, \frac{3}{2}\pi$. (iii) $c=\pm 2/\sqrt{3}$. (iv) $c=-2$.
6. $c=2$. 7. $\{(\frac{7}{3})^{1/2}, (\frac{7}{3})^{3/2}\}$.

Examples XXVIII,

6. (i) $1+3(x-1)+3(x-1)^2+(x-1)^3$.
(ii) $a^n+{}^nC_1a^{n-1}(x-a)+{}^nC_2a^{n-2}(x-a)^2+\ldots\ldots+{}^nC_n(x-a)^n$.
(iii) $\log\sin a+(x-a)\cot a-\frac{1}{2}(x-a)^2\operatorname{cosec}^2 a+\ldots\ldots$
(iv) $1-\frac{1}{2}(x-\frac{1}{2}\pi)^2+\frac{1}{24}(x-\frac{1}{2}\pi)^4-\frac{1}{720}(x-\frac{1}{2}\pi)^6+\ldots\ldots$

Examples XXIX,

8. $x+\frac{1}{3}x^3-\frac{5}{24}x^4+\ldots\ldots$ 9. $x-\frac{1}{2}x^2+\frac{1}{6}x^3-\frac{1}{12}x^4+\frac{1}{24}x^5$.
10. $-\frac{1}{12}$. 12. $x-\frac{1}{2}x^2+\frac{2}{3}x^3$. 13. $x+\frac{1}{6}x^3$. 14. $1+x+\frac{1}{2}x^2-\frac{1}{3}x^3$.

15. $\dfrac{1}{2!}x^2+\dfrac{2^2}{4\,!}x^4+\dfrac{2^2.4^2}{6!}x^6+\ldots\ldots$

17. (i) $x-\dfrac{1}{3!}x^3+\dfrac{1^2.3^2}{5!}x^5-\ldots\ldots$ (ii) $\dfrac{1}{2!}x^2-\dfrac{2^2}{4!}x^4+\dfrac{2^2.4^2}{6!}x^6-\ldots\ldots$

18. None of them can be expanded by the Maclaurin's Theorem in ascending powers of x as the functions and their derivatives are not continuous at $x=0$.

Examples XXX,

1. Min. at $x=\frac{1}{2}$. 2. Max. at $x=1$; Min. at $x=\frac{11}{7}$.

3. $\cos x-h\sin x-\dfrac{h^2}{2!}\cos x+\dfrac{h^3}{3!}\sin x$; $\cdot4848$.

4. $1\cdot0696$. 5. $8\cdot445312$. 6. $-2\cdot532$.

Examples XXXI,

7. $\frac{1}{6}$. 8. $-\frac{1}{3}$. 9. $(1-\log b)/(1+\log b)$. 10. 1.
11. 1. 12. -2. 13. $2a/b$. 14. 1.
15. $\frac{1}{3}$. 16. $\frac{3}{2}$. 17. 2. 18. 2.

Examples XXXII,

1. 1 2. 1. 3. 1. 4. 0. 5. 0. 6. 0.
7. 1. 8. 0. 9. $2/\pi$. 10. 0. 11. 0. 12. $-\frac{1}{2}$.
13. $\frac{1}{2}$. 14. $-\frac{1}{3}$. 15. $-\frac{1}{3}$. 16. 1. 17. 1. 18. 1.
19. 1. 20. 1. 21. 1. 22. $1/e$. 23. $1/e$. 24. 1.
25. 1. 26. $e^{-\frac{1}{2}}$ 27. ae.
28. (i) 1. (ii) $e^{1/3}$. (iii) ∞ if $x\to0+$, 0 if $x\to0-$.

Examples XXXIII,

3. (i) $2(m+n)/(m-n)$. (ii) ∞.
4. $\frac{2}{3}f'''(x)/f''(x)$. 6. $\frac{1}{2}n(n+1)$.
7. $a=-2$, limit $=-1$ 8. $a_1=-4$, $a_2=5$, limit $=1$.
10. $\frac{7}{360}$, $-\frac{1}{45}$. 11. $-\frac{1}{2}$.

Miscellaneous Examples III,

4. $\frac{9}{28}$. 5. $\frac{9}{64}$. 6. $\theta=\frac{1}{3}$.
14. $2(x-\frac{1}{3}x^3+\frac{1}{5}x^5-\ldots\ldots)$. 18. $x^2-\frac{5}{6}x^4+\ldots\ldots$

Examples XXXIV.

1. (*i*) $3x^2-3ay,\ 3y^2-3ax.$

(*ii*) $ae^{ax}\cos(e^{ax}+e^{by}),\ be^{by}\cos(e^{ax}+e^{by}).$

(*iii*) $(y^2-x^2)^{-\frac{1}{2}},\ -\dfrac{x}{y}(y^2-x^2)^{-\frac{1}{2}}.$

(*iv*) $-x(x^2+y^2)^{-\frac{3}{2}},\ -y(x^2+y^2)^{-\frac{3}{2}}$

18. $-\frac{3}{2}.$

Examples XXXV,

6. (*i*) $e^x(\sin y+2t^2\cos y)/t$, where $x=\log t,\ y=t^2$.
(*ii*) $(xte^t-y)/\{t(x^2+y^2)\}$, where $x=\log t,\ y=e^t$.

7. $2xy,\ x^2\ ;\ x(4y^2+xy-2x^2)/(x+2y).$

8. $0\ ;\ \sec^2\theta\ e^{y/x}$, where $x=r\cos\theta,\ y=r\sin\theta$.

10. $2(e^{2\theta}+4)\ ;\ 4re^{2\theta}\ ;\ 4r^2e^{2\theta}.$

11. (*i*) $-x/y,\ -a^2/y^3.$ (*ii*) $-b^2x/a^2y,\ -b^4/a^2y^3.$
(*iii*) $-(y^2+2xy)/(x^2+2xy),\ 6a^3(x^2+xy+y^2)/(x^2+2xy)^3$
(*iv*) $-(x^4-a^3y)/(y^4-a^3x),\ 6a^3xy(x^3y^3+2a^6)/(y^4-a^3x)^3.$

13. $(f_z\phi_x-f_x\phi_z)/(f_y\phi_z-f_z\phi_y),\ (f_x\phi_y-f_y\phi_x)/(f_y\phi_z-f_z\phi_y).$

14. (*i*) $(z-x)/(y-z),\ (x-y)/(y-z).$
(*ii*) $(lz-ny)/(ny-mz),\ (mx-ly)/(ny-mz).$

15. $\{\sin(A-C)+\sin B(\cos C-\cos A)\}/\{\sin(C-B)$
$+\sin A(\cos B-\cos C)\}.$

Examples XXXVII,

4. 0·66 sq. in.

Examples XXXVIII,

1. Min. at $(0, \sqrt[3]{3})$; max. at $(-2, -1)$.
2. Neither max. nor min.
3. Max. at $(0, 4)$.
4. Max. at $(1, 1)$ and $(2, 2)$.
5. Max. at $x=2a$.
6. Max. at $(2, 4)$; min. at $(-7, -5)$.

Examples XXXIX,

1. (*i*) Max.$=\frac{1}{4}ab$. (*ii*) Min.$=a^2b^2/(a^2+b^2)$.

2. Max.$=-2c\sqrt{(ab)}$; min.$=2c\sqrt{(ab)}$.

3. (*i*) Min.$=1/(a^2+b^2)$. (*ii*) Min.$=(a^2+b^2-1)^2/(a^2+b^2)$.

5. (*i*) Max.$=2\cos^2\frac{1}{2}\alpha$; min.$=2\sin^2\frac{1}{2}\alpha$.
(*ii*) Max.$=6$; min.$=1$.

Examples XL,

1. 160, 0. **2.** 70, 0. **3.** 1, 21. **4.** 0, no greatest.

Examples XLI,

1. $y^2=4ax$.
2. $(x^2/a^2)+(y^2/b^2)=1$.
3. $(n-1)^{n-1}x^n+n^n cy^{n-1}=0$.
4. $r^{n/(m-n)}=a^{n/(m-n)}\cos\{n\theta/(m-n)\}$, where (r,θ) are the polar coordinates of (x,y).
5. $a^4/x^2+b^4/y^2=c^4/a^2$.
6. $[f(x,y)]^2+[\varphi(x,y)]^2=[\psi(x,y)]^2$.
7. $y^2=4x+4$.
8. $x^{2/3}+y^{2/3}=k^{2/3}$.
9. $x\pm y=\pm k$.
10. $2g^2x^2=2U^2(U^2-2gy)$.

Examples XLII,

1. (i) $\sqrt{x}+\sqrt{y}=\sqrt{c}$. (ii) $x^{2/3}+y^{2/3}=c^{2/3}$.
(iii) $x^{n/(n+1)}+y^{n/(n+1)}=c^{n/(n+1)}$. (iv) $4xy=c^2$.
(v) $x^m y^n(m+n)^{m+n}=m^m.n^n.c^{m+n}$.
9. (i) $r^2=\{2p-r\cos(\theta-\alpha)\}$.
(ii) $r=a\sqrt{(1+m^2)}\,e^{m(\theta-\tan^{-1}m)}$
(iii) $r^{n/(1-n)}=a^{n/(1-n)}\cos\{n\theta/(1-n)\}$.
10. (i) $r^2(e^2-1)-2ler\cos\theta+l^2=0$.
(ii) $r\cos\theta=a\sin^2\theta$.
(iii) $r^{n/(n+1)}=a^{n/(n+1)}\cos\{n\theta/(n+1)\}$.

Examples XLIII,

1. $c\sec^2\psi$.
2. $4a\cos\psi$.
3. $e\tan\psi$.
4. $2a\sec^3\psi$.
5. $c\sec(x/c)$.
7. $(a^2\sin^2 t+b^2\cos^2 t)^{3/2}/(ab)$.
8. $(c^2+s^2)/c$.
9. $1/\sqrt{2}$.
10. $\frac{1}{3}a$.
11. $1/\{c\cosh^2(x/c)\}$.
16. (i) 1. (ii) $2\sqrt{2}$.
17. For $t=\pm\sqrt{(4^{1/3}-1)}$.
18. (i) Points of numerically greatest curvature are given by $x=\pm\frac{1}{2}(2n+1)\pi$, and points of numerically least curvature by $x=\pm n\pi$, where n is a positive integer.
(ii) Max. at $x=\frac{1}{2}\log\frac{1}{2}$, no min.
30. $a\cos\frac{1}{2}\psi\cot\psi$

Examples XLIV,

1. $\frac{4}{3}a\sin\frac{1}{2}\theta$.
2. $(r^2+a^2m^2-r^2m^2)^{3/2}/(r^2-r^2m^2+2a^2m^2)$.
3. $a^2/3r$.
4. $\sqrt{(r^2-a^2)}$.
5. (i) $4ap/3r$. (ii) $a^2/3r$.
6. (i) a^2b^2/p^3. (ii) a^2b^2/p^3.
7. (i) a^2b^2/p^3. (ii) $2a\operatorname{cosec}^3\psi$. (iii) a.

Examples XLV,

1. $1\frac{1}{2}$.
2. $1/2\sqrt{2}$.
3. $\frac{1}{2}$.
4. $5\sqrt{10}$; ∞.
5. $\frac{1}{2}$.
6. $2\sqrt{2}$; $2\sqrt{2}$.
8. a.

Examples XLVII,

2. $(ax)^{2/3}+(by)^{2/3}=(a^2-b^2)^{2/3}$.

Miscellaneous Examples IV,

1. $\dfrac{dy}{dz}=\dfrac{\sin z\,(c-b\cos z)}{\cos y\,(c-b\sin y)}$.

2. $\dfrac{\partial f}{\partial x}=\dfrac{x^2+2xy-y^2}{(x^2+y^2)+(x+y)^2}$, $\dfrac{\partial f}{\partial y}=\dfrac{y^2+2xy-x^2}{(x^2+y^2)^2+(x+y)^2}$.

5. $A\dfrac{\partial y}{\partial v}$ and $-A\dfrac{\partial x}{\partial v}$, where $A=\left\{\dfrac{\partial x}{\partial u}\dfrac{dy}{dv}-\dfrac{\partial x}{\partial v}\dfrac{\partial y}{\partial u}\right\}^{-1}$.

10. (*i*) $-F_x/F_z$, $-F_y/F_z$.

13. Max. $=-(a^{2/3}+b^{2/3})^{3/2}$, min. $=(a^{2/3}+b^{2/3})^{3/2}$.

15. $5(75)^{1/3}$ miles/hr., Rs. $792\,(75)^{2/3}$.

16. $9(5^{2/3}-2^{2/3})^{3/2}$ ft. 17. (*b*) Max. $=c^2/(a+b)$.

22. Taking the fixed lines as axes, the envelope is $x^2+y^2=l^2$, where l is the length of the sliding line.

23. $2p^2x-2py+(2pq+lp^2+l)=0$, $2l$ being the latus rectum.

25. $x=\pm a$. 26. $\rho=c\cosh^2(x/c)$, $s=c\tan\psi$.

AN INTRODUCTION TO CALCULUS

PART III

DIFFERENTIAL CALCULUS

CHAPTER XIV

ASYMPTOTES

14·1. Algebraic curves. The equation of an algebraic curve of the nth degree is of the form

$$(a_0x^n+a_1x^{n-1}y+\ldots+a_ny^n)+(b_0x^{n-1}+b_1x^{n-2}y+\ldots+b_{n-1}y^{n-1})$$
$$+(c_0x^{n-2}+c_1x^{n-2}y+\ldots+c_{n-2}y^{n-2})+\ldots+(l_0x+l_1y)+m_0=0, \qquad \ldots(1)$$

which may be re-written as

$$x^n\phi_n(y/x)+x^{n-1}\ \phi_{n-1}(y/x)+x^{n-2}\phi_{n-2}(y/x)+\ldots\ldots=0. \qquad \ldots(1')$$

where $\phi_n(y/x)$, $\phi_{n-1}(y/x)$,.........are polynomials in y/x of degrees n, $n-1$,.........at most, respectively.

The equation (1) may also be re-written in either of the following two ways according as it is arranged in descending powers of y or x :

(i) $a_ny^n+(a_{n-1}x+b_{n-1})y^{n-1}+\ldots\ldots$
$$+(a_0x^n+b_0x^{n-1}+\ldots\ldots+l_0x+m_0)=0 \qquad \ldots(2)$$

(ii) $a_0x^n+(a_1y+b_0)x^{n-1}+\ldots\ldots$
$$+(a_ny^n+b_{n-1}y^{n-1}+\ldots\ldots+l_1y+m_0)=0 \qquad \ldots(3)$$

14·2. Branches of a curve. Consider the circle $x^2+y^2=a^2$. Solving for y, we get

$$y=+\sqrt{(a^2-x^2)} \quad \text{or} \quad y=-\sqrt{(a^2-x^2)}.$$

These two equations represent explicitly the two branches of the circle which in ordinary language we call the upper and lower half of the circle. Notice that the circle lies inside the square bounded by the lines $x=\pm a$, $y=\pm a$. We say, therefore, that both branches of the circle are *finite*.

Again, consider the curve $x^2y^2=x^2-y^2$. Solving for y, we get

$$y=+x/\sqrt{(x^2+1)} \qquad \text{or} \qquad y=-x/\sqrt{(x^2+1)}.$$

If $x\to+\infty$, $y\to+1$ and if $x\to-\infty$, $y\to-1$ along the first branch. In the case of the second branch if $x\to+\infty$, $y\to-1$ and if $x\to-\infty$, $y\to+1$.

Here x is capable of taking arbitrarily large values but y remains finite.

In the case of the parabola, $y^2=4ax$, the two branches are $=\pm2\sqrt{a}\sqrt{x}$. As $x\to+\infty$, $y\to+\infty$ along one branch and $y\to-\infty$ along the other branch.

Branches of a curve along which or y or both are capable of taking arbitrarily large values are said to be *infinite.*

A point $P(x, y)$ is said to tend to infinity along an infinite branch of a curve if x or y or both $\to \pm\infty$ while the coordinates of P satisfy the equation to the curve.

14·3. Definition. *A* **rectilinear asymptote** *to an infinite branch of a curve is a straight line, not lying wholly at infinity, such that the distance from it of any point on the infinite branch of the curve tends to zero as the point tends to infinity along the branch.*

We shall use the word asymptote to mean a rectilinear asymptote only unless otherwise stated.

While the definition holds for all curves, we shall consider its application to algebraic curves only.

14 4. Asvmptotes parallel to the axes. Let $P(x, y)$ be any point on the curve $f(x, y)=0$ and let $x=c$ be a straight line parallel to the y-axis. The numerical value of the distance δ of P from this line is given by

$$\delta=|x-c|.$$

If $\delta\to 0$ as $P\to\infty$ along the curve, then the line $x=c$ is an asymptote to the curve. This implies that as $x\to c$, $y\to\infty$ (positive or negative) because at least one coordinate of P must $\to\infty$ if P is to $\to\infty$.

In other words, to find the asymptotes parallel to the y-axis, we have to search for values of c from the equation $f(x, y)=0$ of the curve such that when $x\to c$, $y\to\pm\infty$. The lines $x=c$ are then the asymptotes of the curve parallel to the y-axis.

To put it roughly, $x=c$ is an asymptote if c is a value of x which makes y infinite.

Similarly if $x\to\pm\infty$ when $y\to c$, the line $y=c$ is an asymptote of the curve parallel to the x-axis.

14 41. Algebraic curves. Asymptotes parallel to the y-axis. Let y^m, where $m\leqslant n$, be the highest power of y present in the equation of the algebraic curve of the nth degree. Let the equation be arranged in descending powers of y as

$$A_0(x)y^m+A_1(x)y^{m-1}+\ldots\ldots+A_m(x)=0, \qquad \ldots(1)$$

where $A_0(x)$ is not zero identically and $A_0(x)$, $A_1(x)$,.........are polynomials in x such that the degree of no term in (1) exceeds n.

If $m=n$, *i.e.*, if y^n is present in the equation, then $A_0(x)$ is merely a non-zero constant and, therefore, equation (1) in y cannot have an infinite root for any finite value of x and hence there is no asymptote parallel to the y-axis. Therefore, if the curve is to have any asymptotes parallel to the y-axis, y^n must be missing from the equation of the curve.

Next, let $m<n$. y will have an infinite root only if $A_0(x)=0$. Hence if $A_0(x)$ is a constant, there will again be no asymptote parallel to the y-axis. If $A_0(x)$ contains x, y will be infinite corresponding to every value of x which makes $A_0(x)$ vanish *i.e.* corresponding to every root of the equation $A_0(x)=0$.

If any root of $A_0(x)=0$ is imaginary, then the corresponding asymptote is rejected as we do not consider imaginary asymptotes.

Or we may reason as under :

Let the equation to the curve be arranged in descending powers of y as

$$a_n y^n+(a_{n-1}x+b_{n-1})y^{n-1}+(a_{n-2}x^2+b_{n-2}x+c_{n-2})y^{n-2}+\ldots\ldots \\ +(a_0x^n+b_0x^{n-1}+\ldots\ldots+(x_0+m_0)=0.$$

If $a_n \neq 0$, no finite value of x exists as to make $y \to \infty$. Hence there is no asymptote parallel to the y-axis in this case. If $a_n=0$, then $a_{n-1}x+b_{n-1}=0$ or $x=-b_{n-1}/a_{n-1}$ makes one root in y infinite. Hence $x=-b_{n-1}/a_{n-1}$ or $a_{n-1}x+b_{n-1}=0$ is an asymptote parallel to the y-axis. If there is no term containing y^n and y^{n-1} in the equation then $a_{n-2}x^2+b_{n-2}x+c=0$ gives two asymptotes parallel to the y-axis and so on.

We thus have the following :

Rule. *If, in an equation of the nth degree, the term containing y^n is absent, then the coefficient of the next highest power of y present in the equation, when equated to zero, gives the asymptote or asymptotes parallel to the y-axis, provided this coefficient is not merely a constant.*

14 42. Alegbraic curves. Asymptotes parallel to the x-axis. Arguing as in the previous article, we have the following

Rule. *If, in an equation of the nth degree, the term containing x^n is absent, then the coefficient of the next highest power of x present in the equation, when equated to zero, gives the asymptotes parallel to the x-axis, provided this coefficient is not merely a constant.*

Ex. *Find the asymptotes, parallel to the coordinate axes of the following curves :*

(*i*) $xy^2-x^3=a(x^2+y^2)$. (*ii*) $x^2y^2-y^2=2$. (*iii*) $2x^3+x^2y+y^2=4$.

(*i*) The curve is of the 3rd degree and y^3 is absent from the equation. The highest power of y appearing in the equation is y^2 and its coefficient is $x-a$. Hence $x-a=0$ is the asymptote parallel to the y-axis.

The coefficient of x^3 is a constant, -1. Hence there is no asymptote parallel to the x-axis.

(*ii*) The equation is of the 4th degree and y^4 is absent. The highest power of y appearing in the equation is y^2. Its coefficient is x^2-1. Hence $x^2-1=0$ gives asymptotes parallel to the y-axis. Thus $x=1$, $x=-1$ are the two asymptotes parallel to the y-axis.

Again x^4 is absent from the equation. The highest power of x is x^3. Hence asymptotes parallel to the x-axis are given by equating the coefficient of x^3 to zero. The coefficient is y^2. Thus $y^2=0$, *i.e.*, $y=0$ is an asymptote to the curve. This is the x-axis itself.

(iii) The curve is of the third degree and x^3 is present. Hence there is no asymptote parallel to the x-axis.

Again, y^3 is absent. The highest power of y is y^2. Its coefficient is a constant. Hence there is no asymptote parallel to the y-axis either.

Examples XLVIII

Find the asymptotes, parallel to the axes, of the following curves :

1. $xy^2+x^2y=x^3+y^3$.
2. $(x^2+y^2)x=ay^2$.
3. $x^2y^2=a^2(x^2+y^2)$. (*Panjab, 1958*)
4. $x^2y^2=a^2(x^2-y^2)$.
5. $x^2y^2+y^2=1$.
6. $xy^3+x^3y=a^4$.
7. $xy^2+ay^2=x$.
8. $y^3-x^2y=x^3+1$.
9. $y^3-xy^2=x^3+1$.
10. $(a^2/x^2)+(b^2/y^2)=1$. (*Panjab B.Sc., 1962*)
11. $y=x(x-1)(x-2)$.
12. $y^3=(x-1)^2$.

14·5. Oblique asymptotes. We next consider asymptotes which are not parallel to either coordinate axis. These are called **oblique** asymptotes. The method given below will also enable us to find the asymptotes parallel to the x-axis but these are best found by the method described already.

Let the straight line

$$y=mx+c \qquad \ldots(1)$$

be an oblique asymptote to the curve

$$f(x, y)=0. \qquad \ldots(2)$$

Then both m and c are finite. Let $P(x, y)$ be any point on the branch of the curve to which (1) is an asymptote and let δ be the perpendicular distance of P from (1). Then $\delta \to 0$ as P tends to infinity on the branch of the curve, *i.e.* when x and y tend to infinity under the relationship (2). It should be observed that x and y may tend to plus or minus infinity depending upon the quadrant in which the infinite branch lies. Thus we have to find m and c such that $\delta \to 0$ when $P(x, y)$ tend to infinity on the curve.

Without taking sign into consideration.

$$\delta=\frac{y-mx-c}{\sqrt{(1+m^2)}}, \qquad \ldots(3)$$

and therefore

$$\frac{\delta}{x}=\frac{1}{\sqrt{(1+m^2)}}\left(\frac{y}{x}-m-\frac{c}{x}\right). \qquad \ldots(4)$$

When $x \to \infty$, $\delta \to 0$, $\delta/x \to 0$ and $c/x \to 0$. Hence, we get, in the limit, from (4),

$$0 = \frac{1}{\sqrt{(1+m^2)}} \left(\underset{x \to \infty}{\mathrm{Lt}} \frac{y}{x} - m \right).$$

$$\therefore \underset{x \to \infty}{\mathrm{Lt}} \frac{y}{x} = m \text{ or } m = \underset{x \to \infty}{\mathrm{Lt}} \frac{y}{x}. \qquad \ldots(5)$$

This gives us the slope m of the asymptote.

To find c, substitute this value of m in (3) and then make $x \to \infty$; we get

$$\mathrm{Lt} \frac{y - mx - c}{\sqrt{(1+m^2)}} = \mathrm{Lt}\ \delta = 0.$$

$$\therefore \quad \mathrm{Lt}(y - mx - c) = 0 \text{ or } c = \mathrm{Lt}\ (y - mx). \qquad \ldots(6)$$

This gives us c and thus the asymptote is known.

To sum up: *If an infinite branch of a curve possesses an oblique asymptote $y = mx + c$, the values of m and c are given by the equations*

$$\mathbf{m} = \mathbf{Lt}\ (\mathbf{y}/\mathbf{x}),\ \mathbf{c} = \mathbf{Lt}(\mathbf{y} - \mathbf{mx}),$$

where $x, y \to \infty$, x and y being connected by the relation $f(x, y) = 0$.

Ex. 1. *Examine for asymptotes the curve*

$$x^2 + 3xy + 2y^2 + 3x - 2y + 1 = 0.$$

Since the equation is of the second degree, to find the limit of y/x, we divide by x^2; we get

$$1 + 3\left(\frac{y}{x}\right) + 2\left(\frac{y}{x}\right)^2 + \frac{3}{x} - \frac{2}{x}\left(\frac{y}{x}\right) + \frac{1}{x^2} = 0.$$

Now as $x \to \infty$, $y/x \to m$, $3/x \to 0$, $2/x \to 0$, $1/x^2 \to 0$, and we get, in the limit, the equation

$$1 + 3m + 2m^2 = 0 \text{ whence } m = -1, -\tfrac{1}{2}.$$

This gives two values of m and, therefore, there can be two oblique asymptotes provided the corresponding values of c are finite.

When $m = -1$, $c = \mathrm{Lt}\ (y - mx) = \mathrm{Lt}\ (y + x)$.

and when $m = -\frac{1}{2}$, $c = \mathrm{Lt}\ (y - mx) = \mathrm{Lt}\ (y + \frac{1}{2}x)$.

Now the equation of the curve can be written as

$$(x+y)(x+2y) = 2y - x - 1.$$

Hence, when $m = -1$,

$$c = \underset{x \to \infty}{\mathrm{Lt}}\ (y + x) = \underset{x \to \infty}{\mathrm{Lt}} \frac{2y - 3x - 1}{x + 2y} \qquad \text{[From the equation of the curve]}$$

$$= \underset{x \to \infty}{\mathrm{Lt}} \frac{2(y/x) - 3 - (1/x)}{1 + 2(y/x)} = \frac{2m - 3}{1 + 2m}$$

$$=\frac{-2-3}{1-2}=5,$$

since, when $x\to\infty$, $(y/x)\to m=-1$ and $(1/x)\to 0$.

The corresponding asymptote is $y=-x+5$.

When $m=-\frac{1}{2}$,

$$c=\underset{x\to\infty}{\text{Lt}}\ (y+\tfrac{1}{2}x)=\underset{x\to\infty}{\text{Lt}}\ \frac{2y-3x-1}{2x+2y}$$

[From the equation of the curve]

$$=\underset{x\to\infty}{\text{Lt}}\ \frac{2(y/x)-3-(1/x)}{2+2(y/x)}=\frac{2m-3}{2+2m}$$

$$=\frac{2(-\frac{1}{2})-3}{2+2(-\frac{1}{2})}=-4,$$

since, when $x\to\infty$, $(y/x)\to m=-\frac{1}{2}$ and $(1/x)\to 0$. The corresponding asymptote is $y=-\frac{1}{2}x-4$.

Obviously, there are no asymptotes parallel to either axis.

Hence the curve has only two asymptotes and their equations are $x+y=5$ and $x+2y+8=0$.

Ex. 2. *Find the asymptotes of the curve*

$$(x+y)^2(x+2y+2)=x+9y-2.$$

(Panjab, 1949, '56)

Since the curve is of the 3rd degree, to find the limit of y/x, we divide by x^3 : we get

$$\left(1+\frac{y}{x}\right)^2\left(1+2\frac{y}{x}+\frac{2}{x}\right)=\frac{1}{x^2}+\frac{9}{x^2}\left(\frac{y}{x}\right)-\frac{2}{x^3}.$$

Now as $x\to\infty$, $y/x\to m$, $2/x\to 0$, $1/x^2\to 0$, $2/x^3\to 0$, and we get, in the limit, the equation

$$(1+m)^2(1+2m)=0 \text{ whence } m=-1,\ -1,\ -\tfrac{1}{2}.$$

When $m=-1$,

$$c=\underset{x\to\infty}{\text{Lt}}\ (y-mx)=\underset{x\to\infty}{\text{Lt}}\ (y+x)$$

$$=\underset{x\to\infty}{\text{Lt}}\ \pm\sqrt{\left(\frac{x+9y-2}{x+2y+2}\right)}$$

[From the equation of the curve]

$$=\pm\underset{x\to\infty}{\text{Lt}}\ \sqrt{\left(\frac{1+9(y/x)-(2/x)}{1+2(y/x)+(2/x)}\right)}$$

$$=\pm\sqrt{\left(\frac{1+9m}{1+2m}\right)}=\pm\sqrt{\left(\frac{1-9}{1-2}\right)}=\pm 2\sqrt{2},$$

since when $x\to\infty$, $(y/x)\to m=-1$ and $(2/x)\to 0$. Hence there are two asymptotes corresponding to the value $m=-1$ and their equations are $y=-x\pm 2\sqrt{2}$.

When $m=-\frac{1}{2}$,

$$c=\underset{x\to\infty}{\text{Lt}}\ (y-mx)=\underset{x\to\infty}{\text{Lt}}\ (y+\tfrac{1}{2}x)$$

$$=\underset{x\to\infty}{\text{Lt}}\left[-1+\frac{x+9y-2}{2(x+y)^2}\right]$$

[From the equation of the curve]

$$=\underset{x\to\infty}{\text{Lt}}\left[-1+\frac{(1/x)+9(y/x)(1/x)-(2/x^2)}{2\{1+(y/x)\}^2}\right]$$

$$=-1+\frac{0+9m(0)-0}{2(1+m)^2}=-1,$$

since when $x\to\infty$, $(y/x)\to m=\frac{1}{2}$, $(1/x)\to 0$ and $(2/x^2)\to 0$. Hence the corresponding asymptote is $y=-\frac{1}{2}x-1$.

It is easily seen that there are no asymptotes parallel to the axes.

Thus the curve has the three asymptotes

$$y=-x+2\sqrt{2},\ y=-x-2\sqrt{2} \text{ and } y=-\tfrac{1}{2}x-1.$$

In the above two examples, it should be observed that all the values of m found were real. If any value of m is imaginary, then it is to be rejected for we consider real asymptotes only. Further, it should be observed that when m has been found in the first step, $y-mx$ is a factor of the highest degree terms of the equation of the curve. Thus in Ex. 1, $y+x$ and $y+\frac{1}{2}x$ were factors of $x^2+3xy+2y^2$ and in Ex. 2, $(y+x)^2$ and $y+\frac{1}{2}x$ were factors of $(x+y)^2(x+2y)$.

Ex. 3. *Show that the parabola $y^2=4ax$ has no asymptotes.*

Since the equation is of the 2nd degree, to find the limit of y/x, we divide by x^2; we get

$$\left(\frac{y}{x}\right)^2=\frac{4a}{x},$$

and making $x\to\infty$, we obtain

$$m^2=0 \text{ or } m=0.$$

Thus there is an asymptote parallel to the x-axis if the corresponding value of c is finite. We have

$$b=\underset{x\to\infty}{\text{Lt}}\ (y-mx)=\underset{x\to\infty}{\text{Lt}}\ y=\underset{x\to\infty}{\text{Lt}}\ \pm\sqrt{(4ax)},$$

which is not finite. Hence there is no asymptote parallel to the x-axis.

Also since $y=\pm\sqrt{(4ax)}$, there is no finite value of x which makes y infinite. Hence there is no asymptote parallel to the y-axis also.

Since there is neither an oblique asymptote nor an asymptote parallel to the y-axis, the parabola $y^2=4ax$ has no asymptotes.

It may be remarked that the mere fact that a curve has a branch extending to infinity does not necessarily mean that the curve has an asymptote for, though both branches, $y=\pm\sqrt{(4ax)}$, of the parabola $y^2=4ax$ extend to infinity, it has no asymptote.

The general case of oblique asymptotes for an algebraic curve is discussed in the next section.

Examples XLIX

1. Define an asymptote and state the method for obtaining asymptotes, parallel to the coordinate axes, of a rational algebraic curve. *(Panjab, 1962 S)*

Find the asmyptotes of the following curves :

2. $x^3+y^3=3ax^2$.   3. $y^3=x^2(x+6)$.   4. $y^2(y-x)+ax^2=0$.

5. $y^3-6xy^2+11x^2y-6x^3+x+y=0$. *(Calcutta, 1957)*

6. $y^4-x^4=2bx^2y$.   7. $y^2(x-2a)=x^3-a^3$. *(Lucknow, 1946)*

8. $(x-1)(x-2)(x+y)+x^2+x+1=0$ *(Panjab, 1960 S)*

9. $y^3-xy^2-x^2y+x^3+x^2-y^2=1$. *(Delhi, 1959 ; Patna, 1947)*

10. $(x-y+1)(x-y-2)(x+y)=8x-1$ *(Panjab, 1938)*

Show that the following curves have no asymptotes :—

11. $ay^2=x^2(x+a)$.   12. $y^2(a^2+x^2)=x^2(a^2-x^2)$.

14·6. Oblique asymptotes of an nth degree curve. Let the equation of the curve be written as

$$x^n\phi_n(y/x)+x^{n-1}\phi_{n-1}(y/x)+x^{n-2}\phi_{n-2}(y/x)+\ldots\ldots+\phi_0=0 \qquad \ldots(1)$$

The line $y=mx+c$ is an asymptote if

$$m=\underset{x\to\infty}{\text{Lt}}(y/x),\quad c=\underset{x\to\infty}{\text{Lt}}(y-mx),$$

these limits being calculated from relation (1).

Dividing (1) by x^n, we get

$$\phi_n\left(\frac{y}{x}\right)+\frac{1}{x}\phi_{n-1}\left(\frac{y}{x}\right)+\frac{1}{x^2}\phi_{n-2}\left(\frac{y}{x}\right)+\ldots=0. \qquad \ldots(2)$$

Making $x\to\infty$, and noting that $\frac{1}{x}, \frac{1}{x^2}, \ldots\ldots\ldots\ldots\to 0$ while $\frac{y}{x}\to m$, we get

$$\phi_n(m)=0, \qquad \ldots(3)$$

other terms vanishing in the limit.

(3) gives the slopes of the asymptotes, and $\because$ $\phi_n(m)$ is, in general, of the nth degree, (3) will have n roots corresponding to which there can be n asymptotes.

Let m be one of the roots of (3) and let $y-mx=v$ so that Lt $v=c$ when x and $y\to\infty$ such that $y/x\to m$.

Then $\frac{y}{x}=m+\frac{v}{x}$. Substituting for $\frac{y}{x}$ in (1), we get

$$x^n\phi_n\left(m+\frac{v}{x}\right)+x^{n-1}\phi_{n-1}\left(m+\frac{v}{x}\right)+\ldots\ldots+\phi_0=0.$$

Dividing by x^n and expanding each term by Taylor's theorem, we get

$$\phi_n(m)+\frac{v}{x}\phi_n'(m)+\frac{v^2}{2\,!\,x^2}\phi_n''(m)+\ldots\ldots\ldots$$
$$+\frac{1}{x}\{\phi_{n-1}(m)+\frac{v}{x}\phi_{n-1}'(m)+\ldots\ldots\ldots\}$$
$$+\frac{1}{x^2}\{\phi_{n-2}(m)+\ldots\ldots\ldots\}+\ldots=0,$$

or $$\phi_n(m)+\frac{1}{x}\{v\phi_n'(m)+\phi_{n-1}(m)\}$$
$$+\frac{1}{x^2}\left\{\frac{v^2}{2\,!}\phi_n''(m)+v\phi_{n-1}'(m)+\phi_{n-2}(m)\right\}+\ldots\ldots=0.$$

$\because$ $\phi_n(m)=0$, we have, on multiplying by x,

$$\{v\phi_n'(m)+\phi_{n-1}(m)\}+\frac{1}{x}\left\{\frac{v^2}{2\,!}\phi_n''(m)+v\phi_{n-1}'(m)+\phi_{n-2}m)\right\}$$
$$+\ldots\ldots\ldots=0 \qquad \ldots(4)$$

$\because$ Lt $v=c$ when x and $y\to\infty$, we have in the limit,

$$c\phi_n'(m)+\phi_{n-1}(m)=0. \qquad \ldots(5)$$

$\therefore$ If $\phi_n'(m)\neq 0$, which will be the case if $\phi_n(m)=0$ has no repeated root, we have

$$c=-\phi_{n-1}(m)/\phi'_n(m).$$

Thus the equation of the asymptote corresponding to a (non-repeated) root m of $\phi_n(m)=0$ is

$$y=mx-\frac{\phi_{n-1}(m)}{\phi_n'(m)}.$$

Thus if $m_1, m_2, m_3,$.....................be the non-repeated roots of $\phi_n(m)=0$ and $c_1, c_2, c_3,$.......... be the corresponding values of c, the asymptotes corresponding to these values of m and c are $y=m_1x+c_1$, $y=m_2x+c_2$, etc.

Particular cases. Parallel asymptotes.

Suppose that two of the roots of equation (3) are equal. Let these be m_1, m_1. Then $\phi_n'(m_1)=0$

If $\phi_{n-1}(m_1)\neq 0$, equation (5) fails to give a finite value of c when $m=m_1$ $\therefore$ Lt $(y-m_1x)$ does not exist finitely. Hence there is no asymptote corresponding to these repeated roots.

If, however, $\phi_{n-1}(m_1)=0$, equation (5) becomes an identity. In this case, equation (4) becomes, on multiplying by x

$$\left[\frac{v^2}{2!}\phi_n''(m_1)+v\phi'_{n-1}(m_1)+\phi_{n-2}(m_1)\right]$$

$$+\frac{1}{x}\left[\frac{v^3}{3!}\phi_n'''(m_1)+\frac{v^2}{2!}\phi''_{n-1}(m_1)+v\phi'_{n-2}(m_1)+\phi_{n-3}(m_1)\right]+\ldots=0.$$

On taking limits, as x and $y\to\infty$, we get

$$\frac{c^2}{2!}\phi_n''(m_1)+c\phi_{n-1}'(m_1)+\phi_{n-2}(m_1)=0$$

which gives two values of c, say c_1', c_1''

$\therefore$ $$y=m_1x+c_1' \text{ and } y=m_1x+c_1''$$

are the two parallel asymptotes corresponding to the two equal roots m_1, m_1 of $\phi_n(m)=0$. The discussion can be extended on similar lines, if necessary.

It may be observed that $\phi_n(m)$ is obtained from $x^n\phi_n(y/x)$, *i.e.*, from the terms of the nth degree by putting $x=1$ and $y=m$.

Note 1. Asymptotes corresponding to $m=0$ as a root of $\phi(m)=0$ are parallel to x-axis and are obtained directly by the method of Art. 14·42.

Note 2. When called upon to find the asymptotes of any curve, it is advisable to *first* find the asymptotes parallel to the axes and *then* the oblique asymptotes.

Ex. 1. *Find the asymptotes of the curve*

$$x^3+2x^2y-xy^2-2y^3+4y^2+2xy+y-1=0. \quad \textit{(Panjab, 1946)}$$

The curve is of the 3rd degree. Putting $x=1$ and $y=m$ in the 3rd and 2nd degree terms separately, we get

$$\phi_3(m)=1+2m-m^2-2m^3,$$
$$\phi_2(m)=4m^2+2m.$$

The slopes of the asymptotes are given by $\phi_3(m)=0$,

i.e., $$1+2m-m^2-2m^3=0, \text{ or } (1+2m)(1-m^2)=0,$$

whence $$m=-\tfrac{1}{2},\ 1,\ -1.$$

Also $$\phi_3'(m)=2-2m-6m^2.$$

Now c is given by $c\phi_3'(m)+\phi_2(m)=0$, *i.e.*, by

$$c(2-2m-6m^2)+(4m^2+2m)=0,$$

whence $$c=-\frac{2m^2+m}{1-m-3m^2}.$$

For $m=1,\ -1,\ -\frac{1}{2}$, we get $c=1,\ 1,\ 0$ respectively. Hence the asymptotes of the curve are

$$y=x+1,\ y=-x+1,\ y=-\tfrac{1}{2}x.$$

Ex. 2. *Find the asymptotes of the curve*

$$x^3+4x^2y+5xy^2+2y^3+2x^2+4xy+2y^2-x-9y+1=0.$$

The curve is of the 3rd degree and x^3 and y^3 are both present

in the equation. Hence there are no asymptotes parallel to the axes.

Here $\phi_3(m)=1+4m+5m^2+2m^3,$

$$\phi_2(m)=2+4m+2m^2,$$

$$\phi_1(m)=-1-9m.$$

The slopes of the asymptotes are the roots of $\phi_3(m)=0$.

i.e., of $1+4m+5m^2+2m^3=0,$

or of $(1+2m)(1+m)^2=0.$

$\therefore$ $m=-1,\ -1,\ -\frac{1}{2}.$

Also $\phi_3'(m)=4+10m+6m^2.$

Now c is given by $c\phi_3'(m)+\phi_2(m)=0$

i.e.,
$$c(4+10m+6m^2)+(2+4m+2m^2)=0 \qquad \ldots(i)$$

For $m=-\frac{1}{2}$, (*i*) gives $c=-\dfrac{2+4(-\frac{1}{2})+2(-\frac{1}{2})^2}{4+10(-\frac{1}{2})+6(-\frac{1}{2})^2}=-1.$

Hence the corresponding asymptote is

$$y=-\tfrac{1}{2}x-1 \quad \text{or} \quad x+2y+2=0,$$

When $m=-1$, (*i*) fails to give c as it takes the form

$$c.0+0=0.$$

c is now obtained from the equation

$$\frac{c^2}{2!}\phi_3''(m)+c\phi_2'(m)+\phi_1(m)=0$$

or
$$\frac{c^2}{2}(10+12m)+c(4+4m)+(-1-9m)=0.$$

For $m=-1$, this becomes,

$$-c^2+8=0 \quad \text{whence } c=\pm 2\sqrt{2}.$$

Hence the corresponding asymptotes are

$$y=-x\pm 2\sqrt{2}, \quad \text{or} \quad x+y\pm 2\sqrt{2}=0.$$

14·61. A few observations. (1) $\phi_n(m)=0$, which gives the directions of the asymptotes, is, in general, of the nth degree and has n roots, real or imaginary. Hence there are, in general, n asymptotes, real or imginary, to a curve of the nth degree.

(2) The equations of the asymptotes depend, in general, on the terms of the nth and $(n-1)$th degrees. Hence all curves of the nth degree having the same nth and $(n-1)$th degree terms, have generally the same asymptotes.

(3) The equation $x^n\phi_n(y/x)=0$ (obtained by equating the nth degree terms to zero) is homogeneous of the nth degree and, therefore, represents n straight lines. The slopes of these lines are the roots of $\phi_n(m)=0$ which also give the slopes of the asymptotes.

Hence $x^n\phi_n(y/x)=0$ represents n straight lines through the origin parallel to the asymptotes of the curve.

(4) Every odd degree equation has at least one real root. Hence if n is odd, $\phi_n(m)=0$ will have at least one real root and, therefore, the curve will have one real asymptote. An odd degree curve, therefore, must have at least one infinite branch and cannot be a closed curve.

(5) Imaginary roots of an algebraic equation always occur in conjugate pairs. Hence an even degree curve has either an even number of real asymptotes or no asymptotes at all.

If all the roots of $\phi_n(m)=0$ are imaginary, the curve must be a closed curve.

Examples L

Find the asymptotes of the following curves :

1. $y^3-x^2y+2y^2+4y+x=0.$
2. $x^2y+xy^2+xy+y^2+3x=0.$ *(Panjab, 1960)*
3. $(a-x)y^2=(a+x)x^2.$
4. $x(y+b)^2=y(x+a)^2.$
5. $y^2(x-b)=x^3+a^3.$ *(Panjab, Sept. 1950)*
6. $x^3+3x^2y-xy^2-3y^3+x^2-2xy-3y^2+4x+5=0.$ *(Panjab, 1945)*
7. $y^2(a^2-x^2)=x^4.$ *(Aligarh, 1935)*
8. $x^3-4xy^2-3x^2+12xy-12y^2+8x+2y+4=0.$ *(Allahabad, 1942)*
9. $y^3-2y^2x-yx^2+2x^3+y^2-6xy+5x^2-2y+2x+1=0.$ *(Patna, 1945)*
10. $x^3-x^2y-xy^2+y^3-3x+y+7=0.$
11. $x^3+3x^2y-4y^3-x+y+3=0.$ *(Panjab, 1961)*
12. $4x^3-3xy^2-y^3+2x^2-xy-y^2=1.$ *(Sagar, 1948)*
13. $x^3-2x^2y+xy^2+x^2-xy+2=0.$ *(Agra, 1951)*
14. $x^3y-2x^2y^2+xy^3=a^2x^2+b^2y^2.$ *(Panjab, 1943)*
15. $x^2(x-y)^2+a^2(x^2-y^2)-a^2xy=0.$ *(Delhi ; Panjab, 1951)*

14·62. **Miscellaneous methods.** I. If the equation of a curve of the nth degree can be put in the form

$$(ax+by+c)P_{n-1}+F_{n-1}=0,$$

where P_r and F_r denote rational algebraic expressions which contain rth and lower degree terms but no higher, then the asymptote parallel to $ax+by=0$ is given by

$$ax+by+c=-\text{Lt}\frac{F_{n-1}}{P_{n-1}}$$

where x and $y\to\infty$ in such a way that $\text{Lt}(y/x)=-a/b$.

Ex. 1. *Find the asymptotes of the curve*

$$(x+y+1)(x^2+y^2-xy)-3xy+x^2+y^2+2x-3y+5=0.$$

The asymptote parallel to $x+y=0$ is given by

$$x+y+1=\text{Lt}\frac{3xy-x^2-y^2-2x+3y-5}{x^2+y^2-xy},$$

where x and $y\to\infty$ in such a way that Lt $(y/x)=-1$,

$$=\text{Lt}\frac{3\left(\frac{y}{x}\right)-1-\left(\frac{y}{x}\right)^2-\frac{2}{x}+3\frac{y}{x^2}-\frac{5}{x^2}}{1+\left(\frac{y}{x}\right)^2-\frac{y}{x}}$$

$$=\text{Lt}\frac{-3-1-1}{1+1+1}=-\frac{5}{3}.$$

$\therefore$ $3x+3y+8=0$ is an asymptote.

Since x^2+y^2-xy cannot be broken into real factors, the other two asymptotes are imaginary.

Hence $3x+3y+8=0$ is the only asymptote.

Note. Since, when x and $y\to\infty$,

$$\text{Lt}\frac{y^r}{x^s}=\text{Lt}\frac{1}{x^{s-r}}\left(\frac{y}{x}\right)^r=0,$$

provided $r<s$, the working can be greatly simplified in practice by neglecting terms in the numerator and denominator which are of a degree lower than that of the denominator. Thus in the example solved above, the asymptote parallel to $x+y=0$ may be taken as

$$x+y+1=\text{Lt}\frac{3xy-x^2-y^2}{x^2+y^2-xy}.$$

II. With the same notation as above, if the equation of a curve can be put in the form

$$(ax+by+c)P_{n-1}+F_{n-2}=0,$$

$ax+by+c=0$ is itself, an asymptote provided P_{n-1} does not contain any factor of the form $ax+by+k$.

If, however, $ax+by+k$ is a factor of P_{n-1}, the equation of the curve should be put in the form

$$(ax+by)^2P_{n-2}+(ax+by)Q_{n-2}+F_{n-2}=0,$$

where P_r, Q_r and F_r stand for expressions containing terms of the rth and lower degrees but of no higher degree. Then the asymptotes parallel to $ax+by=0$ are given by

$$(ax+by)^2+(ax+by)\text{ Lt }\frac{Q_{n-2}}{P_{n-2}}+\text{Lt}\frac{F_{n-2}}{P_{n-2}}=0$$

where x and $y\to\infty$ such that Lt $(y/x)=-a/b$.

Ex. 2. *Find the asymptotes of the curve*

$$(x+y+1)^2(x^2+y^2-xy)+3xy-7x^2-2y^2-7x+8=0.$$

The asymptotes parallel to $x+y=0$ are given by

$$(x+y+1)^2+\text{Lt}\frac{3xy-7x^2-2y^2-7x+8}{x^2+y^2-xy}=0,$$

[where x and $y \to \infty$ so that $y/x \to -1$]

i.e., by $$(x+y+1)^2+\text{Lt}\frac{3xy-7x^2-2y^2}{x^2+y^2-xy}=0,$$

i.e., by $$(x+y+1)^2=-\text{Lt}\frac{3(y/x)-7-2(y/x)^2}{1+(y/x)^2-(y/x)}$$
$$=-\frac{-3-7-2}{1+1+1}=4.$$

$\therefore$ The two asymptotes are $x+y+1=\pm 2$ or
$$x+y+3=0 \text{ and } x+y-1=0.$$

Ex. 3. *Find the asymptotes of*
$$(x-y)^2(x^2+y^2)-10(x-y)x^2+12y^2+2x+y=0.$$

The equation is of the 4th degree.

$\because$ $x^2+y^2=0$ represents an imaginary line-pair, the two asymptotes parallel to these lines are imaginary. The two asymptotes parallel to $x-y=0$ are given by
$$(x-y)^2-10(x-y)\ \text{Lt}\frac{x^2}{x^2+y^2}+\text{Lt}\frac{12y^2}{x^2+y^2}=0,$$
[where x and $y \to \infty$ such that Lt $y/x=1$]

i.e., $$(x-y)^2-10(x-y)\ \text{Lt}\ \frac{1}{1+(y/x)^2}+\text{Lt}\frac{12(y/x)^2}{1+(y/x)^2}=0$$

i.e., $$(x-y)^2-10(x-y)\ \frac{1}{1+1}+\frac{12(1)}{1+1}=0,$$

i.e., $$(x-y)^2-5(x-y)+6=0,$$

i.e., $$(x-y-2)(x-y-3)=0.$$

Thus the two parallel asymptotes are
$$x-y-2=0 \text{ and } x-y-3=0.$$

14·63. **Asymptotes by inspection.** If the equation of a curve be of the form
$$F_n+F_{n-2}=0$$
where F_r is of degree r at the most, then every linear factor of F_n equated to zero will give an asymptote provided that no two asymptotes so obtained are parallel are coincident.

Let $ax+by+c$ be a non-repeated linear factor of F_n, and let $F_n=(ax+by+c)P_{n-1}$. Then the equation of the curve may be written as
$$(ax+by+c)P_{n-1}+F_{n-2}=0.$$

Hence by II Art. 14·62 above, $ax+by+c=0$ is an asymptote.

Cor. If $u_r \equiv a_rx+b_ry+c_r$, the curve
$$u_1u_2\ldots\ldots\ldots u_n+F_{n-2}=0$$
has $u_1=0$, $u_2=0, \ldots\ldots, u_n=0$ as asymptotes, provided that no two lines so obtained are parallel.

Conversely, a curve having $u_1=0, u_2=0, \ldots\ldots\ldots\ldots, u_n=0$ as asymptotes must have its equation of the form

$$u_1u_2\ldots\ldots u_n+F_{n-2}=0.$$

Ex. 1. *Find the asymptotes of the curve*

$$xy(x^2-y^2)+2x^2+2y^2+1=0.$$

This is of the form $F_n+F_{n-2}=0$ $(n=4)$. The linear factors of F_4 are $x, y, (x-y)$ and $(x+y)$. Since none of them is repeated, the four asymptotes of the curve are

$$x=0,\ y=0,\ x-y=0 \text{ and } x+y=0.$$

Ex. 2. *Find the asymptotes of the curve*

$$(x-y+1)(x+2y-3)(3x+4y-7)+5x-y+1=0.$$

By Cor. Art. 14·63 the three asymptotes of the curve are

$$x-y+1=0,\ x+2y-3=0 \text{ and } 3x+4y-7=0.$$

Examples LI

Find the asymptotes of the following curves :—

1. $xy(x-y)=a(x^2+a^2)$. 2. $y^2(x^2-a^2)=x$. (*Delhi, 1957*)
3. $(x+y-1)(x+3y-1)(3x-y+2)+3x+4y+5=0$.
4. $(x^2-y^2)(2x+3y)(3x+2y)+2x^2+3y^2-6=0$.
5. $y^3+x^2y-2xy^2-y+1=0$. (*Panjab, 1954 S*)
6. $(x+y)(x-y)(2x-y)-4x(x-2y)+4x=0$. (*Panjab B.Sc., 1961*)
7. $y^3-2xy^2-x^2y+2x^3+2x^2-3xy+x-2y+1=0$.
8. $(y-x)(y-2x)^2+(y+3x)(y-2x)+2x+2y-1=0$, (*Delhi, 1954*)
9. $(x+y)^2(x+2y+2)=x+9y-2$. (*Panjab, 1956 ; Delhi, 1958*)
10. $(x-y)^2(x-2y)(x-3y)-2a(x^3-y^3)-2a^2(x+y)(x-2y)=0$.
11. $x^2y-xy^2+xy+y^2+x-y=0$. (*Panjab, 1955*)
12. $(x^2-y^2)(x+2y+1)+x+y+1=0$. (*Delhi, 1956*)

14·7. Intersection of a curve and its asymptotes. *To prove that an asymptote of an algebraic curve of the nth degree meets the curve in at least two points at infinity.*

Let the equation of the algebraic curve of the nth degree be

$$x^n\phi_n(y/x)+x^{n-1}\phi_{n-1}(y/x)+x^{n-2}\phi_{n-2}(y/x)+\ldots\ldots=0. \qquad \ldots(1)$$

Let

$$y=my+c \qquad \ldots(2)$$

be an asymptote, then m and c are given by

$$\phi_n(m)=0 \text{ and } c\phi_n'(m)+\phi_{n-1}(m)=0. \qquad \ldots(3)$$

We assume that no other asymptote of the curve is parallel to $y=mx+c$.

Substituting for y from (2) in (1), the abscissae of the points of intersection of the curve and the asymptotes are given by

$$x^n\phi_n\left(m+\frac{c}{x}\right)+x^{n-1}\phi_{n-1}\left(m+\frac{c}{x}\right)+\ldots=0.$$

Expanding each term by Taylor's Theorem and re-arranging, we get

$$x^n\phi_n(m)+x^{n-1}\{c\phi_n'(m)+\phi_{n-1}(m)\}$$
$$+x^{n-2}\left\{\frac{c^2}{2\,!}\phi_n''(m)+c\phi_{n-1}'(m)+\phi_{n-2}(m)\right\}+\ldots=0. \quad \ldots(4)$$

The coefficients of x^n and x^{n-1} both vanish by virtue of equations (3) above. Hence two roots of equation (4) are infinite. Thus every asymptote of the curve meets the curve in at least two points at infinity.

14·71. A straight line cuts a curve of the nth degree in n points, real or imaginary. $\therefore$ an asymptote cuts such a curve in n points two of which lie at infinity. Thus there are, in general, $n-2$ finite points of intersection of an asymptote and a curve of the nth degree. Now a curve of the nth degree possesses n asymptotes, real or imaginary. Hence these n asymptotes meet the curve in $n(n-2)$ finite points besides the $2n$ points at infinity.

Assuming that the curve has no parallel asymptotes, its equation may be put in the form

$$F_n+F_{n-2}=0, \qquad \ldots(i)$$

so that
$$F_n=0 \qquad \ldots(ii)$$

represents the n asymptotes of the curve, real or imaginary.

The points of intersection satisfy (*i*) and (*ii*), $\therefore$ they also satisfy

$$(F_n+F_{n-2})-F_n=0, \ \textit{i.e.,}\ F_{n-2}=0. \qquad \ldots(iii)$$

Hence the $n(n-2)$ finite points of intersection of a curve of the nth degree and its n asymptotes lie on a curve of the $(n-2)$th degree. In particular :

(*i*) The $4\times2=8$ points of intersection of a curve of the fourth degree and its four asymptotes lie on a curve of the second degree, *viz.*, a conic.

(*ii*) The $3\times1=3$ points of intersection of a cubic curve and its three asymptotes lie on a curve of the first degree *i.e.*, a straight line.

Ex. *Show that the asymptotes of the curve*

$$(x^2-y^2)(y^2-4x^2)-6x^3+5x^2y+3xy^2-2y^3-x^2+3xy-1=0$$

cut the curve again in eight points which lie on a circle.

The curve is of the fourth degree and there are no asymptotes parallel to the axes.

Now
$$\phi_4(m)=(1-m^2)(m^2-4),$$
$$\phi_3(m)=-6+5m+3m^2-2m^3.$$

The slopes of the asymptotes are given by $\phi_4(m)=0$

i.e.,
$$(1-m^2)(m^2-4)=0. \qquad \therefore\ m=\pm1,\ \pm2.$$

Also c is given by $c\phi_4'(m)+\phi_3(m)=0$,

i.e.,
$$c(10m-4m^3)+(-6+5m+3m^2-2m^3)=0,$$

whence $c=0, -1, 0$ and -1 corresponding to $m=1, -1, 2$ and -2 respectively

$\therefore$ the four asymptotes are

$$y=x,\ y=-x-1,\ y=2x,\ y=-2x-1.$$

The joint equation of the four asymptotes is

$$(y-x)(y+x+1)(y-2x)(y+2x+1)=0,$$

or $F_4\equiv 5x^2y^2-4x^4-y^4-2y^3+3xy^2+5x^2y-6x^3-y^2+3xy-2x=0.$

The equation of the curve may be written as

$$5x^2y^2-4x^4-y^4-2y^3+3xy^2+5x^2y-6x^3-y^2+3xy-2x+(x^2+y^2-1)=0$$

which is of the form $F_4+F_2=0$.

Hence the $4(4-2)$, *i.e.*, 8 points of intersection of the curve and its four asymptotes lie on the curve $F_2=0$ *i.e.*, on the circle

$$x^2+y^2-1=0.$$

14·72. Equation of a curve with given asymptotes. The method of writing down the equation of a curve when its asymptotes are given is illustrated by the following example. [Also ref. Cor. § 14·63 of this Chapter].

Ex. *Find the equation of the hyperbola having $x+y-1=0$ and $x-y+2=0$ as its asymptotes and passing through the origin.*

The general equation of a curve having the given lines as asymptote is

$$(x+y-1)(x-y+2)+c=0. \qquad [F_2+F_0=0]$$

$\because$ this passes through the origin,

$$-2+c=0 \quad \text{or} \quad c=2.$$

$\therefore$ the required equation is

$$(x+y-1)(x-y+2)+2=0.$$

Examples LII

1. Show that the asymptotes of the cubic

$$x^3-2y^3+xy(2x-y)+y(x-y)+1=0$$

cut the curve in three points which lie on the line

$$x-y+1=0. \qquad \textit{(Sagar, 1948)}$$

2. Show that the asymptotes of the cubic

$$x^2y-xy^2+xy+y^2+x-y=0$$

cut the curve again in three points which lie on the line $x+y=0$.

(Panjab, 1940)

3. Show that the eight points of intersection of the curve

$$xy(x^2-y^2)+x^2+y^2=a^2$$

and its asymptotes lie on a circle.

4. Show that the four asymptotes of the curve

$$xy(x^2-y^2)+25y^2+9x^2=144$$

cut the curve again in eight points which lie on an ellipse of eccentricity 4/5.

5. Find the equation of the quartic curve which has $x=0$, $y=0$, $y=x$, $y=-x$ as asymptotes, which passes through (a, b) and cuts its asymptotes again in eight points lying on the circle

$$x^2+y^2=a^2. \qquad \textit{(Bombay, 1947)}$$

6. Show that the eight points of intersection of the curve

$$x^4-5x^2y^2+4y^4+x^2-y^2+x+y+1=0$$

and its asymptotes lie on a rectangular hyperbola. *(Allahabad, 1946)*

7. Show that the equation of the cubic curve which has the same asymptotes as the curve

$$x^3-6x^2y+11xy^2-6y^3+x+3y+5=0$$

nd which passes through the points (0, 0), (0, 1) and (1, 0) is

$$x^3-6x^2y+11xy^2-6y^3-x+6y=0.$$

8. Find the equation of the hyperbola having the same ymptotes as

$$x^2-5xy+6y^2+5x-11y+9=0$$

ıd passing through (1, 1).

14·8. Method of expansion. If it is possible to express y rom the equation of a curve in the form

$$y=mx+c+\frac{\alpha}{x}+\frac{\beta}{x^2}+\frac{\gamma}{x^3}+\ldots\ldots \qquad \ldots(i)$$

then $y=mx+c$ is an asymptote to the curve.

From (i), $\dfrac{y}{x}=m+\dfrac{c}{x}+\dfrac{\alpha}{x^2}+\ldots\ldots$

$\therefore \qquad \text{Lt}\dfrac{y}{x}=m$ when x and $y\to\infty$

Again from (i)

$$y-mx=c+\frac{\alpha}{x}+\frac{\beta}{x^2}+\ldots\ldots\ldots$$

$\therefore$ Lt $(y-mx)=c$ when x and $y\to\infty$ so that $y/x\to m$.

Hence $\qquad y=mx+c$ is an asymptote.

Cor. A straight line is an asymptote to a curve if the difference between the ordinates of the curve and the line corresponding to a common abscissa approaches zero as a limit when the abscissa tends to infinity.

14·81. The problem of finding an asymptote to a curve is thus

reduced to the problem of expanding y from the equation of the curve in the form

$$y=mx+c+\frac{\alpha}{x}+\frac{\beta}{x^2}+\frac{\gamma}{x^3}+\ldots\ldots$$

If the equation of a curve is given explicitly, or if it is possible easily to express y explicitly, the expansion of y in descending powers of x is readily obtained by means of the Binomial Theorem. The method also applies when x is expressed explicitly in terms of y.

If, however, it is not possible readily to expand y in descending powers of x as above, we assume the above expansion for y and substitute this series for y in the equation of the curve treating the resulting equation as an identity. By equating the coefficients of different powers of x, equations are obtained to determine $m, c, \alpha, \beta, \ldots\ldots\ldots\ldots$

14·82. Position of a curve relative to its asymptotes.

Let
$$y=mx+c+\frac{\alpha}{x}+\frac{\beta}{x^2}+\ldots\ldots \qquad \ldots(1)$$

represent the curve (or one of its branches). Then

$$y=mx+c \qquad \ldots(2)$$

is an asymptote to the curve.

Let y_c and y_a denote the ordinates of the curve and the asymptote respectively corresponding to any value of x. Then

$$y_c-y_a=\frac{\alpha}{x}+\frac{\beta}{x^2}+\ldots\ldots$$

The sign of the right member of this equation and, therefore, of y_c-y_a is determined by the first term α/x or, if $\alpha=0$, by the next term β/x^2, and so on, when x is sufficiently large.

The curve lies above or below the asymptote according as

$$y_c > \text{ or } < y_a$$

i.e., according as
$$y_c-y_a > \text{ or } < 0,$$

i.e., according as
$$\frac{\alpha}{x} > \text{ or } < 0.$$

Hence if x is positive (*i.e.*, for branches of the curve in the first and the fourth quadrants) the curve lies above or below the asymptote according as α is positive or negative.

Again if x is negative (*i.e.*, for branches of the curve in the second and third quadrants) the curve lies above or below the asymptote according as α is negative or positive.

It may also be noted that α/x changes sign with x, *i.e.*, it has one sign if $x\to+\infty$ and another when $x\to-\infty$. Hence the curve will lie above the asymptote at one end and below it at the other.

If, however, $\alpha=0$, the sign of the right-member and, therefore,

of $y_c - y_a$ depends upon β/x^2 and, therefore, on the sign of β, as x^2 remains positive whether $x \to +\infty$ or $-\infty$.

Thus the curve lies above or below the asymptote according as β is positive or negative.

Again, since β/x^2 does not change sign with x, the curve lies on the same side of the asymptote at either end.

The discussion can be extended on similar lines if β is also zero, and so on.

Ex. 1. *Find the asymptotes of the hyperbola*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

and discuss the position of the hyperbola relative to its asymptotes.

Solving the equation of the hyperbola for y, we get

$$y = \pm \frac{bx}{a}\left(1 - \frac{a^2}{x^2}\right)^{\frac{1}{2}} = \pm \frac{bx}{a}\left(1 - \frac{a^2}{2x^2} - \frac{a^4}{8x^4} - \dots\right)$$

$$\therefore \quad y = \frac{b}{a}x - \frac{ab}{2}\frac{1}{x} - \frac{a^2b}{8}\frac{1}{x^2} - \dots\dots \qquad \dots(i)$$

or

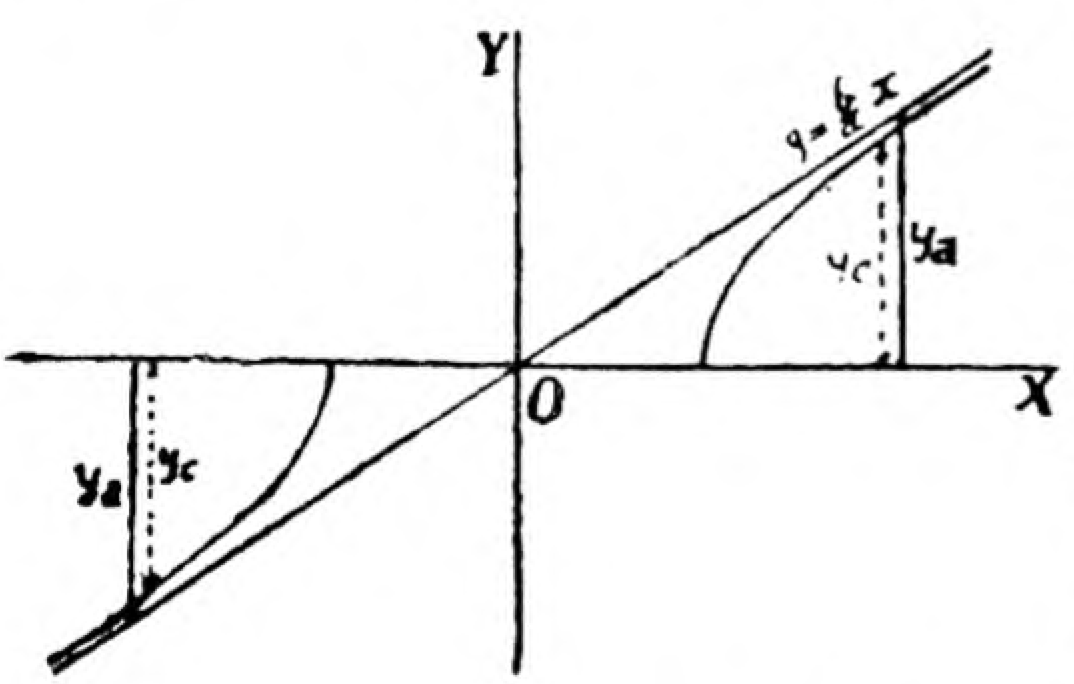
$$y = -\frac{b}{a}x + \frac{ab}{2}\frac{1}{x} + \frac{a^2b}{8}\frac{1}{x^2} + \dots\dots \qquad \dots(ii)$$

Thus the corresponding asymptotes of the hyperbola are respectively

$$y = \frac{b}{a}x, \qquad \dots(iii)$$

and

$$y = -\frac{b}{a}x. \qquad \dots(iv)$$



Let y_c denote the ordinate of any point of the hyperbola and y_a the corresponding ordinate of the asymptote to that branch. [The figure shows branch (*i*) and the corresponding asymptote (*iii*).]

Then from (*i*) and (*iii*),

$$y_c - y_a = -\frac{ab}{2}\frac{1}{x} - \frac{a^2b}{8}\frac{1}{x^2} - \dots\dots$$

Now for positive values of x, $-(ab/2x)$ is negative $\therefore$ $y_c - y_a$ is negative, *i.e.*, the ordinate of the curve is less than the corresponding ordinate of the asymptote. Hence the curve lies below the asymptote in the first quadrant.

Again, for negative values of x, $-(ab/2x)$ is positive, $\therefore$ y_c-y_a is positive, *i.e.*, the curve lies above the asymptote in the third quadrant.

The position of branch (*ii*) relative to the corresponding asymptote (*iv*) may be discussed similarly.

Ex. 2. *Find the asymptote of the curve*

$$x^3+y^3=3axy, \qquad \text{...(1)}$$

and discuss its position relative to the curve.

Let $$y=mx+c+\frac{\alpha}{x}+\frac{\beta}{x^2}+\dots\dots \qquad \text{...(2)}$$

represent an infinite branch of the curve. Substituting for y from (2) in (1), we get

$$x^3+\left(mx+c+\frac{\alpha}{x}+\frac{\beta}{x^2}+\dots\dots\right)^3=3ax\left(mx+c+\frac{\alpha}{x}+\frac{\beta}{x^2}+\dots\right)$$

Equating the coefficients of x^3, we get

$$m^3+1=0 \text{ whence } m=-1,$$

which is the only real root showing that there can be only one asymptote.

Equating the coefficients of x^2, x and the constant term, we get,

$$3m^2c=3am. \qquad \text{...(3)}$$
$$3mc^2+3m^2\alpha=3ac, \qquad \text{...(4)}$$

and $$c^3+3m^2\beta+6mc\alpha=3a\alpha, \qquad \text{...(5)}$$

Substituting $m=-1$ and solving, we get

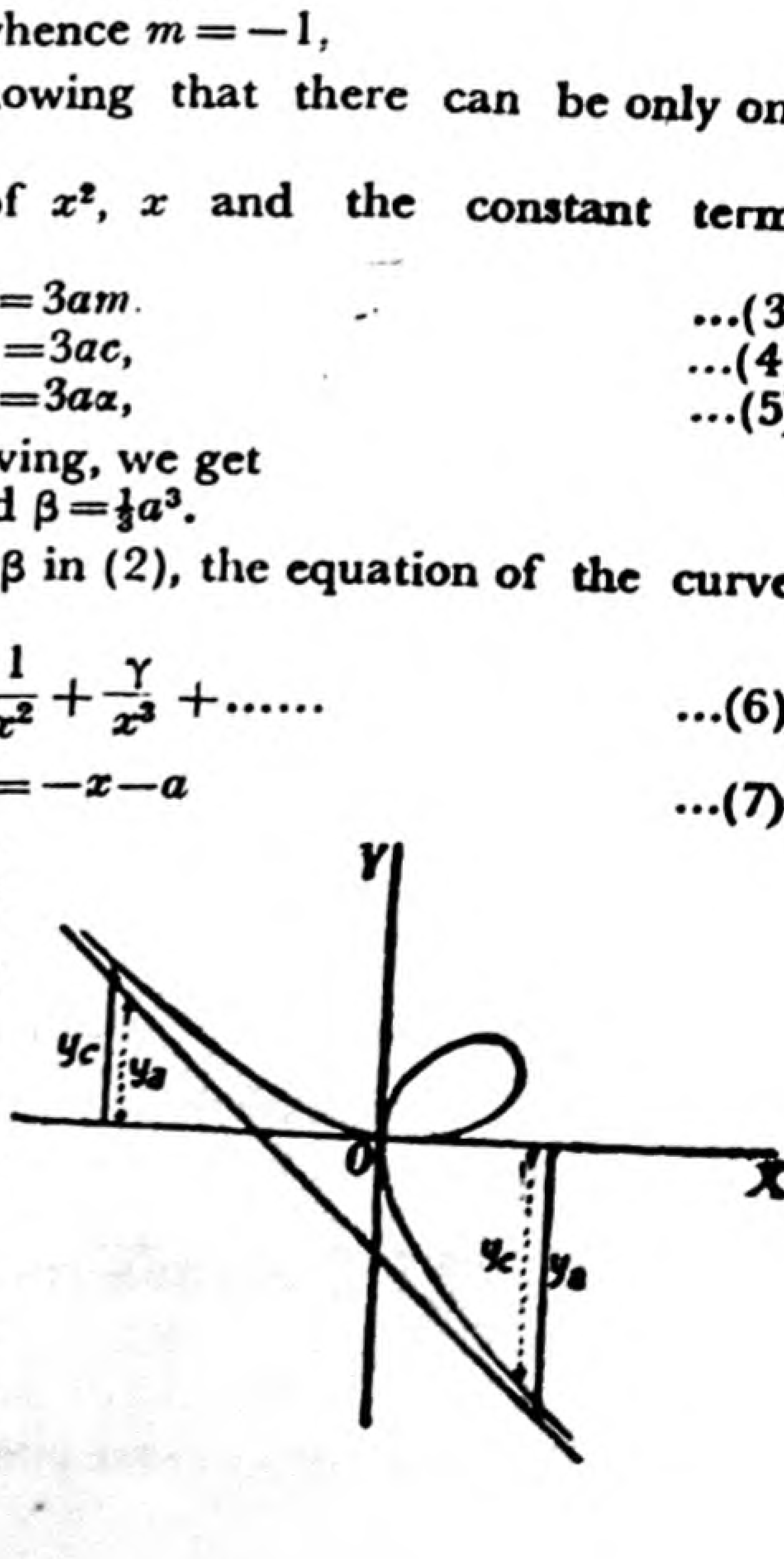
$$c=-a,\ \alpha=0, \text{ and } \beta=\tfrac{1}{3}a^3.$$

Substituting for m, c, α, and β in (2), the equation of the curve may be written as

$$y=-x-a+\frac{a^3}{3}\,\frac{1}{x^2}+\frac{\gamma}{x^3}+\dots\dots \qquad \text{...(6)}$$

Hence $$y=-x-a \qquad \text{...(7)}$$

is the only asymptote of the curve.



If y_c denotes the ordinate of any point on the curve and y_a the corresponding ordinate of the asymptote, then from (6) and (7),

$$y_c-y_a=\frac{a^3}{3}\,\frac{1}{x^2}+\frac{\gamma}{x^3}+\dots\dots\dots,$$

Since $a^3/3x^2$ is positive whether x is positive or negative, y_c-y_a is positive whether $x\to\infty$ or $-\infty$. Hence the curve lies above the asymptote at either end.

The curve and the asymptote have been drawn in the attached figure.

14·83. Method of approximation. If, for any infinite branch of a curve, it is possible to expand y in descending powers of x as

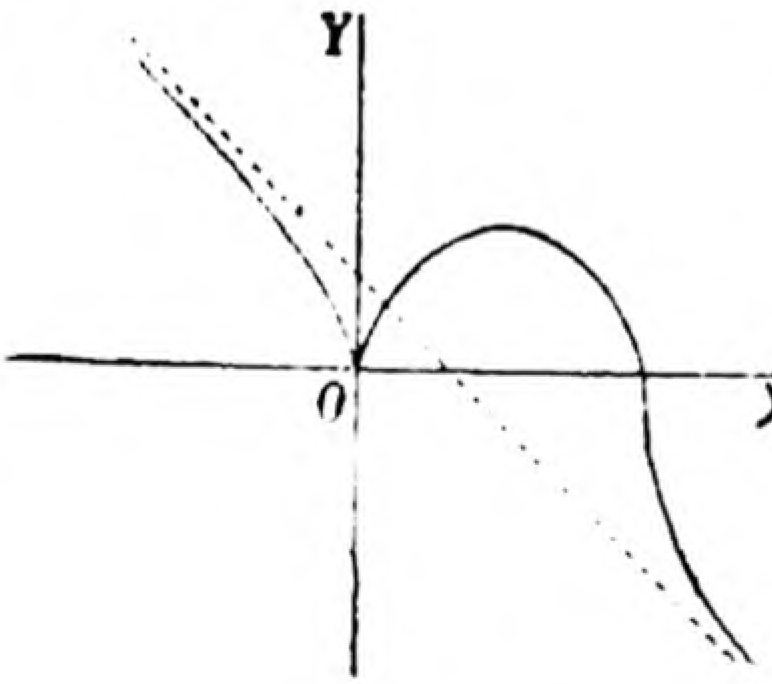
$$y=mx+c+\frac{\alpha}{x}+\frac{\beta}{x^2}+\frac{\gamma}{x^3}+\ldots\ldots$$

then it is easy to see that

$$y=mx+c \qquad \text{(asymptote)}$$
$$y=mx+c+\alpha/x$$
$$y=mx+c+\alpha/x+\beta/x^2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

are successively closer and closer approximations to the curve at infinity.

A linear approximation more removed than the asymptote may be taken as $y=mx$ itself (provided $c\neq 0$). Now for very large values of x and y, it is the highest degree terms which matter. Hence any approximation of the curve at infinity must come out of these. For successive closer approximations we successively associate terms of the lower degrees with them. The following examples are suggestive and indicate how we may proceed to get approximations to the curve at infinity.



Ex. 1. *Discuss the form of* $x^3+y^3=3xy^2$ *at infinity and find how the curve is situated relative to its asymptote.*

The highest degree terms equated to zero give $x^3+y^3=0$ which represents the curve at infinity. The only real factor of the left-member being $x+y$, we have as a first approximation

$$y+x=0.$$

For a second approximation,

$$y+x=\frac{3ax^2}{x^2-xy+y^2}=\frac{3ax^2}{x^2-x(-x)+(-x)^2} \qquad [\text{Putting } y=-x]$$
$$=a,$$

giving $x+y-a=0$ as an asymptote.

A third approximation is obtained by writing $(a-x)$ for y: thus

$$y+x=\frac{3ax^2}{x^2-x(a-x)+(a-x)^2}=a\left\{1-\frac{a}{x}+\frac{a^2}{3x^2}\right\}^{-1}$$
$$=a+\frac{a^2}{x},$$

omitting $1/x^2$ and higher order infinitesimals.

$\because$ a^2 is always positive, the curve lies above the asymptote when x is positive and very large and it lies below the asymptote at the other end.

Ex. 2. *Discuss the position of the curve* $(x-a)(x-b)y^2=a^2x^2$, $a>b>0$, *relative to its asymptotes.*

$x=a$, $x=b$, $y=\pm a$ are the four asymptotes, y is infinite corresponding to $x=a$ and $x=b$; similarly x is infinite when $y=\pm a$. Denote the corresponding points at infinity on the curve by (a, ∞), (b, ∞), (∞, a), and $(\infty, -a)$ respectively. Then we see that :

(*i*) near (a, ∞)

$$x-a=\frac{a^2x^2}{(x-b)y^2}=\frac{a^4}{(a-b)y^2} \qquad \text{[Putting } x=a$$

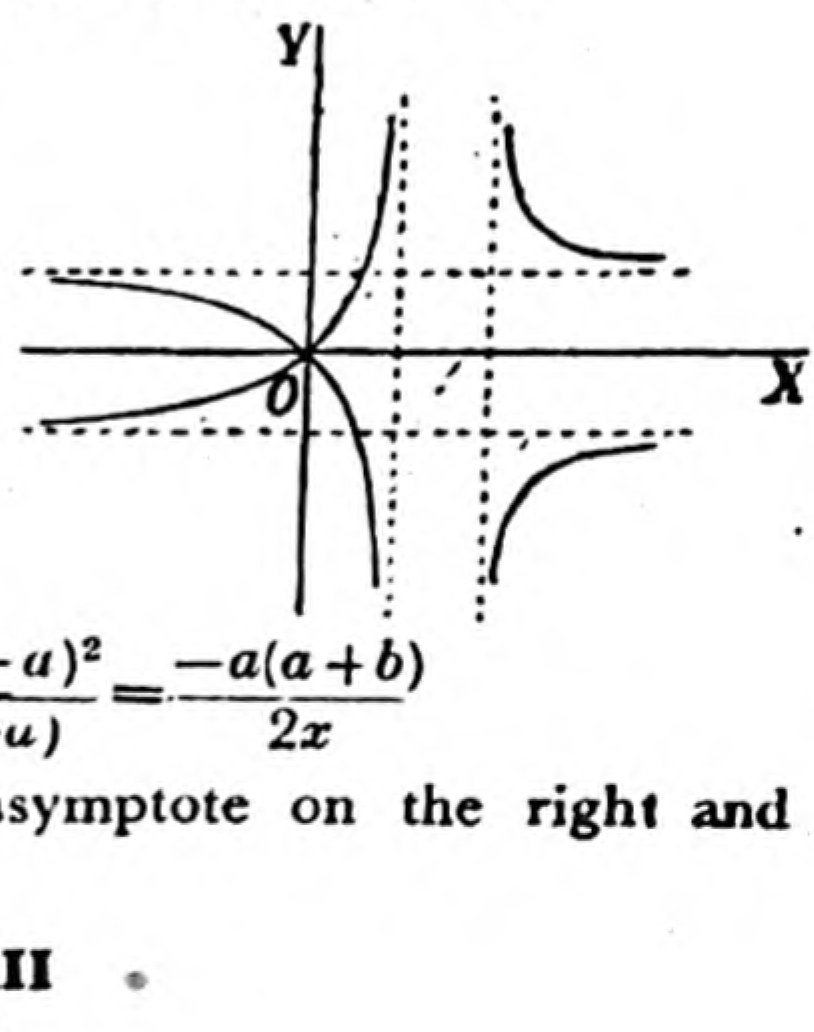
showing that the curve lies to the right of the asymptote $x=a$.

(*ii*) near (b, ∞),

$$x-b=\frac{a^2x^2}{(x-a)y^2}=\frac{a^2b^2}{(b-a)y^2}=-\frac{a^2b^2}{(a-b)y^2}$$

showing that the curve lies to the left of the asymptote $x=b$.

Again, writing the equation as $x^2(y^2-a^2)=[(a+b)x-ab]y^2$, we observe that :



(*iii*) near (∞, a),

$$y-a=\frac{(a+b)xy^2}{x^2(y+a)}=\frac{a(a+b)}{2x}.$$

showing that the curve lies above the asymptote $y-a=0$ on the right and below it on the left of the y-axis.

(*iv*) $(\infty, -a)$.

$$y+a=\frac{(a+b)xy^2}{x^2(y-a)}=\frac{(a+b)(-a)^2}{x(-a-a)}=\frac{-a(a+b)}{2x}$$

showing that the curve lies below the asymptote on the right and above it on the left of the y-axis.

Examples LIII

1. Find the oblique asymptotes of

$$(x-1)y^2=x^3$$

and discuss the position of the curve relative to its asymptotes.

2. Find the position with regard to its asymptotes of the curve

$$x^3-y^3+a^2(x-2y)=0.$$

3. Find the asymptotes of the curve

$$(x+3a)y^2=x(x-a)(x-2a)$$

and determine on which side of its asymptotes the curve lies.

4. Determine the position of the curve

$$(x+y)(x-y)^2=a^2x$$

with regard to its asymptotes.

5. Determine the position of the curve

$$x(y-x)^2=8a^3$$

with regard to its asymptotes.

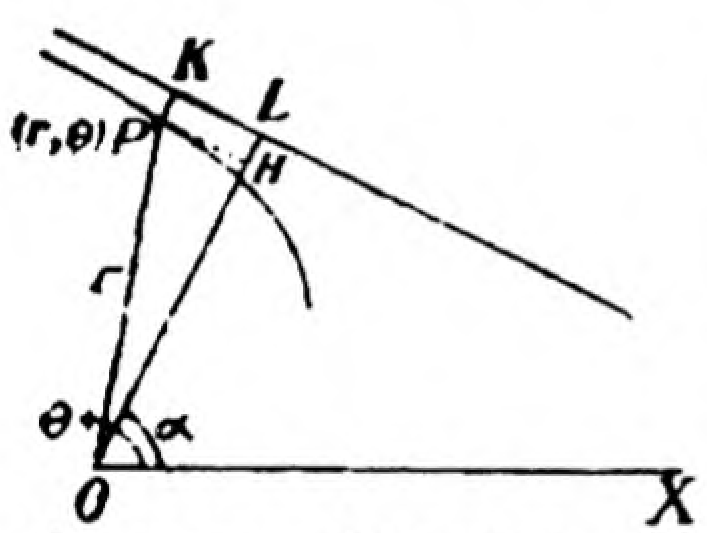
14·9. Asymptotes in polar coordinates. Let the equation of the curve be

$$r=f(\theta). \qquad ...(1)$$

The equation of a line in polar coordinates is

$$p=r\cos(\theta-\alpha), \qquad ...(2)$$

where p is the length of the perpendicular from the origin on the line and α is the angle which this perpendicular makes with the initial line.



Let $P(r, \theta)$ be any point on an infinite branch of the curve and let KL be the line (2). Further let $PK=\delta$ be the perpendicular from P on the line and let $OL=p$ be the perpendicular from the origin on the line KL. Draw $PH \perp OL$.

The line (2) is an asymptote of the curve (1) if $\delta \to 0$ as P tends to infinity on the curve. Now P tends to infinity on the curve if $r \to \infty$. From (1) let $\theta \to \theta_1$ as $r \to \infty$ on the infinite branch in question. Then from the figure

$$\delta=PK=HL=OL-OH=p-r\cos(\theta-\alpha)$$

$$\therefore \qquad \frac{\delta}{r}=\frac{p}{r}-\cos(\theta-\alpha). \qquad ...(3)$$

When $r \to \infty$ on the branch of the curve, $\delta \to 0$, $p/r \to 0$ and $\theta \to \theta_1$. Hence, in the limit, we get from (3),

$$\cos(\theta_1-\alpha)=0. \qquad ...(4)$$

$$\therefore \qquad \theta_1-\alpha=\tfrac{1}{2}\pi \quad \text{or} \quad \alpha=\theta_1-\tfrac{1}{2}\pi.$$

This determines the value of α for the asymptote, and therefore the direction of the asymptote.

Again, since $\delta=PK=HL \to 0$ as $r \to \infty$ on the curve, therefore

$$p=OL=\underset{r\to\infty}{\text{Lt}}\ OH=\underset{r\to\infty}{\text{Lt}}\ r\cos(\theta-\alpha)=\underset{u\to 0}{\text{Lt}}\ \frac{\cos(\theta-\alpha)}{u}.$$

which is of the form 0/0, since $\cos(\theta-\alpha)$ and $u=1/r$ both tend to zero when $r \to \infty$. Also when $r \to \infty$, $\theta \to \theta_1$, hence by the method for evaluating the limits of indeterminate forms,

$$p = \underset{\theta \to \theta_1}{\text{Lt}} \frac{\cos(\theta - \alpha)}{u} = \underset{\theta \to \theta_1}{\text{Lt}} \frac{-\sin(\theta - \alpha)}{\frac{du}{d\theta}}$$

$$= -\frac{\sin \frac{1}{2}\pi}{\underset{\theta \to \theta_1}{\text{Lt}} \left(\frac{du}{d\theta}\right)} = \underset{\theta \to \theta_1}{\text{Lt}} \left(-\frac{d\theta}{du}\right). \qquad \text{[by (4)]}$$

This determines p and thus substituting the values of p and α in (2), the equation of the asymptote is

$$\underset{\theta \to \theta_1}{\text{Lt}} \left(-\frac{d\theta}{du}\right) = r \cos\{\theta - (\theta_1 - \tfrac{1}{2}\pi)\} = r \sin(\theta_1 - \theta).$$

If there are more than one infinite branches of the curve, let $\theta_2, \theta_3, \ldots\ldots$ be the other values of θ which make r infinite (or u zero); we can then calculate the corresponding values of α and p by proceeding as above and obtain all the other asymptotes.

We thus arrive at the following :

Rule. *To find the asymptotes of the curve $r = f(\theta)$, write the equation in the form $1/u = f(\theta)$ and find the limit of θ as $u \to 0$. Let θ_1 be this limit, or one of the limits if more than one such limits exist. Find $(-d\theta/du)$ and its limit when $\theta \to \theta_1$. Then the corresponding asymptote is*

$$r \sin(\theta_1 - \theta) = \underset{\theta \to \theta_1}{\text{Lt}} \left(-\frac{d\theta}{du}\right).$$

It should be noted that $(-d\theta/du)$ is the length of the polar sub-tangent of the curve. Thus the perpendicular from the origin on the asymptote is the limiting value of the polar sub-tangent when the point of contact of the tangent recedes to infinity along the curve.

Ex. 1. *Find the asymptotes of the curve*
$r = a \sec \theta + b \tan \theta.$ (*Panjab, 1953*)

$\because \; r = a \sec \theta + b \tan \theta = \dfrac{a + b \sin \theta}{\cos \theta}$, the equation of the curve may be written as

$$u = \frac{\cos \theta}{a + b \sin \theta} \qquad \ldots(1)$$

When $u \to 0$, $\cos \theta \to 0$, $\therefore \; \theta \to 2m\pi \pm \frac{1}{2}\pi$, where m is an integer or zero.

Differentiating (1) *w.r.* to θ.

$$\frac{du}{d\theta} = -\frac{a \sin \theta + b}{(a + b \sin \theta)^2}, \text{ or } -\frac{d\theta}{du} = \frac{(a + b \sin \theta)^2}{a \sin \theta + b}.$$

When $\theta \to \dfrac{\pi}{2}$, $\left(-\dfrac{d\theta}{du}\right) \to a + b$. $\quad \therefore$ the corresponding asymptote is

$$r \sin(\pi/2 - \theta) = a + b, \text{ i.e., } r \cos \theta = a + b. \qquad \ldots(2)$$

When $\theta \to -\frac{\pi}{2}$, $\left(-\frac{d\theta}{du}\right) \to b-a$. $\therefore$ the corresponding asymptote is

$$r \sin(-\pi/2-\theta)=b-a, \quad \text{or} \quad r\cos\theta = a-b. \qquad \ldots(3)$$

These two asymptotes correspond to $m=0$.

Let us now consider the direction corresponding to $\theta=\frac{3}{2}\pi$, then $(-d\theta/du) \to b-a$ and the corresponding asymptote is

$$r \sin(\tfrac{3}{2}\pi-\theta)=b-a \quad i.e., \; r\cos\theta=a-b,$$

which is the same as (3).

It will be seen that (2) and (3) are the only two different asymptotes, asymptotes corresponding to other values of m coinciding with one or the other.

Ex. 2. *Show that the curve $r(1-\cos\theta)=a$ has no asymptotes.* (*Panjab*, **1944**)

The equation of the curve may written as

$$au=1-\cos\theta. \qquad (i)$$

When $u \to 0$, $\cos\theta \to 1$, $\therefore \theta \to 2m\pi$.

Differentiating (1) *w.r.* to θ.

$$a\frac{du}{d\theta}=\sin\theta, \quad \therefore -\frac{d\theta}{du}=-\frac{a}{\sin\theta}.$$

When $\theta \to 2m\pi$, $\left(-\frac{d\theta}{du}\right) \to \infty$. Thus the curve cannot have any asymptote.

Examples LIV

1. Establish a formula for finding asymptotes of curves in polar coordinates and hence or otherwise find asymptotes of $r\tan 3\theta=a$. (*Panjab, B.Sc. 1962 S*)

2. What do you understand by the asymptote of a curve? State the method for obtaining asymptotes of the curve $r=f(\theta)$. Find the asymptotes of $r\sin 2\theta=a$. (*Panjab*, **1962**)

Find the asymptotes of the following curves :

3. $r\theta=a$. (*Panjab, 1950 S ; 1962 S*) 4. $r=a\theta/(\theta-1)$.
5. $r\sin n\theta=a$. (*Delhi, 1957 ; Panjab, 1954 S*)
6. $r=a\tan\theta$. (*Panjab, B.Sc. 1961 ; B.A. 1960 S*)
7. $r^2\theta=a^2$. 8. $r^n\sin n\theta=a^n$. 9. $r=a+b\cot n\theta$.
10. $r\cos 2\theta=a\sin 3\theta$. 11. $r=a\operatorname{cosec}\theta+b$. (*Nagpur*, **1939**)
12. $\log\theta=\frac{r}{a}$. 13. $\log\theta=\frac{a}{r}$. 14. $\frac{l}{r}=1+e\cos\theta$.
15. Show that there is an infinite number of parallel asymptotes to the curve

$$r=a/(\theta\sin\theta)+b,$$

and show that their distances from the pole are in **Harmonical** Progression. (*Benares*, **1935**)

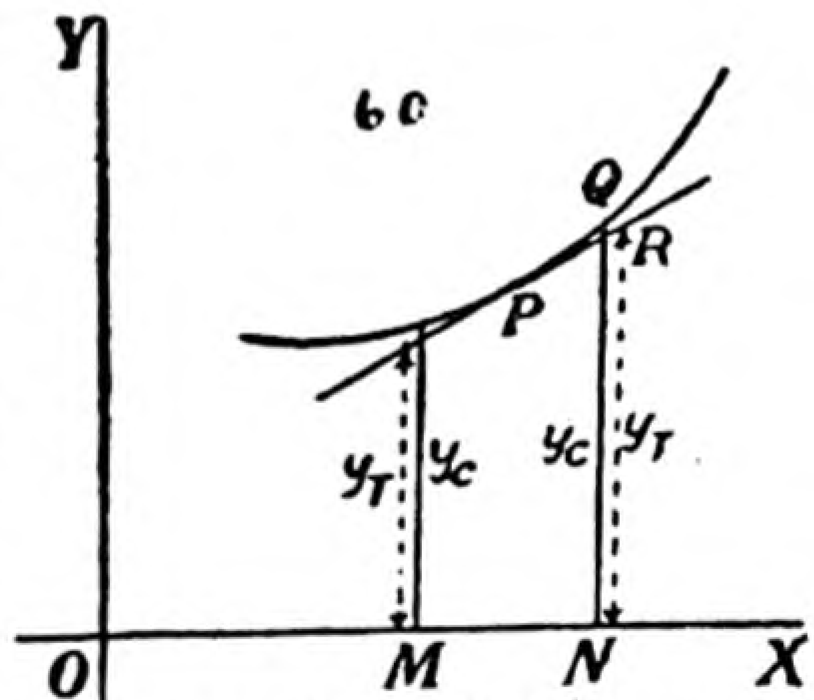
CHAPTER XV

SINGULAR POINTS

15·1. In ordinary language 'singular' means 'extraordinary', or unusual. In the present Chapter, we shall consider points which are, in a way, extraordinary. Such points have some peculiarity about them and as such are called singular points. We shall consider only the more important of the class of singular points such as points of inflexion and double points.

15·2. Concavity, Convexity. A curve is said to be **concave upwards** (or **in the positive y-direction**) at a point if it lies above the tangent on both sides of the point. It is said to be **convex upwards** at the point if it lies below the tangent on both sides of the point.

Let $P(x, y)$ be any point on the curve $y=f(x)$. Draw the tangent at P. If y_c denote the ordinate of any point on the curve in the immediate neighbourhood of P and y_t the corresponding ordinate of the tangent (*i.e.*, the two ordinates correspond to the same abscissa), the curve will be concave or convex in the positive y-direction according as $y_c >$ or $< y_t$ on both sides of P.



A curve or a certain arc thereof is said to be *concave or convex in the positive y direction* if it is concave or convex in that direction at every point on it.

It may be noted that y_c and y_t are taken with their proper signs so that the definition holds whether the curve lies above or below the x-axis.

It has been assumed that the tangent to the curve at P is not parallel to the y-axis so that $f'(x)$ is finite at the point. If the tangent at P is parallel to y-axis, concavity or convexity at P may be discussed with regard to the positive x-direction

15·21. Condition for concavity and convexity. *To determine the condition that the curve may be concave or convex in the positive y-direction.*

Let $P(x, y)$ be any point on the curve

$$y=f(x), \tag{i}$$

then the coordinates of Q may be expressed as $[x, f(x)]$.

Then the equation of the tangent at P is

$$Y-f(x)=f'(x)\,[X-x],$$

or

$$Y=f(x)+f'(x)\,[X-x]. \tag{ii}$$

Let $Q[x+h, f(x+h)]$ be a neighbouring point on the curve. Let the ordinate NQ through Q meet the tangent in R.

Let y_c and y_t denote the ordinates NQ, NR of the curve and the tangent respectively corresponding to the abscissa $x+h$. Then

$$y_c = f(x+h)$$

$$y_t = f(x) + hf'(x) \qquad \text{from (ii)}$$

$$\therefore \quad y_c - y_t = f(x+h) - f(x) - hf'(x)$$

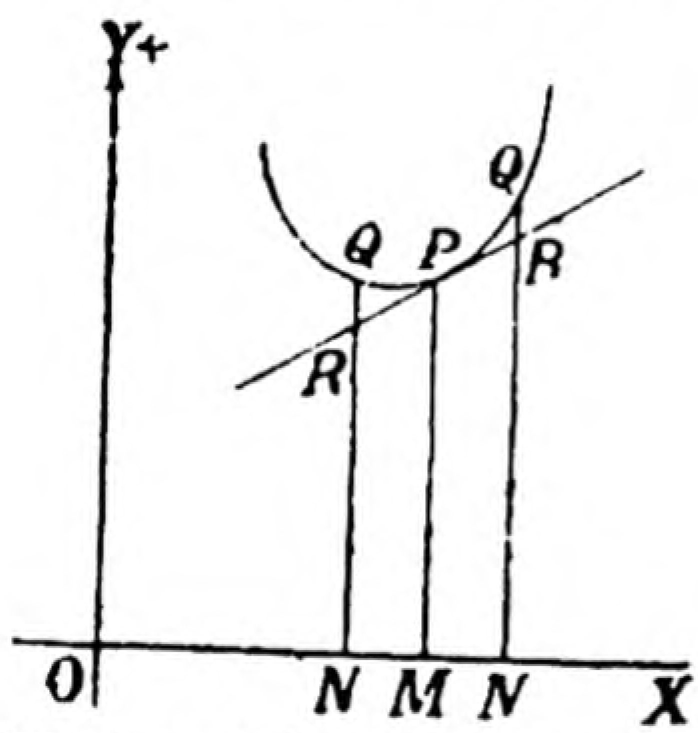
$$= f(x) + hf'(x) + \frac{h^2}{2\,!} f''(x+\theta h) - f(x) - hf'(x),$$

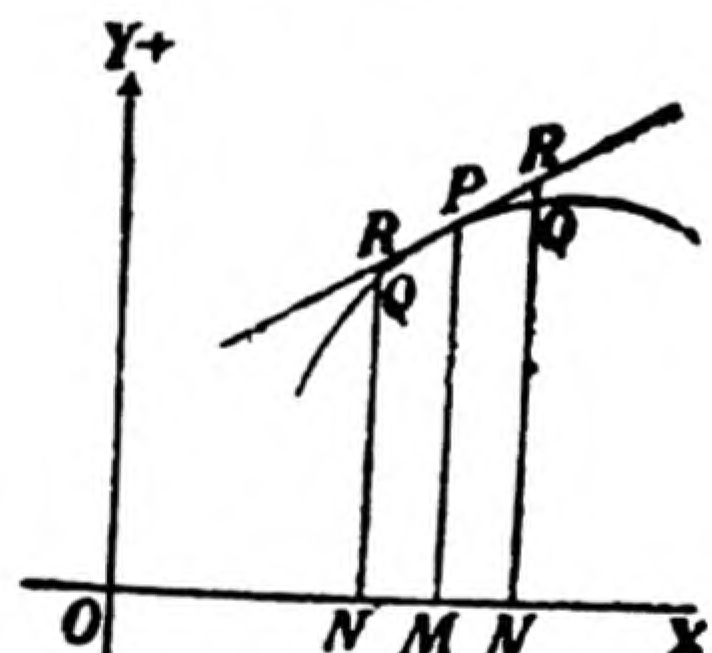
where $0<\theta<1$, by Taylor's theorem.

$$\therefore \quad y_c - y_t = \frac{h^2}{2\,!} f''(x+\theta h).$$

Assuming that $f''(x)$ is continuous and not zero, we note that $f''(x+\theta h)$ has the same sign as $f''(x)$ for arbitrarily small values of h. Hence, ($\because h^2$ is positive whether h is positive or negative) the sign of $y_c - y_t$ depends upon that of $f''(x)$.

The curve is **concave** upward at P if $y_c - y_t$ is positive and retains that sign in the neighbourhood of P, *i.e.*, if $f''(x)$ is positive.





The curve is **convex** upward at P if $y_c - y_t$ is negative and retains that sign in the neighbourhood of P, *i.e.*, if $f''(x)$ is negative.

Thus the curve is concave or convex in the positive y-direction according as $f''(x)$ is positive or negative.

Ex. 1. *Find the intervals in which the curve*

$$y = (\cos x + \sin x)e^x$$

is concave upwards, $0 \leqslant x \leqslant 2\pi$.

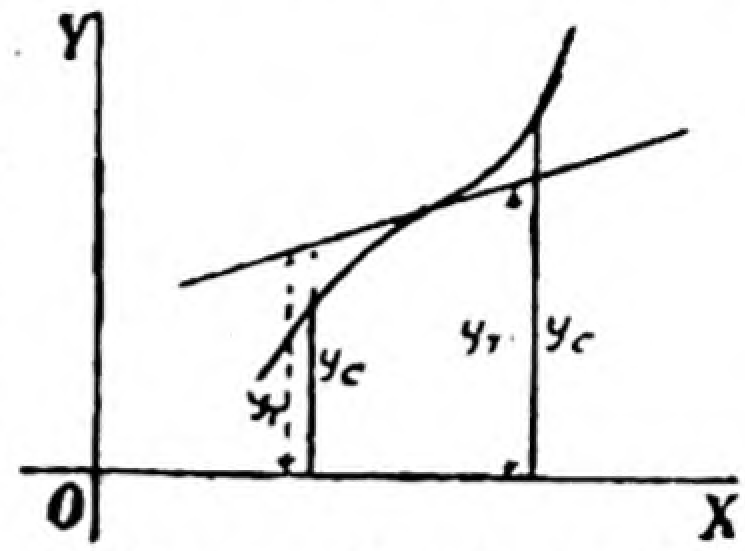
Here
$$y_1 = 2e^x \cos x,$$

$$y_2 = 2e^x (\cos x - \sin x).$$

Now e^x is positive for all calues of x, $\therefore y_2$ is positive, *i.e.*, the curve is concave in the positive y-direction when $\cos x - \sin x > 0$, *i.e.*, when $0 \leqslant x < \pi/4$ and again when $\frac{5}{4}\pi < x \leqslant 2\pi$.

15·22. Points of inflexion. A point on a curve is said to be **a point of inflexion** if the curve changes from concavity to convexity or *vice versa* in passing through the point.

From the definition it is clear that the curve lies above the tangent on one side of the point and below it on the other side. At a point of inflexion, therefore, the curve crosses its tangent. This property of the point of inflexion is also sometimes taken as its definition. If y_c, y_t denote the ordinate of the curve and the corresponding ordinate of the tangent respectively in the immediate neighbourhood of a point of inflexion P, $y_c > y_t$ on one side of P and $y_c < y_t$ on the other side of P. Thus $y_c - y_t$ changes sign in passing through the point P.



15·23. Condition for the existence of a point of inflexion. At a point of inflexion, the curve changes from concavity to convexity or *vice versa*. Hence $f''(x)$ is positive on one side of the point and negative on the other, and since $f''(x)$ has been assumed to be continuous at x, it must be zero at the point.

Thus *the necessary and sufficient conditions for the existence of a point of inflexion are* : (1) $f''(x)=0$ *at the point and* (2) $f''(x)$ *changes sign in passing through the point.*

Cor. 1. A point of inflexion of $f(x)$ is a point at which $f'(x)$ exists and has an extreme value.

Cor. 2. If $(x-c)$ is a multiple factor of an even order of $f'(x)$ then $x=c$ is an inflexion of $y=f(x)$.

Cor. 3. If $f''(x)$ is continuous, a point of inflexion of the curve $y=f(x)$ exists between every pair of consecutive maxima and minima of $f(x)$.

Let $f(x)$ be maximum at $x=a$ and minimum at $x=b$, the two being consecutive extreme values of $f(x)$. $\because$ $f(x)$ is maximum at $x=a$, $f''(a)$ is negative and $\because$ $f(x)$ is minimum at $x=b$, $F''(b)$ is positive. Since $f''(x)$ is assumed to be continuous, it must be zero at least at one point between a and b and change sign as x passes through this value. Hence there is an inflexion between a and b.

Note. It should be observed that $f'(x)$ may have any value at a point of inflexion. If $f'(x)$ vanishes at such a point, it is called a point of *stationary inflexion.*

Ex. 1. *Find the points of inflexion on the curve*

$$y=(x-1)^3(x-2)^4.$$

Here

$$y_1=(x-1)^2(x-2)^3(7x-10),$$

$$y_2=6(x-1)(x-2)^2(7x^2-20x+14).$$

Hence $y_2=0$ gives

$$x=1,\ 2 \text{ or } \tfrac{1}{7}(10\pm\sqrt{2}),$$

and then can we write

$$y_2=42(x-1)(x-2)^2(x-\alpha)(x-\beta)$$

where $\alpha=\frac{1}{7}(10+\sqrt{2})$, $\beta=\frac{1}{7}(10-\sqrt{2})$.

It is easily seen that y_2 changes sign as x passes through each of the values 1, α and β and y_2 does not change sign, when x passes through the value 2. Hence the curve has points of inflexion at $x=1$, α and β and the coordinates of the points of inflexion are

$$(1,0),\ \{\alpha, (\alpha-1)^3(\alpha-2)^4\},\ \{\beta, (\beta-1)^3(\beta-2)^4\}.$$

Ex. 2. *Find the points of inflexion on the curve*

$$x=a(2\theta-\sin\theta),\ y=a(2-\cos\theta).$$

Here $$\frac{dy}{d\theta}=a\sin\theta,\quad \frac{dx}{d\theta}=a(2-\cos\theta).$$

$$\therefore\quad \frac{dy}{dx}=\frac{dy}{d\theta}\Big/\frac{dx}{d\theta}=\frac{\sin\theta}{2-\cos\theta}.$$

$$\therefore\quad \frac{d^2y}{dx^2}=\frac{d}{d\theta}\left(\frac{\sin\theta}{2-\cos\theta}\right)\cdot\frac{d\theta}{dx}=\frac{2\cos\theta-1}{(2-\cos\theta)^3}.$$

$\dfrac{d^2y}{dx^2}=0$ gives $\cos\theta=\frac{1}{2}$, $\therefore\ \theta=2n\pi\pm\frac{1}{3}\pi$, where n is an integer.

It is easy to see that $\dfrac{d^2y}{dx^2}$ changes sign as θ passes through each of these values. Hence there are points of inflexion corresponding to every value of θ above. The coordinates of the points of inflexion are

$$\left[a\left(4n\pi\pm\frac{2\pi}{3}\pm\frac{\sqrt{3}}{2}\right),\ \frac{3a}{2}\right].$$

15·24. **General conditions.** *If $f'(c)=f''(c)=\ldots\ldots=f^{n-1}(c)=0$ and $f^n(c)\neq 0$, $f^n(x)$ being continuous at $x=c$, then $f(x)$ has a point of inflexion at $x=c$ if and only if n is odd.*

With the notation of the previous articles,

$$y_c-y_t=f(c+h)-f(c)-hf'(c)$$
$$=\frac{h^n}{n!}f^n(c+\theta h)\qquad \text{[Expanding } f(c+h) \text{ by Taylor's Theorem].}$$

Since $f^n(x)$ is continuous at $x=c$ and $f^n(c)\neq 0$, $f^n(c+\theta h)$ has the same sign as $f^n(c)$ for all sufficiently small values of h.

$\therefore\ y_c-y_t$ changes sign with h if and only if n is odd.

Hence $f(x)$ has a point of inflexion at $x=c$ if and only if n is odd.

Cor. 1. If n is even, the curve is concave upwards if $f^n(c)$ is positive and convex upwards if $f^n(c)$ is negative.

Cor. 2. If $(x-c)$ is a multiple factor of $f(x)$ of an odd order ($\geqslant 3$), then $x=c$ gives an inflexion of the curve $y=f(x)$.

There is no point of inflexion corresponding to a multiple factor of an even order.

Examples LV

1. Show that $y=e^x$ is everywhere concave upwards and $y=\log x$ is everywhere convex upwards.

2. Examine $y=\cos x$ for concavity and convexity in the range $(0, 2\pi)$. Also find the points of inflexion in this range.

3. Find the ranges of values of x in which the curve

$$y=x^4-10x^3+36x^2+24x+11$$

is (*i*) concave and (*ii*) convex upwards.

Also find its points of inflexion.

4. Show that every point in which the sine curve $y=c \sin (x/a)$ meets the axis of x is a point of inflexion on the curve.

(*Lucknow, 1950*)

5. Find the points of inflexion on the following curves :

(*i*) $y(a^2+x^2)=x^3$ (*Panjab, 1953 S*)

(*ii*) $y=(x-2)^6(x-3)^5$. (*Mysore, 1937*)

(*iii*) $x=(y-1)(y-2)(y-3)$. (*iv*) $x^3-axy-b^2y=0$.

(*v*) $y^2(a^2-x^2)=a^3x$.

(*vi*) $y^2=x(x+1)^2$. (*Delhi Hons. 1947*)

6. Show that the points of inflexion upon $x^2y=a^2(x-y)$ are given by $x=0$, $x=\pm a\sqrt{3}$. (*Patna, 1937*)

7. Show that the origin is a point of inflexion on the curve

$$a^{m-1}y=x^m$$

if m be odd and greater than 2.

8. Show that the points of inflexion of the curve

$$y^2=(x-a)^2(x-b)$$

lie on the line $3x+a=4b$. (*Lucknow, 1948 ; Panjab, 1960*)

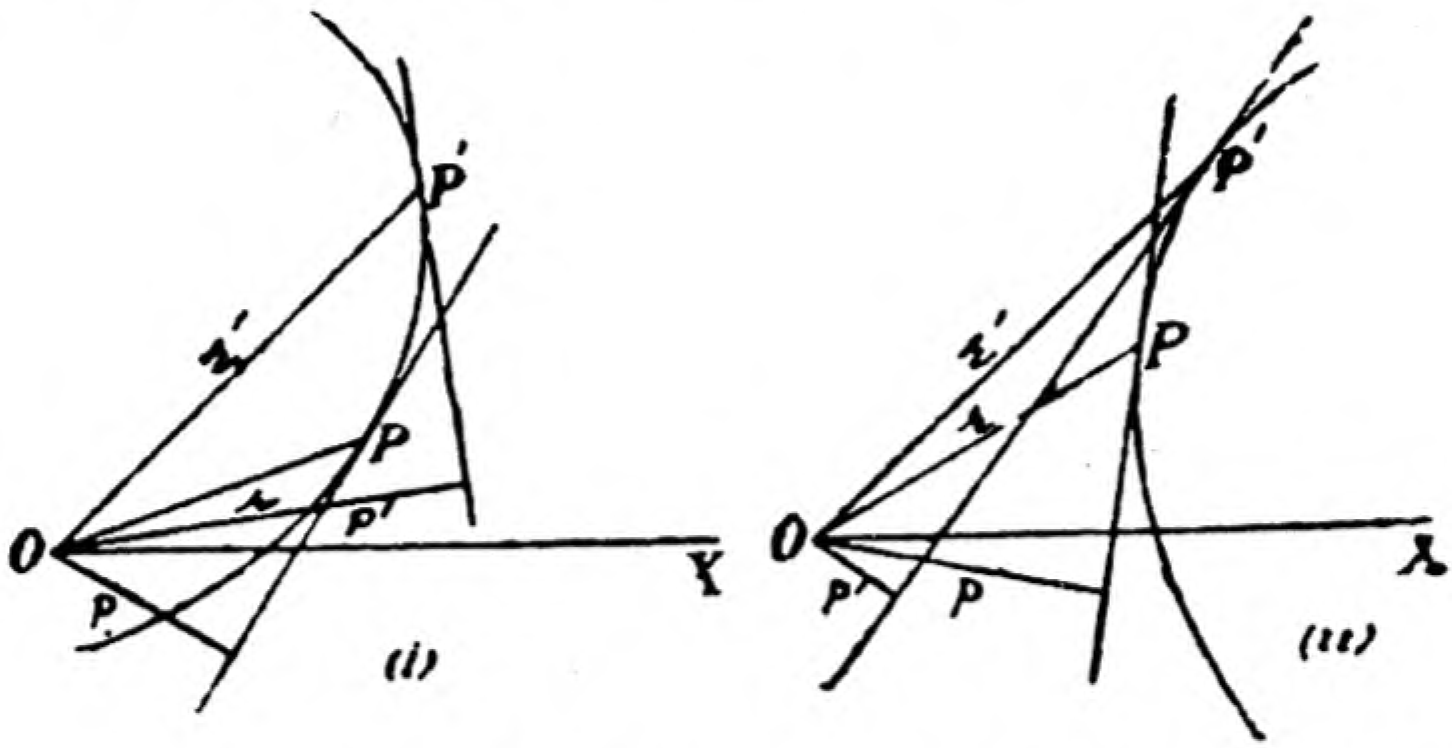
9. Find the points of inflexion on the curve

$$x=a \tan t,\ y=\tfrac{1}{2}a \sin 2t.$$

10. Show that the abscissa of the point of inflexion on the curve $x=a-b \cos \theta$, $y=a\theta-b \sin \theta$ is $(a^2-b^2)/a$.

15·3. Concavity or convexity with respect to a point. *At a point P of itself, a curve is said to be concave or convex towards any point O in its plane according as the point O and the part of the curve in the immediate neighbourhood of P lie on the same or opposite sides of the tangent at P.*

In the case of polar coordinates, concavity or convexity is usually considered with regard to the pole.



Let p be the length of the perpendicular from the origin on the tangent at any point $P(r, \theta)$ on the curve.

Then the curve is concave at P towards the pole [Fig. (*i*)], if p increases as r increases, *i.e.*, if $\dfrac{dp}{dr}$ is positive.

Similarly, the curve is convex at P towards the pole if $\dfrac{dp}{dr}$ is negative.

Also, if $\dfrac{dp}{dr}$ is zero at P and changes sign at the point, P must be a point of inflexion.

15·31. Alternative criterion for the existence of a point of inflexion. Let $r=f(\theta)$ be any curve and φ be the angle which the tangent at any point makes with the radius vector. Then if r_1, r_2 denote the first and second derivatives of r *w.r.* to θ, we have

$$\tan\varphi=r\frac{d\theta}{dr}=\frac{r}{r_1} \text{ and } \therefore \sin\varphi=\frac{r}{\sqrt{(r^2+r_1^2)}}$$

Next $p=r\sin\varphi=\dfrac{r^2}{\sqrt{(r^2+r_1^2)}}$

$$\text{and } \therefore \frac{dp}{dr}=\frac{dp}{d\theta}\cdot\frac{d\theta}{dr}=\frac{d}{d\theta}\left(\frac{r^2}{\sqrt{(r^2+r_1^2)}}\right)\cdot\frac{1}{r_1}$$

$$=\frac{r(r^2+2r_1^2-rr_2)}{(r^2+r_1^2)^{3/2}}. \qquad \ldots(i)$$

Again, if $u=\dfrac{1}{r}$, then $\dfrac{du}{d\theta}=-\dfrac{1}{r^2}\dfrac{dr}{d\theta}=-\dfrac{r_1}{r^2}$,

$$\text{and} \qquad \frac{d^2u}{d\theta^2}=-\frac{r^2r_2-r_1.2rr_1}{r^4}=\frac{2r_1^2-rr_2}{r^3}.$$

$$\therefore \qquad u+\frac{d^2u}{d\theta^2}=\frac{1}{r}+\frac{2r_1^2-rr_2}{r^3}=\frac{r^2+2r_1^2-rr_2}{r^3} \qquad \ldots(ii)$$

From (i) and (ii), it follows that $\frac{dp}{dr}$ and $u+\frac{d^2u}{d\theta^2}$ have the same sign and vanish together when

$$r^2+2r_1^2-rr_2=0.$$

Hence applying the result of the previous article, the curve $r=f(\theta)$ is concave or convex towards the pole at a point according as $r^2+2r_1^2-rr_2$ or $u+\frac{d^2u}{d\theta^2}$ is positive or negative at that point and has an inflexion at the point if $r^2+2r_1^2-rr_2$ or $u+\frac{d^2u}{d\theta^2}$ vanishes at the point and changes sign at the point.

In practice, it is useful to discuss the sign of $\frac{dp}{dr}$ when the pedal equation to a curve is given and that of $u+\frac{d^2u}{d\theta^2}$ when the polar equation to the curve is given.

Ex. *Find the points of inflexion on the curve*

$$r(\theta^2-1)=a\theta^2.$$

Here $au=1-\frac{1}{\theta^2}$, $\therefore a\frac{du}{d\theta}=\frac{2}{\theta^3}$ and $a\frac{d^2u}{d\theta^2}=-\frac{6}{\theta^4}$.

$$\therefore\quad a\left(u+\frac{d^2u}{d\theta^2}\right)=1-\frac{1}{\theta^2}-\frac{6}{\theta^4}=\frac{1}{\theta^4}(\theta^4-\theta^2-6)$$

$$=\frac{1}{\theta^4}(\theta^2-3)(\theta^2+2)$$

Putting $u+\frac{d^2u}{d\theta^2}=0$, we get

$$(\theta^2-3)(\theta^2+2)=0.$$

$\therefore\ \theta=\pm\sqrt{3}$ are the only real values of θ satisfying the equation.

$\because\ \theta^2-3$ changes sign as θ passes through each of the values $\pm\sqrt{3}$, it follows that $u+\frac{d^2u}{d\theta^2}$ changes sign at each of these points, since the other two factors $1/\theta^4$ and (θ^2+2) are positive.

Hence there are points of inflexion for $\theta=\pm\sqrt{3}$. The corresponding value of r in each case is easily seen to be $3a/2$.

Examples LVI

1. Find the points of inflexion on the curve $a^2=r^2\theta$.

2. Show that the points of inflexion on the curve $r=b\theta^n$ are given by $r=b[-n(n+1)]^{n/2}$.

3. Determine whether the spiral $r\cosh\theta=a$ is convex or concave towards the pole.

Double points

15·4. A point on a curve is called a *double point* if two branches of the curve pass through it. It is called a *triple point* if three branches of the curve pass through it. In general, it is called a *multiple point* of the rth order if r branches of the curve pass through the point.

Multiple points on a curve are another example of singular points which, as was pointed out earlier, include points of inflexion, etc., within their scope. In a restricted sense, however, the expression 'singular points' is applied to a class of points on the curve $f(x, y)=0$ for which $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$. The discussion of singular points as such is beyond the scope of this book. Even among multiple points we shall confine our attention primarily to double points.

15·41. Classification of double points. Since two branches of a curve pass through a double point, there must be two tangents to the curve at the point, one to each of the two branches.

If the two tangents be real and different, the double point is called a **node** (Fig. 1), if the two tangents be real and coincident, it is called a **cusp** (Fig. 2), and if the two tangents are imaginary so that there are no real points on the curve in the neighbourhood of the double point, it is called a **conjugate point** (Fig. 3). A conjugate point is also called an **isolated point** on the curve.

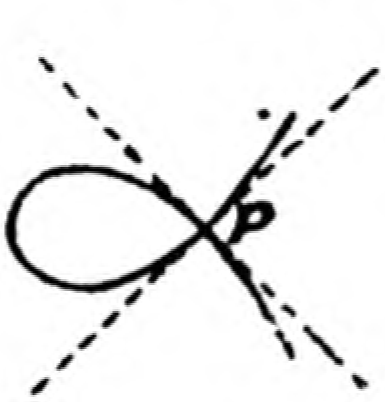


Fig. 1. Node

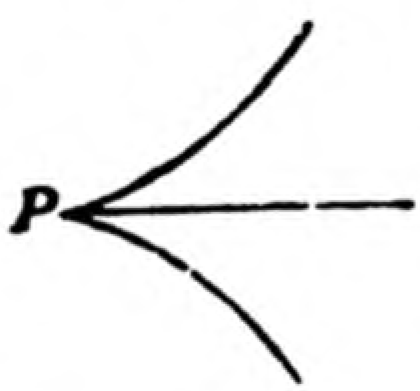


Fig. 2. Cusp.

Fig. 3. Conjugate point.

15·42. Conditions for the existence of a double point. *To find the conditions for the existence of a double point on a curve and to discuss its nature.*

Let the equation to the curve be

$$f(x, y)=0. \qquad ...(1)$$

Then

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}=0. \qquad ..(2)$$

Equation (2), being of the first degree in $\frac{dy}{dx}$, determines $\frac{dy}{dx}$ uniquely unless $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ both vanish at the point.

Thus the necessary conditions for the existence of a double point are

$$\frac{\partial f}{\partial x}=0 \text{ and } \frac{\partial f}{\partial y}=0. \qquad ...(3)$$

Such of the simultaneous solutions of equations (3) which satisfy equation (1) determine double or multiple points on the curve.

Differentiating (2) again *w. r.* to x, we get

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y \partial x}\cdot\frac{dy}{dx}+\left(\frac{\partial^2 f}{\partial x \partial y}+\frac{\partial^2 f}{\partial y^2}\frac{dy}{dx}\right)\frac{dy}{dx}+\frac{\partial f}{\partial y}\frac{d^2 y}{dx^2}=0.$$

$\because \dfrac{\partial f}{\partial y}=0$ at a double point, the values of $\dfrac{dy}{dx}$ at such a point are given by the quadratic

$$\frac{\partial^2 f}{\partial x^2}+2\frac{\partial^2 f}{\partial x \partial y}\frac{dy}{dx}+\frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dx}\right)^2=0. \qquad ...(4)$$

The point (x, y) will be a double point (and not a multiple point of a higher order) if

$$\frac{\partial^2 f}{\partial x^2},\ \frac{\partial^2 f}{\partial x \partial y} \text{ and } \frac{\partial^2 f}{\partial y^2}$$

do not all vanish simultaneously at the point. In particular, it will be a node, a cusp or a conjugate point according as the roots of (4) are real and distinct, coincident, or imaginary, *i.e.*, according as

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \gtreqless \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2}.$$

Ex. 1. *Examine $a^2y^2=a^2x^2-4x^3$ for singular points*

Here $$f(x, y)=4x^3+a^2y^2-a^2x^2=0 \qquad ...(i)$$

is the equation to the curve. Hence

$$\frac{\partial f}{\partial x}=12x^2-2a^2x,\ \frac{\partial f}{\partial y}=2a^2y. \qquad ...(ii)$$

At a singular point, $\dfrac{\partial f}{\partial x}=0=\dfrac{\partial f}{\partial y}$.

$$\therefore\quad 12x^2-2a^2x=0, \text{ and } 2a^2y=0. \qquad ...(iii)$$

The solutions of these two equations are $(0, 0)$ and $(\frac{1}{6}a^2, 0)$. The latter does not satisfy equation (i), and is, therefore, discarded.

Hence $(0, 0)$ is the only singular point on the curve.

Again, from (ii) on differentiating partially *w. r.* to x and y, we get

$$\frac{\partial^2 f}{\partial x^2}=24x-2a^2,\ \frac{\partial^2 f}{\partial y^2}=2a^2 \text{ and } \frac{\partial^2 f}{\partial x \partial y}=0.$$

At $(0, 0)$, $\frac{\partial^2 f}{\partial x^2}=-2a^2$, $\frac{\partial^2 f}{\partial y^2}=2a^2$ and $\frac{\partial^2 f}{\partial x \partial y}=0$.

$$\therefore \qquad \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2-\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2}=4a^2,$$

which is positive. Hence the origin is a node.

This could have been seen directly as soon as it was known that the origin is the *only* singular point. The tangents at the origin are given by $y^2-x^2=0$ which represents two real and distinct lines $y=\pm x$.

Ex. 2. *Examine the nature of the double points on the curve*

$$x^4-4y^3-12y^2-8x^2+16=0. \qquad \ldots(1)$$

Here $\qquad f(x, y)=x^4-4y^3-12y^2-8x^2+16$

$$\therefore \qquad \frac{\partial f}{\partial x}=4x^3-16x, \quad \frac{\partial f}{\partial y}=-12y^2-24y. \qquad \ldots(A)$$

Equating these to zero, we get

$$4x^3-16x=0, \qquad \ldots(2)$$

$$-12y^2-24y=0. \qquad \ldots(3)$$

From (2), $x=0, 2, -2$, and from (3), $y=0, -2$. $\therefore$ the simultaneous solutions of (2) and (3) are

$$(0, 0), (0, -2), (2, 0), (2, -2), (-2, 0), (-2, -2).$$

By actual substitution, we find that of these six points only the three points $(2, 0)$, $(-2, 0)$, and $(0, -2)$ satisfy the equation to the curve.

Differentiating equations (A) again partially *w.r.* to x and y, we get

	At (2, 0)	*At* (−2, 0)	*At* (0, −2)
$\frac{\partial^2 f}{\partial y^2}=12x^2-16=$	32 ,	32 ,	−16
$\frac{\partial^2 f}{\partial y^2}=-24y-24=$	−24 ,	−24 ,	24
$\frac{\partial^2 f}{\partial x \partial y}=0 \quad =$	0 ,	0	0

At $(2, 0)$, $\left(\frac{\partial^2 f}{\partial x \partial y}\right)-\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{dy^2}=768$, which is positive. Hence $(2, 0)$ is a node.

At $(-2, 0)$, $\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2-\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2}=768$, which is positive. Hence $(-2, 0)$ is a node.

At $(0, -2)$, $\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2-\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2}=384$, which is positive. Hence $(0, -2)$ is also a node.

15·43. Tangents at the origin. Let the curve pass through the origin and let its equation be

$$u_n+u_{n-1}+\ldots\ldots+(lx^2+mxy+ny^2)+(ax+by)=0, \qquad \ldots(i)$$

where u_n denotes the sum of the terms of the nth degree in x and y. Let the Maclaurin expansion of y in ascending power of x be

$$y=px+\frac{qx^2}{2\,!}+\ldots\ldots \qquad \ldots(ii)$$

where $\quad p=\dfrac{dy}{dx}$ at (0, 0) and $q=\dfrac{d^2y}{dx^2}$ at (0, 0).

Substituting the expression (ii) for y in the equation of the curve and observing that u_r will give rise to rth and higher degree terms, we get

$$\ldots\ldots+lx^2+mx(px+\frac{qx^2}{}+\ldots\ldots)+n(px+\frac{qx^2}{2\,!}+\ldots\ldots)^2$$
$$+ax+b(px+\frac{qx^2}{2\,!}+\ldots\ldots)\equiv 0.$$

Equating coeff. of x to zero, we get

$$a+bp=0, \qquad \therefore\ p=-a/b.$$

$\therefore$ equation of the tangent at the orign is

$$y=-\frac{a}{b}.x \text{ or } ax+by=0.$$

If the equation of the curve does not contain terms of the first degree in x and y, so that $a=0$, $b=0$, and is of the form

$$u_n+v_{n-1}+\ldots\ldots+(lx^2+mxy+ny^2)=0,$$

we have, on substituting the expression (ii) for y,

$$\ldots\ldots+lx^2+mx(px+\frac{qx^2}{2\,!}+\ldots\ldots)+n(px+\frac{qx^2}{2\,!}+\ldots)^2=0.$$

Equating coefficient of x^2 to zero, we get

$$l+mp+np^2=0.$$

This, being a quadratic, has two roots in p. If p denotes either of the roots, the corresponding tangent is $y=px$.

Hence eliminating p between the last two equations, we get

$$l+m\left(\frac{y}{x}\right)+n\left(\frac{y}{x}\right)^2=0,\ i.e.,\ lx^2+mxy+ny^2=0,$$

which is the joint equation of the two tangents at the origin.

From the above, we observe the following rule to write down the equation of the tangent or tangents at the origin.

Rule. *The terms of the lowest degree in x and y equated to zero give the equation of the tangent or tangents (real or imaginary) at the origin.*

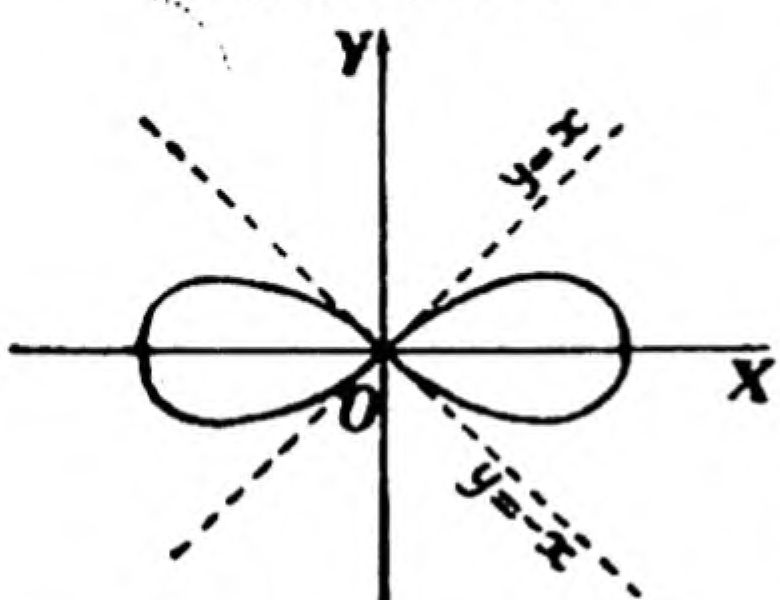
15·44 Nature of the origin. *To find whether the origin is a node, a cusp or a conjugate point. We have seen that if a curve passes through*

the origin, the equation to the tangent or tangents thereat is obtained by equating to zero the lowest degree terms in the equation to the curve.

(i) If the lowest degree terms in the equation of a curve are of the first degree, we shall get only one tangent line to the curve at the origin which, therefore, cannot be a double point.

(ii) The origin will be a double point only if the lowest degree terms in the equation to the curve are of the second degree. Further the origin will be a node, a cusp or a conjugate point according as these terms break up into real and distinct factors, form a perfect square or break up into imaginary factors.

Illustrations.



(i) If the equation to the curve be

$$x^4+y^4-a^2(x^2-y^2)=0,$$

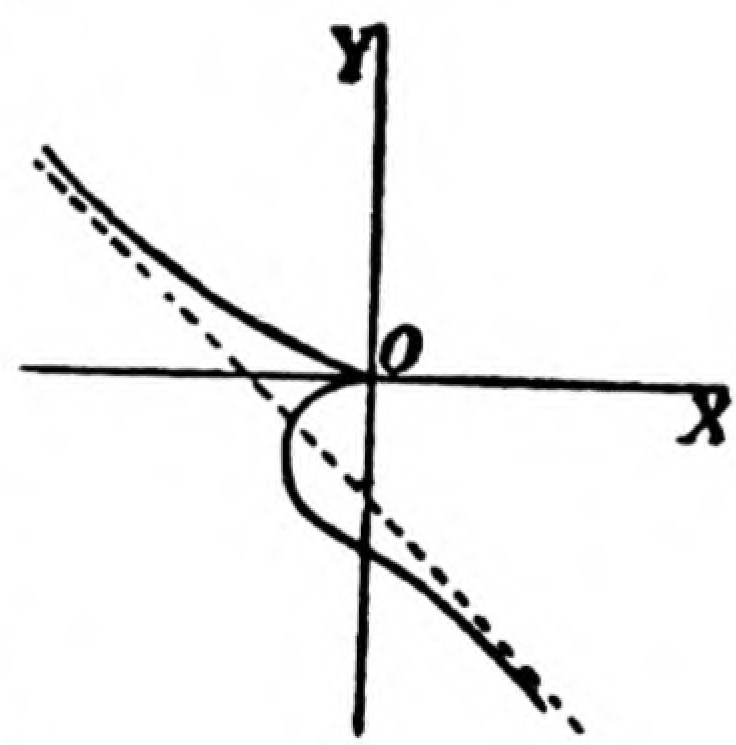
the tangents at the origin are given by the equation

$$x^2-y^2=0$$

which represents two real distinct straight lines, *viz.*, $x-y=0$ and $x+y=0$. Hence the origin is a node.

(ii) If the equation to the curve be

$$x^3+y^3+ay^2=0$$

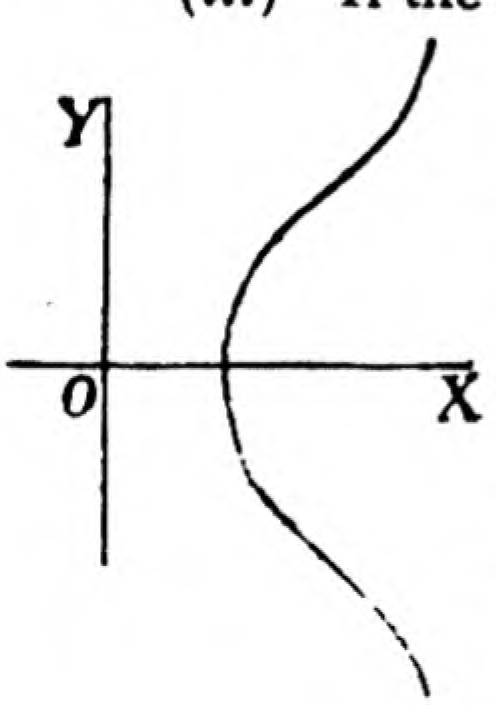


the tangents at the origin are given by

$$y^2=0$$

which represents a *pair* of coincident lines. Hence the origin is a cusp.

(iii) If the equation to the curve be



$$x^3-(x^2+y^2)=0,$$

the tangents at the origin are given by

$$x^2+y^2=0$$

which represents a pair of imaginary lines. Hence the origin is on isolated point on the curve.

Ex. *Prove that $(a, 4a)$ is a double point on the curve*

$$y(x-a)^2=x(y-4a)^2$$

and find the equations of the tangents at the point.

Shift the origin to $(a, 4a)$, keeping the directions of the axes unchanged. If (x, y) transforms into (X, Y), we have

$$x=X+a, \qquad y=Y+4a.$$

Hence the transformed equation is

$$(Y+4a)X^2=(X+a)Y^2 \quad \text{or} \quad X^2Y-XY^2+a(4X^2-Y^2)=0.$$

Tangents at the new origin are

$$4X^2-Y^2=0 \qquad i.e., \ Y=\pm 2X.$$

These being real and distinct, the new origin is a node. Referred to the original axes, the equations of the tangents at the double point are

$$y-4a=\pm 2(x-a)$$

i.e., $$2x+y-6a=0 \quad \text{and} \quad 2x-y+2a=0.$$

Examples LVII

Find the position and nature of the double points on the following curves :—

1. $4y^2=x^2(4-x^2)$.
2. $x^3-4x^2+4x-2y^2=0$.
3. $ay^2=x^3-bx^2$.
4. $y^3=x^3+ax^2$. *(Delhi, 1950)*
5. $(x^2+y^2)^2=x^2-y^2$.
6. $x^3+y^3=3axy$. *(Panjab, 1957)*
7. $xy^2-ax^2+2a^2x-a^3=0$. *(Panjab, 1960 S)*
8. $x^4-2y^3-3y^2-2x^2+1=0$. *(Panjab, 1951, '62 S)*
9. $x^4+y^3+2x^2+3y^2=0$. *(Panjab, 1949)*
10. $y(y-6)=x^2(x-2)^3-9$. *(Delhi, 1958 ; Allahabad, 1941)*
11. $(x+y)^3-\sqrt{2}(y-x+2)^2=0$. *(W. Panjab, 1949)*
12. $(2y+x+1)^2=4(1-x)^5$. *(Panjab, 1946 ; Allahabad, 1942)*

13. Show that the origin is a node, a cusp or a conjugate point on the curve

$$y^3=ax^2+bx^3$$

according as $a>,=$ or <0. *(Panjab, B.Sc. 1960 S.)*

14. Prove that the curve

$$y^2=(x-a)^2(x-b)$$

has, at $x=a$, a node if $a>b$, a cusp if $a=b$ and a conjugate point if $a<b$. *(Bombay, 1948 ; Panjab, 1960)*

15. Examine the nature of the origin on the curve

$$x^7+2x^4+2x^2y+x^2+2xy+y^2=0. \qquad \textit{(Agra, 1941)}$$

16. Find the equations of the tangents to the curve

$$x^2y^2=(a^2-y^2)(b+y)^2$$

at its double point.

15·5. Form of a curve near the origin. Let $y=mx$ be a tangent to the curve $y=f(x)$ at the origin. Suppose it is possible to expand y in ascending powers of x in the form

$$y=mx+bx^2+cx^3+\ldots\ldots$$

Then $$y_c-y_t=bx^2+cx^3+\ldots\ldots \qquad \ldots(1)$$

By taking x small enough, the sign of the expression on the right of (1) can be made to depend upon that of bx^2. If $b>0$, $y_c>y_t$ in the neighbourhood of the origin whether x be positive or negative. Hence the curve lies above the tangent on either side of the origin. Similarly, if $b<0$, the curve lies below the tangent on either side of the origin.

If $b=0$, $c\neq 0$, the sign of y_c-y_t is the same as that of cx^3 for sufficiently small values of x. Evidently cx^3 changes sign with x so that the curve lies above the tangent on one side of the origin and below it on the other. The origin, in this case, is a point of inflexion and the curve crosses its tangent. Actually, if $c>0$, the curve lies above the tangent for positive values of x and below it for negative values. If $c<0$, the position is reserved.

Ex. *Find the form of the curve*

$$y^2(a^2+x^2)=x^2(a^2-x^2)$$

near the origin.

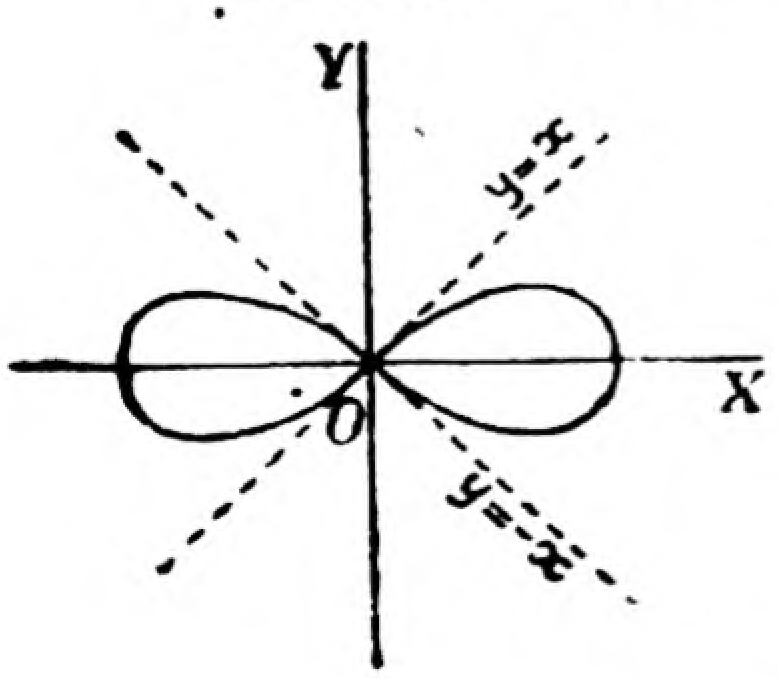
From the equation to the curve, we have

$$y=\pm x\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}}\left(1+\frac{x^2}{a^2}\right)^{-\frac{1}{2}}$$

$$=\pm x\left(1-\frac{x^2}{2a^2}-\frac{x^4}{8a^4}+\ldots\right)\left(1-\frac{x^2}{2a^2}+\frac{3x^4}{8a^4}-\ldots\right)$$

$$=\pm x\left(1-\frac{x^2}{a^2}+\frac{x^4}{2a^4}-\ldots\right)$$

Thus the two branches of the curve are



$$y=x-\frac{x^3}{a^2}+\frac{x^6}{2a^4}+\ldots\ldots \qquad \ldots(i)$$

and $$y=-x+\frac{x^3}{a^2}-\frac{x^5}{2a^4}-\ldots\ldots \qquad \ldots(ii)$$

Considering (*i*), we find that :

(*a*) $y=x$ is a tangent to this branch of the curve at the origin.

(*b*) The coefficient of x^2 is zero and that of x^3 is negative. Thus y_2 is zero at the origin and y_3 is negative.

The origin is a point of inflexion and the curve is convex in the positive y-direction to the right of the origin and, therefore, lies below the tangent.

To the left of the origin, the curve is concave in the upward direction and lies above the tangent.

Considering the second branch (*ii*) similarly, we find that the curve is concave to the right of the origin and convex to the left and hence lies above the tangent for positive values of x and below it for negative values.

15·51. Form of a curve near the origin. (*Contd.*)

Method of approximation. If a branch of a curve passes through the origin and it is possible to expand y in ascending powers of x as

$$y=ax+bx^2+cx^3+\ldots\ldots$$

then it is easy to see that

$$\begin{aligned} &y=ax && \text{(tangent at the origin)}\\ &y=ax+bx^2\\ &y=ax+bx^2+cx^3\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

are successively closer and closer approximations to the curve near the origin. Evidently, for very small values of x and y, it is terms of the least degree which matter. Hence any approximation of the curve near (0, 0) must come out of these and for successive closer approximations, we successively associate terms of higher degrees with them. The following examples will illustrate how we may proceed to get the approximations of the curve near (0, 0). It may also be pointed out that to get the approximate form of the curve at any other point, we may first shift the origin to that point and then proceed as below.

Ex. 1. *Disuss the form of the curve* $y^2(x-4)=x^2(y-1)$ *near the origin.*

Writing the equation in the form

$$xy(x-y)+(4y^2-x^2)=0,$$

we observe that the first approximation gives $4y^2-x^2=0$ which equation represents the two tangents at the origin *viz.*,

$$2y=\pm x, \quad \text{or} \quad y=\pm\tfrac{1}{2}x.$$

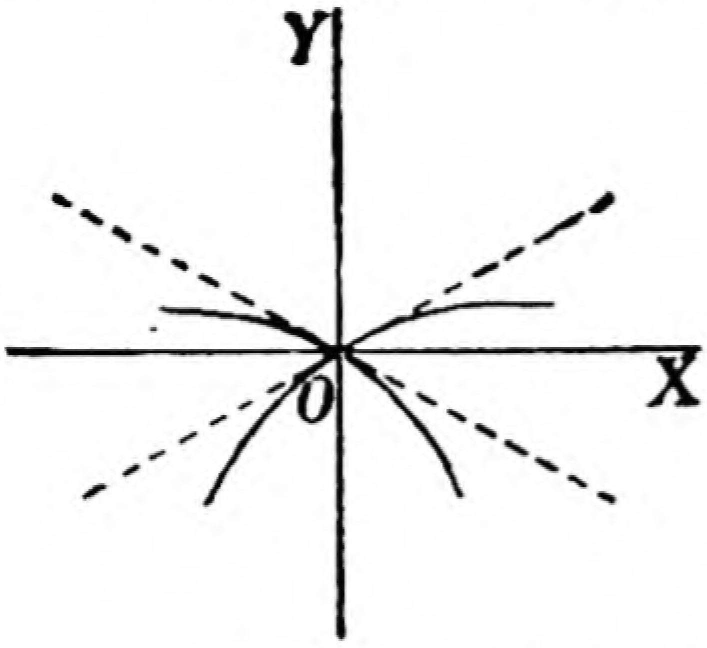
A second approximation for the branch to which $2y=x$ is a tangent is obtained from

$$\begin{aligned} 2y-x&=\frac{xy(y-x)}{2y+x}\\ &=\frac{x(\frac{1}{2}x)(\frac{1}{2}x-x)}{2(\frac{1}{2}x)+x} \qquad \text{[Writing } \tfrac{1}{2}x \text{ for } y]\\ &=-\tfrac{1}{8}x^2 \end{aligned}$$

i.e., $$y=\tfrac{1}{2}x-\tfrac{1}{16}x^2$$

showing that near (0, 0) the curve lies below the tangent for positive and negative values of x.

Similarly, a second approximation to the branch to which $2y=-x$ is a tangent is obtained as



$$y=-\tfrac{1}{2}x-\tfrac{3}{16}x^2$$

showing again that near (0, 0) the curve lies below the tangent both for positive and negative values of x.

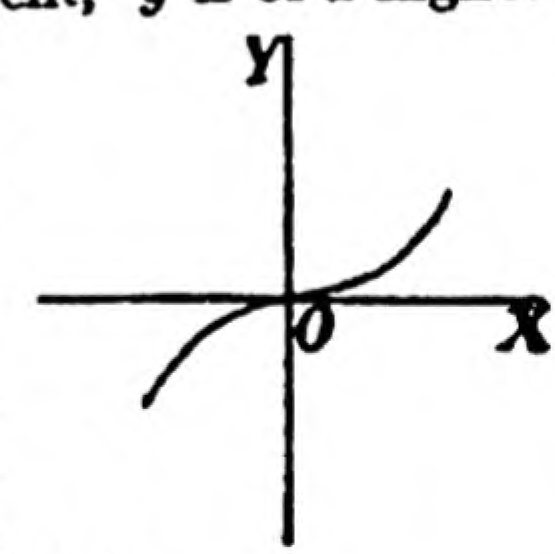
Note. If $y=0$ is a tangent to a curve at the origin, a closer approximation is $y=lx^{\alpha}$ $(\alpha>1)$. Hence y^m will be an infinitesimal of a higher order than x^m $(m>1)$ and may be neglected while considering the branch to which $y=0$ is a tangent.

Similar remarks apply to the case in which $x=0$ is a tangent at the origin.

Ex. 2. *Discuss the form of the curve $x^4+y^4=4a^2xy$ near the origin.*

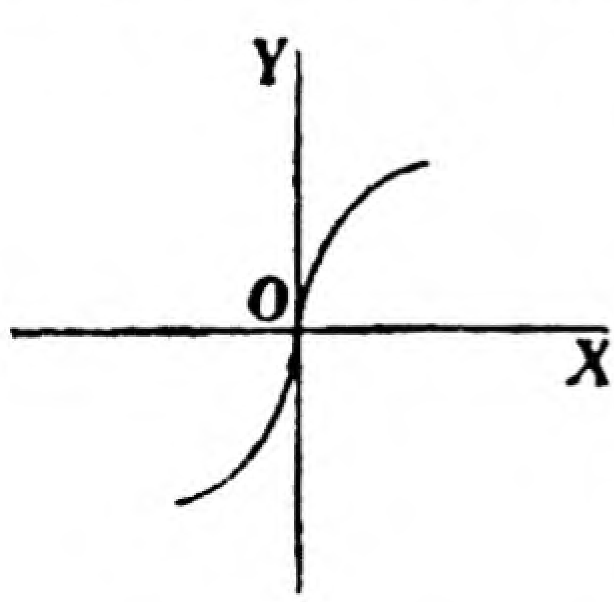
$y=0$, $x=0$ are tangents to the curve at the origin which, therefore, is a node.

For the branch to which $y=0$ is a tangent, y is of a higher order of smallness than x. Hence y^4 is an infinitesimal of a higher order than x^4. Thus omitting y^4, the corresponding branch approximates to



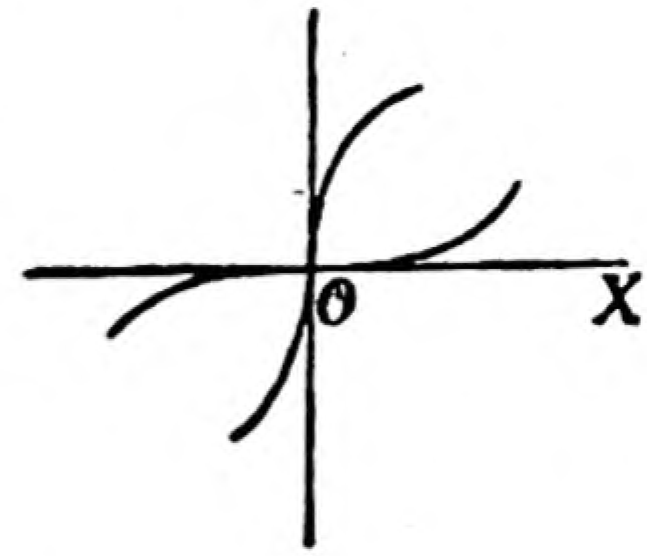
$$x^4=4a^2xy \quad \text{or} \quad 4a^2y=x^3.$$

Thus the curve lies above the tangent $y=0$ for positive values of x and below it for negative values. Since the curve crosses its tangent at the origin, there is a point of inflexion here.



Again, for the branch to which $x=0$ is a tangent, x is of a higher order of smallness than y. Hence x^4 is an infinitesimal of a higher order than y^4. Thus omitting x^4, the corresponding branch approximates to $y^4=4a^2xy$ or $4a^2x=y^3$ showing that the curve lies to the right of the tangent $x=0$ in the first quadrant and to the left of it in the third quadrant. Thus the curve crosses this tangent also at the origin.

Combining the two results, we come to the conclusion that the shape of the curve near the origin is as shown in the figure.



Examples LVIII

Discuss the form near the origin of the following curves :—

1. $y^2(a-x)=x^2(a+x)$.
2. $x^4-2x^3y+x^3-2y^3-2x^2+xy+y^2=0$.
3. $x^4+y^4=x+y$. 4. $x^3(y^2-8)=y^3(x^2-1)$.

15·6. Cusps : their classification.

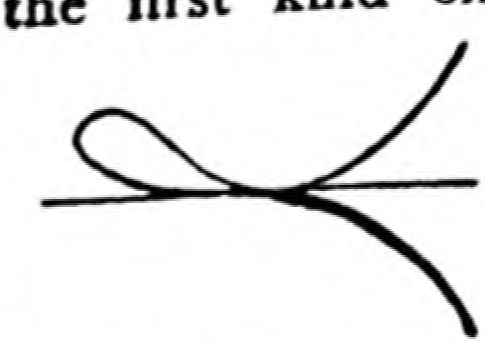
We know that the two branches of a curve have a common tangent at a cusp. The following figures illustrate how the branches of the curve can lie with regard to this common tangent. The cusp is named according to this relationship.

(*i*) A cusp may be single or double. It is said to be single if the curve does not extend beyond the point of contact ; otherwise it is said to be **double.**

(*ii*) A cusp is said to be of the **first kind** or a **ceratoid cusp** if the two branches lie on opposite sides of the tangent ; one in which the two branches lie on the same side of the tangent is called a cusp of the **second kind** or a **ramphoid cusp**.

Single cusp (1st kind) Single cusp (2nd kind) Double cusp (1st kind) Double cusp (2nd kind)

It may happen that a double cusp may be of the first kind on one side of the point of contact and of the second kind on the other side. Such a point is called **a point of osculinflexion.**



Point of osculinflexion.

A point of osculinflexion is a point of inflexion and a cusp combined together, so to say.

Ex. 1. *Find the form near the origin of the curve*

$$y^2-2x^2y+x^4-x^6=0$$

and discuss the nature of the cusp.

Solving the equation as a quadratic in y, we get

$$y=\frac{2x^2\pm\sqrt{\{4x^4-4(x^4-x^5)\}}}{2}=x^2\pm x^{5/2}$$

Thus the equations to the two branches of the curve are

$$y=x^2+x^{5/2} \text{ and } y=x^2-x^{5/2}.$$

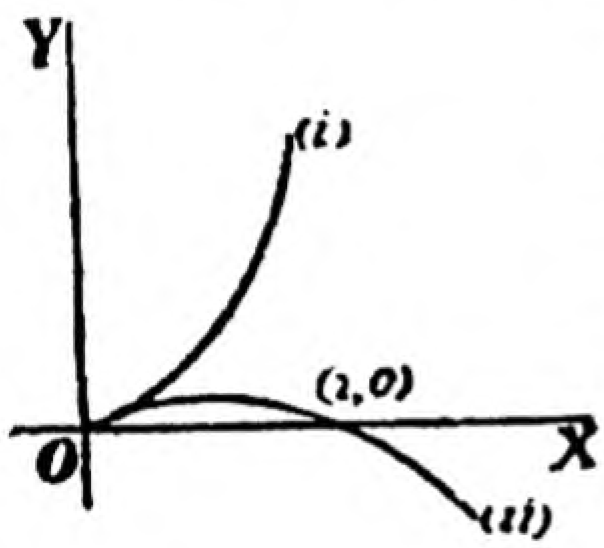
From the equation to the curve, we find that tangents at the origin are $y^2=0$. This equation represents a pair of coincident lines. Hence the origin is a cusp.

When x is positive and very small, $x^2>x^{5/2}$, $\therefore$ for both branches, y is positive. Hence both branches lie above the x-axis which is the cuspidal tangent at the origin.

Again, x cannot be negative, because in that case, the corresponding values of y are imaginary for either branch. Hence no part of the curve lies to the left of the y-axis.

Thus the origin is a single cusp of the second kind.

Note. For positive values of x, the ordinate of the first branch is always positive and increases, $\therefore$ the first branch always lies above the x-axis. For the second branch, the ordinate is positive only so long as $x^2>x^{5/2}$, *i.e.*, $1>\sqrt{x}$, *i.e.*, $1>x$. At $x=1$, $y=0$ for this branch, and for $x>1$, the values of y are negative. Thus this branch lies above the x-axis for $0<x<1$, crosses the x-axis at $x=1$ and subsequently lies below the x-axis. The form of the curve, therefore, is as shown in the diagram.



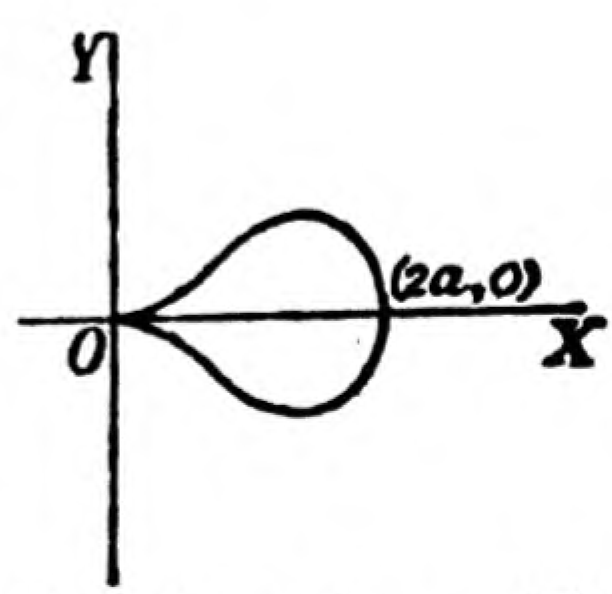
Ex. 2. *Find the nature of the cusp on the curve* ...(i)

$$a^4y^2=x^5(2a-x).$$

The curve passes through the origin where the tangents are given by $y^2=0$, which represents a pair of coincident lines. Hence the origin is a cusp.

Again from the equation to the curve,

$$a^2y=\pm x^2\sqrt{(2ax-x^2)}.$$



Thus there are two branches of the curve and their equations are

$$a^2y=+x^2\sqrt{(2ax-x^2)} \quad \text{or} \quad a^2y=+x^2\sqrt{\{a^2-(x-a)^2\}} \quad \text{... (ii)}$$

and $$a^2y=-x^2\sqrt{(2ax-x^2)} \quad \text{or} \quad a^2y=-x^2\sqrt{\{a^2-(x-a)^2\}} \quad \text{...(iii)}$$

When x is negative, the corresponding values of y for both branches are imaginary. Hence no part of the curve lies to the left of the y-axis. $\therefore$ The cusp is a single one.

Again, when x is positive and very small, the ordinate of the first branch is positive and that of the second branch is negative. Hence the two branches lie on opposite sides of the x-axis which is the cuspidal tangent. Hence the origin is a single cusp of the first kind.

Ex. 3. *Examine the curve*

$$(y-1)^2=(x-3)^3 \qquad \ldots(1)$$

for singularities.

Here $$f(x, y)=(y-1)^2-(x-3)^3.$$

$$\frac{\partial f}{\partial x}=-3(x-3)^2, \quad \frac{\partial f}{\partial y}=2(y-1).$$

$\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$ give $x=3$, $y=1$. The solution also satisfies (1). Hence (3, 1) is a multiple point.

Shift the origin to the point (3, 1). Then if (x, y) transforms into (X, Y), we have

$$x=X+3, \qquad y=Y+1.$$

Hence the equation of the cure becomes $Y^2=X^3$.

Tangents at the new origin are given by $Y^2=0$ which represents a pair of coincident lines.

Thus the new origin is a cusp.

Again from the equation to the curve, $Y=\pm X^{3/2}$. Hence the equations to the two branches of the curve are

$$Y=X^{3/2} \qquad \ldots(2)$$

and

$$Y=-X^{3/2} \qquad \ldots(3)$$

Now X cannot be negative since the corresponding values of Y are imaginary. $\therefore$ no part of the curve, lies to the left of the Y-axis. Hence the cusp is a single cusp.

Again, when X is positive, the ordinate of the first branch is positive and that of the second branch is negative. Thus the two branches of the curve lie on opposite sides of the X-axis which is the cuspidal tangent. Hence the cusp is of the first kind.

Thus with respect to the original axes, we may say that the given curve has a single cusp of the first kind at (3, 1).

Examples LIX

Find the form of the curve near the origin and state the nature of the cusp :

1. $y^2=x^3$.
2. $y^4=a^2x^2$.
3. $y^3=x^2(x-2a)$.
4. $x^2(x-y)=y^2$.
5. $x^3+y^3-ay^2=0$.
6. $(y-2x^2)^2=x^7$.

Examine the form of the curve near the origin in the following cases :—

7. $x^5+16x^2y-64y^2=0$. 8. $x^4-2x^2y+2xy^2+y^2=0$.
9. $x^4-2x^2y-xy^2+y^2=0$. 10. $x^4+y^4=cx^2y$.

11. Show that the curve

$$x^3+2x^2+2xy-y^2+5x-2y=0$$

has a cusp of the first kind at the point $(-1, -2)$. *(Panjab, 1945)*

12. Show that the curve $y^3=(x-a)^2(2x-a)$ has a single cusp of the first kind at the point $(a, 0)$. *(Aligarh, 1946)*

Miscellaneous Examples V

1. Find the asymptotes of the following curves :

(i) $x^3+3x^2y-xy^2-3y^3+x^2-2xy+3y^2+4x+5=0$. *(Panjab, 1945)*
(ii) $x^3-4xy^2-3x^2+12xy-12y^2+8x+2y+4=0$. *(Panjab, 1947)*
(iii) $y^2(x-b)=x^3+a^3$. *(Panjab, 1950)*
(iv) $x^2y-xy^2+xy+y^2+x-y=0$. *(Panjab, 1955)*
(v) $(x^2-a^2)y^2=x^2(x^2-4a^2)$. *(Panjab, 1959)*
(vi) $x^2y=x^3+x+y$. *(Calcutta, 1955)*

2. Show that the asymptotes of the curve

$$x^2y^2-x^2-y^2-x-y+1=0$$

form a square through two of whose angular points the curve passes. *(Panjab, 1953)*

3. Find all the asymptotes of the curve

$$3x^3+2x^2y-7xy^2+2y^3-14xy+7y^2+4x+5y=0.$$

Show that the asymptotes meet the curve again in three points which lie on a straight line, and find the equation of this line. *(Delhi, Hons., 1952)*

4. Find the equation of the cubic curve which touches the y-axis at the origin, passes through the point $(2, 0)$ and the lines $y=2x$, $y=x\pm 1$ for asymptotes.

5. A cubic curve, having a cusp at $(0, 0)$ with the x-axis as the cuspidal tangent, has the three linear asymptotes.

$$a_rx+b_ry+1=0,\ r=1, 2, 3.$$

Prove that

$$\Sigma\frac{1}{a_1}=0 \text{ and } \Sigma b_1(a_2+a_3)=0. \qquad \textit{(I.A.S., 1952)}$$

6. Find the equation of the cubic curve having

$$x=0,\ y=0,\ x+2y-6=0$$

for its asymptotes and having a double point at $(-1, 2)$. Find the equations of the tangents at the double points.

7. Find the asymptotes of

(*i*) $r \sin \theta = 2 \cos 2\theta$. (*Agra, 1950*)

(*ii*) $r\theta \cos \theta = a \cos 2\theta$.

8. Show that every asymptote of the curve

$$r = a \sec m\theta + b \tan m\theta$$

touches one of two fixed circles.

9. Find the points of inflexion on the curves :

(*i*) $y = 3x^4 - 4x^3 + 1$.

(*ii*) $a^2y^2 = x^2(a^2 - x^2)$. (*Delhi, Hons., 1955*)

(*iii*) $x = t - 1,\ y = t^3 + 1$. (*iv*) $x = \tan \theta,\ y = \cos^2\theta$.

10. (*i*) Show that no conic section can have a point of inflexion.

(*ii*) Show that $y = (\log x)^n$, where n is a positive integer greater than unity, has one or two points of inflexion according as n is even or odd.

11. For the curve

$$y = x^3 + bx^2 + c$$

where $b < 0$, show that the point of inflexion is equidistant from the maximum and minimum points. (*Panjab, B Sc. 1962 S*)

12. Show that the line joining the two points of inflexion of the curve

$$(x - a)y^2 = (x + a)x^2$$

subtends an angle $\frac{1}{3}\pi$ at the origin.

13. Show that the cubic curve cannot have more than one double point unless it degenerates.

14. Find the coordinates of the double point and of the inflexions of the plane curve :

$$x : y : 1 = (t + 1)^3 : t^3 : (t - 1)^3$$

and show that the inflexions are collinear. (*M.T.I., 1948*)

15. Find the double points on the curves :

(*i*) $(x - 2)^2 = y(y - 1)^2$. (*Delhi, 1959*)

(*ii*) $x^2(x - y) + y^2 = 0$. (*Delhi, 1951*)

(*iii*) $x^2y^2 = (a + y)^2(b^2 - y^2)$, distinguishing between the cases $b < a$ and $b > a$. (*Delhi, Hons., 1953*)

(*iv*) $x^4 + y^2 - 4a^2xy = 0$. (*Panjab, 1959*)

(*v*) $a^2y^2 - b^2x^2 - x^2y^2 = 0$. (*Panjab, 1956*)

16. Show that the curve

$$y^2 = (x - a)^2(2x - a)$$

has a single cusp of the first kind at the point $(a, 0)$.

17. Find the multiple points on the curve

$$ay^2 - x^3 + bx^2 = 0$$

and discuss their nature. (*Panjab, 1962*)

18. Find the nature of the double points on the curve

$$(y + 1)^2 = (x - 1)^2(x - 4)$$

and show that it has two real points of inflexion. (*Panjab, 1961*)

CHAPTER XVI

CURVE TRACING

16·1. It is often of considerable interest to find the general shape of a curve represented by a given equation. analytical geometry, the student must have discussed the shapes of the curves represented by equations of the second degree, particularly those represented by the standard equations like $y^2=4ax$, $\frac{x^2}{a^2}\pm\frac{y^2}{b^2}=1$, etc. He will, therefore, be conversant with the elementary notions of symmetry, intercepts on the axes, etc. The aim of the present chapter is to apply the methods of the Calculus in supplying additional information in the form of points of inflexion and of maximum or minimum values, double points, asymptotes, etc., and to lay down certain rules so that it may be possible to get a sketch of the graph without having to know a large number of points on the curve. It is evident that a thorough knowledge of the previous chapters is essential before one can take to curve tracing. The procedure laid down in the following section will be helpful in drawing a fairly large number of curves which one generally comes across. The order in which the various steps are indicated is by no means rigid or exclusive and can be varied to suit the exigencies of the case.

16·2. I. Symmetry. (*i*) *If the equation of a curve remains unchanged when y is changed into $-y$, the curve is symmetrical about the x-axis.* For, in this case, if a point (x, y) lies on the curve, then the point $(x, -y)$ also lies on the curve and *vice versa.*

In the case of algebraic curves, it implies that the equation should contain only even powers of y. Thus $y^2=4x$ is symmetrical about the x-axis.

(*ii*) *A curve is symmetrical about the y-axis when its equation remains unchanged when x is changed into $-x$,* for if a point (x, y) lies on the curve, then the point $(-x, y)$ also lies on the curve and *vice versa.* For example, $y(a^2+x^2)=a^2-x^2$ is symmetrical about the y-axis.

When a curve is symmetrical about both axes, its form need be known in the first quadrant only. The complete graph can then be drawn by symmetry.

(*iii*) *If the equation of the curve remains unchanged when* **both** *x and y are changed into $-x$, $-y$ respectively, the curve is symmetrical in opposite quadrants.* For examples,

$$xy=c^2, \qquad xy^3+x^3y+a^2(x^2-y^2)=0$$

are symmetrical in opposite quadrants.

Every curve which is symmetrical about both axes is *ipsofacto* symmetrical in opposite quadrants but the converse is not true. As will be seen, neither of the two curves whose equations are given above, are symmetrical about either axis.

(iv) *If the equation of the curve remains unchanged when x and y are interchanged the curve is symmetrical about the line $y=x$.* For, the points (x, y) and (y, x) are symmetrical about the line $y=x$ and if (x, y) lies on the curve, so does (y, x) and *vice versa*. Hence the curve is symmetrical about $y=x$. Thus the curve $x^3+y^3=3axy$ is symmetrical about the line $y=x$.

Similarly, if the equation of a curve remains unchanged when x and y are replaced by $-y$ and $-x$ respectively, then the curve is symmetrical about the line $y=-x$.

II. Nature of the origin. Notice if the curve passes through the origin. It will be so in the case of algebraic curves if the absolute term is missing from the equation. If the curve passes through the origin, write down the equation of the tangent or tangents thereat. In case there are two or more tangents, find the nature of the singularity at the origin. Also find the position of the curve relative to the tangents at the origin. This can be done by the methods given in an earlier chapter.

III. Intersection with the axes. Find the points of intersection of the curve and the axes of coordinates. Find the tangents at these points, if necessary, and the position of the curve relative to these tangents. If possible, find some other points on the curve, for example, its points of intersection with $y=x$, or $y=-x$.

IV. Asymptotes. Find the asymptotes of the curve and the relative positions of the curve and the asymptotes.

V. Regions containing the curves. Find regions to which the curve is confined. This is usually done by solving for x or for y separately and considering both positive and negative values of x. Values of x (or y) which make y (or x) imaginary are to be ruled out. Again such values of x and y which make the left and right members of an equation opposite in sign are to be rejected. For example, consider the equation.

$$y^2=\frac{(x-1)^3}{2-x}.$$

If x is less than 1, y^2 is negative so that y is imaginary.

Again, y^2 is imaginary if x is greater than 2.

Thus the curve lies entirely inside the region bounded by the two parallels $x=1$ and $x=2$.

Again, consider the curve $x^4+y^4=4a^2xy$.

We observe that the left-hand member is positive for all values of x and y. Hence so must be the right-hand member. Thus x and y must both be of the same sign. Therefore no part of the curve

can lie in the second or fourth quadrant where x and y are of opposite signs.

It is also sometimes useful to change to polar coordinates for this purpose or to obtain the parametric equations of the curve. For example, changing to polar coordinates, the equation $(x^2+y^2)^2=a^2(x^2-y^2)$ transforms into $r^2=a^2\cos 2\theta$ and since $\cos 2\theta$ cannot be greater than 1, it follows that $r^2 \leqslant a^2$, so that the curve lies inside a circle of radius a with its centre at the pole.

Again, from the parametric representation of

$$x^{2/3}+y^{2/3}=a^{2/3} \quad viz., \quad a=x\cos^3 t,\ y=a\sin^3 t,$$

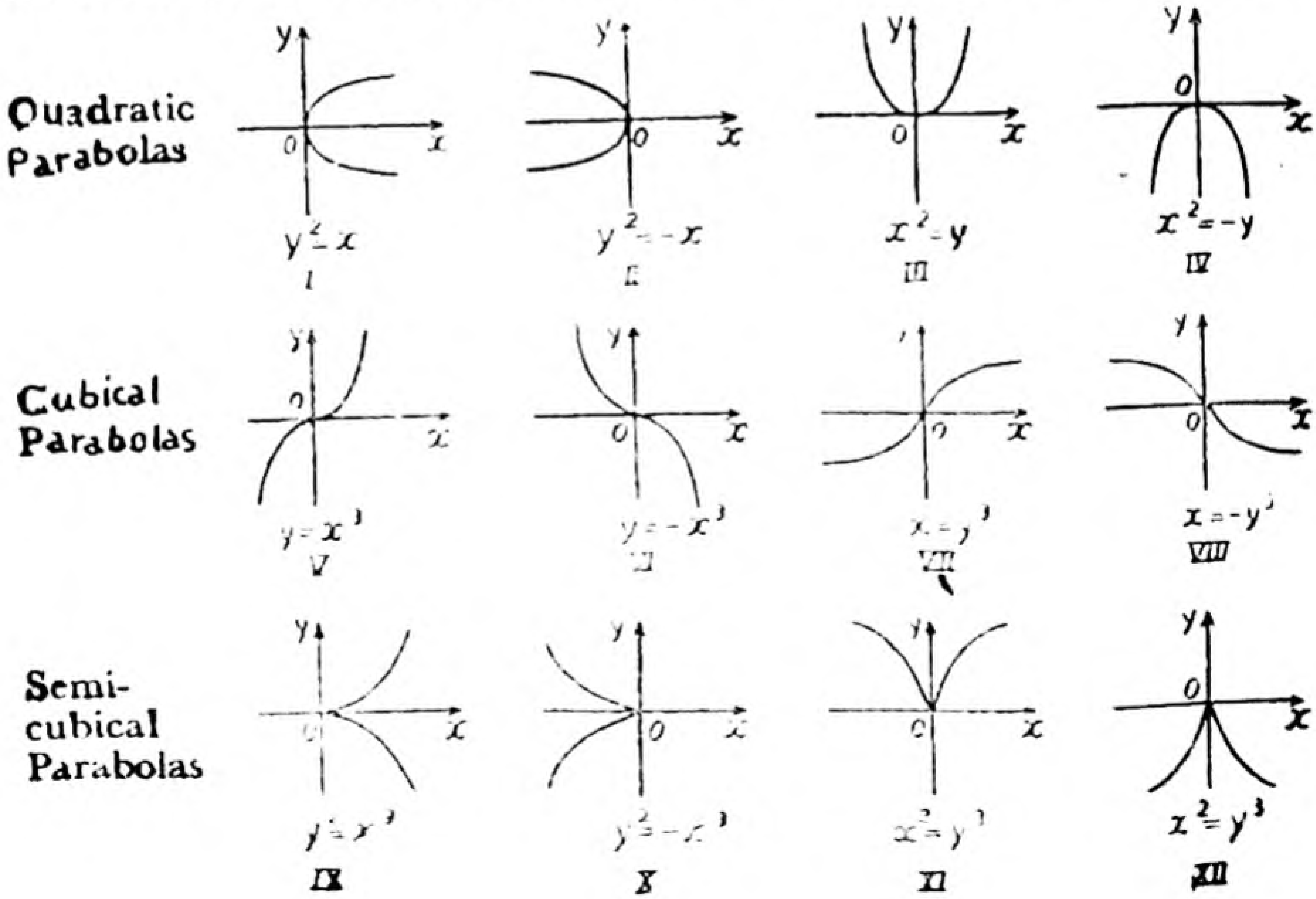
it is obvious that x and y cannot be greater than a numerically. Thus the whole curve lies inside the square formed by the lines

$$x=\pm a,\ y=\pm a.$$

VI. Stationary values. Find $\dfrac{dy}{dx}$, if convenient, and find the points at which the tangent is parallel to the x-axis or y-axis, discussing the behaviour of y (*i.e.* whether y is increasing or decreasing) between consecutive points.

VII. Inflexions and other singularities. Find points of inflexion and other singular points, if any, and in the case of the latter, examine their nature. This may not be done if it involves tedious calculations.

VIII. Approximations. For small values of x and y, it is terms of the lower degrees that matter and an approximate form of



the curve near the origin is got out of these. For large values of x and y, we fall upon terms of the higher degrees and get an approximation of the curve out of them. For example, the curve $y^2=x+x^3$ approximates to $y^2=x$ near the origin and to $y^2=x^3$ for large values of x and y. For this purpose it is important to know the forms of some elementary parabolas which are given on previous page.

We discuss below the tracing of two of these parabolas

(*a*) $\mathbf{y=x^3}$.

(*i*) The curve is symmetrical in opposite quadrants.

(*ii*) The curve passes through the origin. The tangent at the origin is $y=0$. If x is positive, y is positive and if x is negative y is negative. The curve, therefore, lies above the tangent in the first quadrant and below it in the third. Hence the origin is a point of inflexion.

(*iii*) Since x and y must both be of the same sign, there is no part of the curve in the second and fourth quadrants where x and y are of opposite signs.

(*iv*) The curve crosses the axes only at the origin.

(*v*) There are no asymptotes.

(*vi*) $\dfrac{dy}{dx}=3x^2$, which is positive for every x; therefore y increases with x numerically and tends to infinity with x.

(*vii*) $\dfrac{dy}{dx}=0$ only at the origin.

(*viii*) $\dfrac{d^2y}{dx^2}=6x$, which is positive for all positive values of x and is negative for negative values of x.

The curve is, therefore, concave upwards in the first quadrant and convex upwards in the third.

There is obviously an inflexion at the origin, since $\dfrac{d^2y}{dx^2}$ vanishes at the point and changes sign. [*Cf.* (*ii*) above].

Hence the form is as shown at V in the attached figure.

(*b*) $\mathbf{y^2=x^3}$.

(*i*) the curve is symmetrical *w.r.* to the x-axis only.

(*ii*) The curve passes through the origin and tangents at the origin are $y^2=0$ which represents a pair of coincident lines. Hence the orign is a cusp. Moreover, x cannot be negative, $\therefore$ there is a single cusp at the origin.

For any positive value of x, there are two equal and opposite alues of y. $\therefore$ the cusp is of the first kind.

(*iii*) The curve crosses the axes only at the origin and at no er point.

(*iv*) The curve has no asymptotes.

(*v*) The two branches of the curve are

$$y=\pm x^{3/2}.$$

Consider the branch $y=x^{3/2}$. For it

$$\frac{dy}{dx}=\tfrac{3}{2}x^{1/2},\ \frac{d^2y}{dx^2}=\tfrac{3}{4}.\frac{1}{\sqrt{x}}.$$

(*vi* $\frac{dy}{dx}=0$ only at the origin.

(*vii*) $\frac{d^2y}{dx^2}$ is positive for all positive values of x.

$\therefore$ the branch of the curve above the x-axis is concave upward.

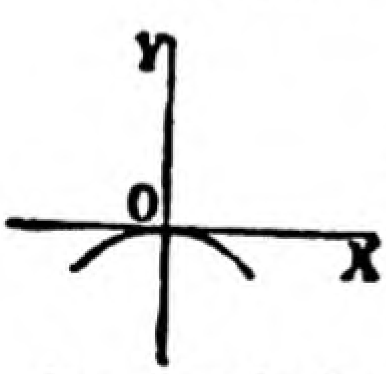
(*viii*) As x increases, y increases there being no limit to the increase of either. Hence the curve extends to infinity. Also y increases more rapidly than x for $x>1$.

Also $\frac{dy}{dx}\to\infty$ as $x\to\infty$, so that for large values of x the curve tends to be infinite in the direction of the y-axis.

Thus the form of the curve is as shown at IX in the above figure.

16·3 Curves of the form $y=f(x)$ and $y=\frac{f(x)}{g(x)}$, *where $f(x)$ and $g(x)$ are polynomials in x.*

Ex. 1. *Trace the curve $y=x^3-3ax^2$.*



(*i*) There is no symmetry of any kind.

(*ii*) The curve passes through the origin and the equation of the tangent at the origin is $y=0$. Near (0, 0), the curve is approximately of the form $y=-3ax^2$, showing that (*a*) the curve lies below the tangent on either side of the origin, and (*b*) the shape of the curve at the origin is like that of a parabola with its axis parallel to the y-axis and pointing downwards.

(*iii*) Where the curve meets the x-axis, $y=0$.

$$\therefore \qquad x^3-3ax^2=0 \text{ or } x=0 \text{ or } 3a.$$

(*iv*) The curve has no asymptotes. For lange values of x, however, the curve behaves like $y=x^3$.

(*v*) A value of y exists for every value of x.

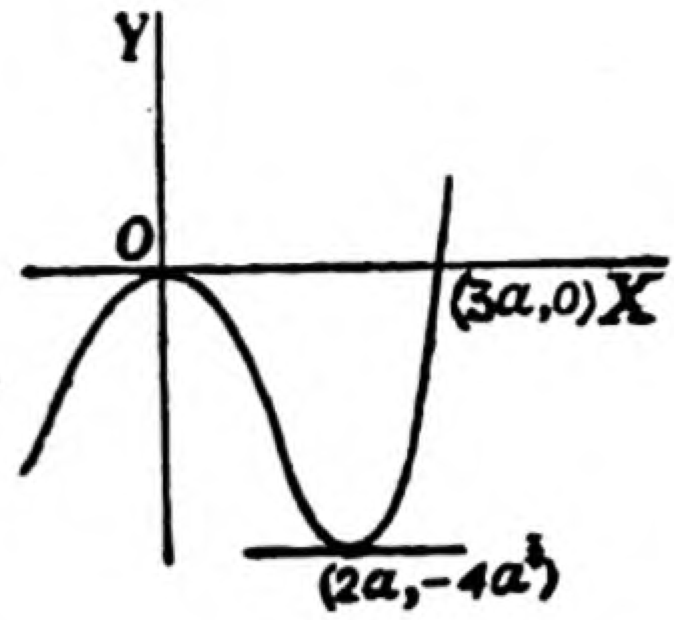
Also $y\to\infty$ as $x\to\infty$, and $y\to-\infty$ as $x\to-\infty$

Writing the equation of the curve in the form

$$y=x^2(x-3a),$$

it is obvious that y is negative for $x<3a$ and positive for $x>3a$.

(*vi*) $\frac{dy}{dx}=3x(x-2a)$. Hence, $\frac{dy}{dx}$ vanishes at $x=0$ and $x=2a$. When $x=0$, $y=0$; when $x=2a$, $y=-4a^3$.



(*vii*) $\frac{d^2y}{dx^2}=6(x-a)$ which vanishes as $x=a$ and changes sign.

$\therefore$ $x=a$ gives a point of inflexion.

$\frac{d^2y}{dx^2}$ is positive for $x>a$ and negative for $x<a$. $\therefore$ The curve is concave upward for $x>a$, and convex upward for $x<a$.

The form of the curve is as shown in the diagram.

Ex. 2. *Trace the curve* $y(1-x^2)=x^2$. ((*Panjab*, *1960*)

(*i*) The curve is symmetrical *w.r.* to y-axis.

(*ii*) The curve passes through the origin.

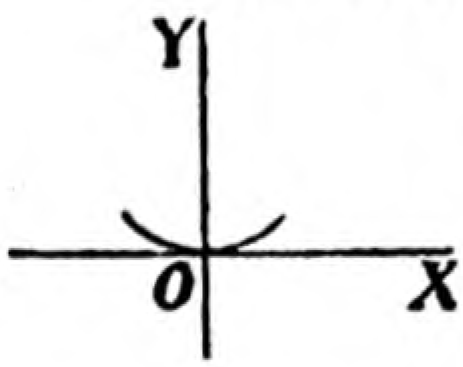
Writing the equation in the form

$$x^2y+x^2-y=0,$$

we see that $y=0$ is the tangent at the origin.

Near (0, 0), the first approximation is the tangent $y=0$.

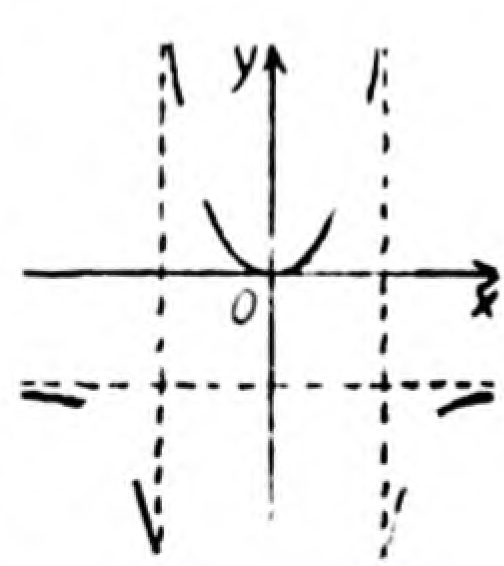
The second approximation is $x^2-y=0$, or $y=x^2$ [neglecting the first term which becomes an infinitesimal of order 4].



Thus the curve lies above the tangent and approximates the parabola $x^2=y$ near the origin.

(*iii*) The curve does not meet the axes at any other point besides the origin.

(*iv*) $x=\pm 1$ and $y=-1$ are its three asymptotes.

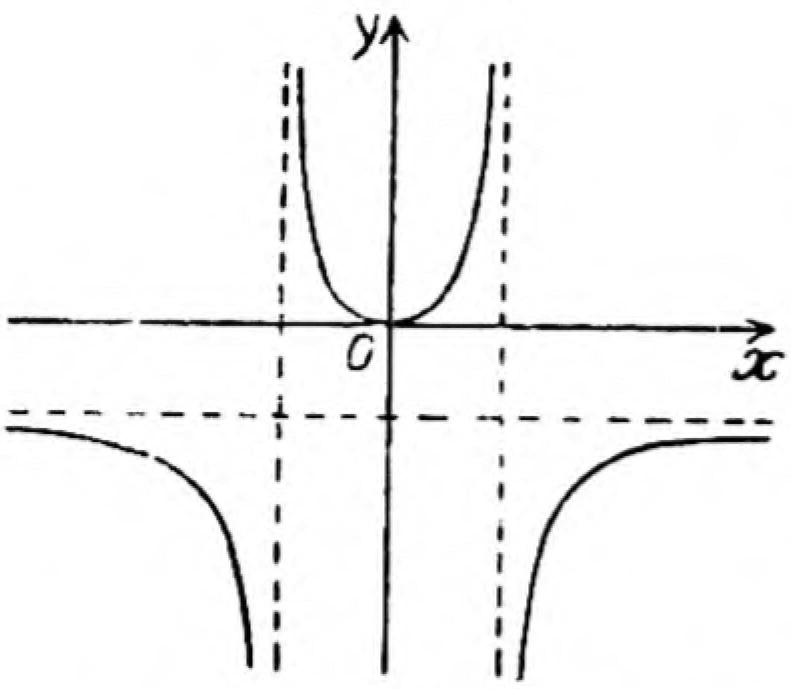


(*v*) Writing the equation in the form

$$x^2=\frac{y}{1+y},$$

we observe that y cannot lie between 0 and -1.

$\therefore$ The asymptote $y=-1$ must be approached from below.



y is positive for x numerically less than 1 and negative for x numerically greater than 1. Hence (*a*) the asymptote $x=1$ is approached from the left at the top and from the right from below, and (*b*) the asymptote $x=-1$ is approached from the right as the top and from the left from below.

Hence the shape of the curve is as shown in the diagram.

Ex. 3. *Trace the curve*

$$y=\frac{(x-1)(x-2)}{(x-3)(x-4)}.$$

(*i*) There is no symmetry.

(*ii*) The curve does not pass through the orgin.

(*iii*) It crosses the x-axis at $(1, 0)$ and $(2, 0)$ and the y-axis at $(0, \frac{1}{6})$.

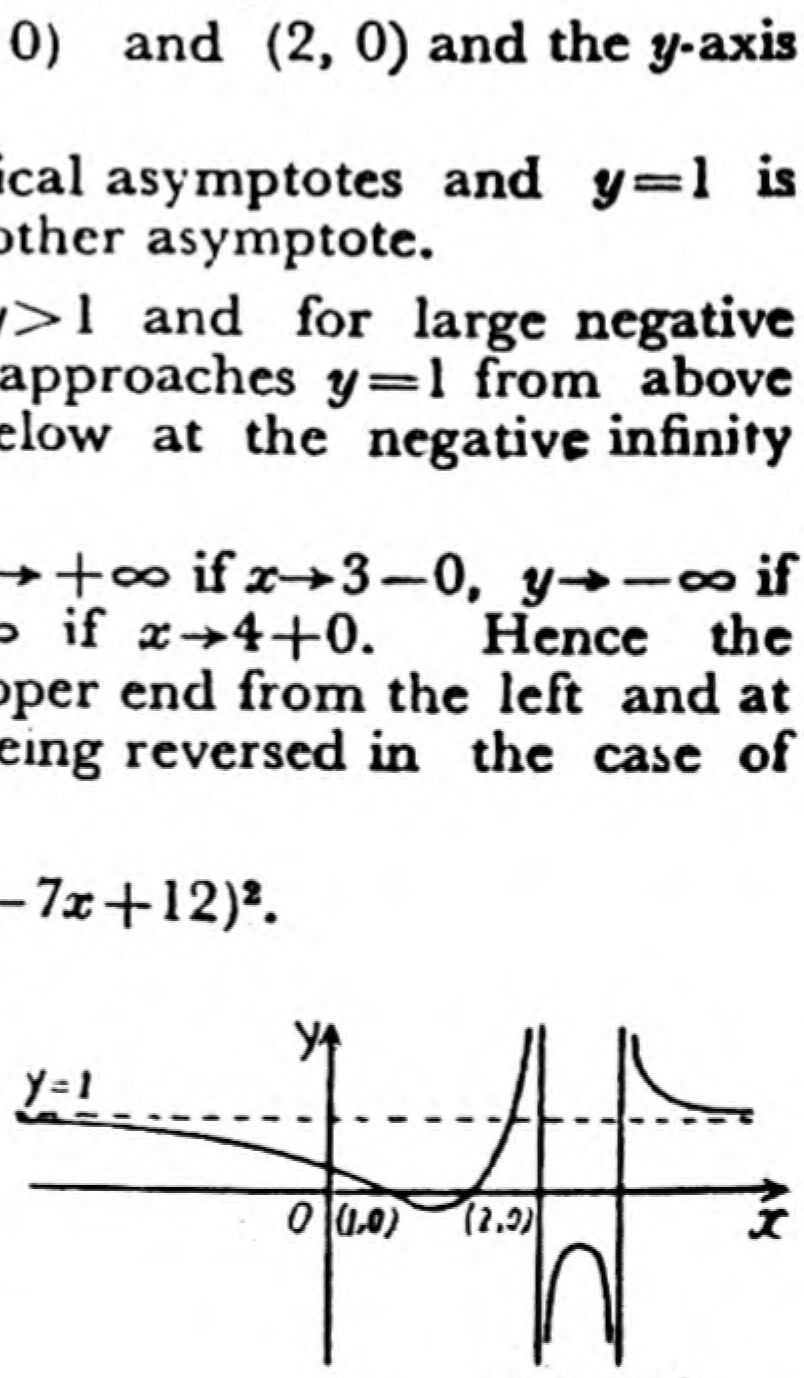
(*iv*) $x-3=0$, $x-4=0$ are the vertical asymptotes and $y=1$ is the horizontal asymptote. There is no other asymptote.

For large positive values of x, $y>1$ and for large negative values of x, $y<1$. Therefore, the curve approaches $y=1$ from above at the positive infinity end and from below at the negative infinity end.

(*v*) y exists for every value of x, $y\to+\infty$ if $x\to3-0$, $y\to-\infty$ if $x\to3+0$, $y\to-\infty$ if $x\to4-0$, $y\to+\infty$ if $x\to4+0$. Hence the asymptote $x=3$ is approached at its upper end from the left and at its lower end from below, this order being reversed in the case of $x=4$.

(*vi*) $\dfrac{dy}{dx}=-2(2x^2-10x+11)/(x^2-7x+12)^2$.

$\therefore$ $\dfrac{dy}{dx}=0$ at $x=\frac{1}{2}(5\pm\sqrt{3})$ and changes sign at each of these points. These values of x give extreme values of y, a minimum at $x=\frac{1}{2}(5-\sqrt{3})$ and a maximum at $x=\frac{1}{2}(5+\sqrt{3})$.



Hence the form of the curve is as shown in the diagram.

16·31. Curves of the form $\mathbf{y^2=f(x)}$ or $\mathbf{y^2=\dfrac{f(x)}{g(x)}}$ *where $f(x)$ and $g(x)$ are polynomials in x.*

Ex. 1. *Trace the curve* $a^4y^2=a^2x^4-x^6$.

(*i*) The curve is symmetrical *w.r.t.* both axes.

(*ii*) It passes through the origin and tangents at the origin are $y^2=0$. Hence there is a cusp at the origin.

As an approximation near (0, 0), we have

$$a^4y^2=a^2x^4 \text{ or } x^2=\pm ay.$$

the + sign corresponding to the upper and the $-$sign to the lower branch. Thus near the origin, the curve approximates to two parabolas touching each other at their vertices and having their axes in opposite directions along the y-axes.

(*iii*) The curve crosses the x-axis at (0, 0), $(\pm a, 0)$.

It crosses the y-axis only at the origin.

(*iv*) The curve has no asymptotes.

(*v*) The equation may be written as

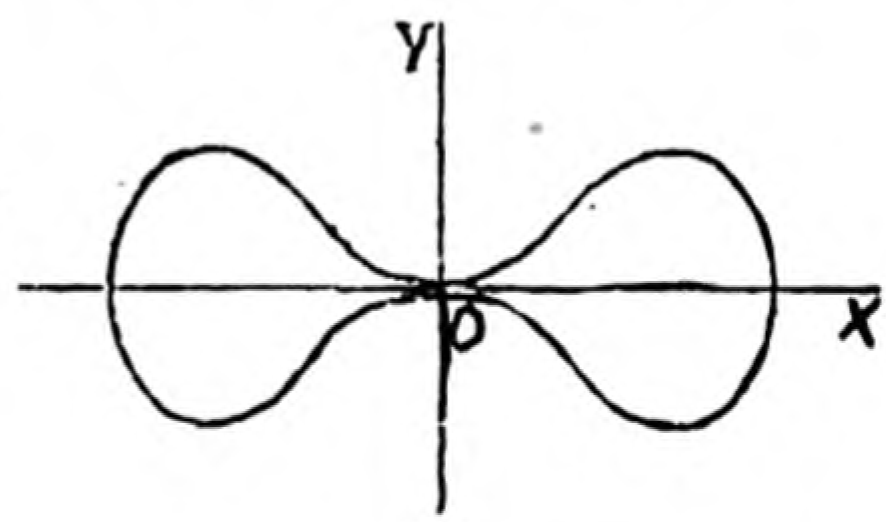
$$a^4y^2=x^4(a^2-x^2).$$

$\therefore$ x cannot be greater than a numerically.

(*vi*) $\dfrac{dy}{dx}=\dfrac{4a^2x^3-6x^5}{2a^4y}=\dfrac{x(2a^2-3x^2)}{a^2.\ (a^2-x^2)}$.

Besides $x=0$ (which, as we have already seen, gives a cusp), $\dfrac{dy}{dx}$ vanishes at the points for which $x=\pm\sqrt{\tfrac{2}{3}}a=\pm\cdot8a$ approximately and then $y=\pm\cdot4a$ appromately,

Thus the shape of the curve is as shown in the diagram.



Ex. 2. *Trace the curves*

(*a*) $y^2=x^2.\dfrac{a+x}{a-x}$.

(*b*) $y^2=x^2.\dfrac{x+a}{x-a}$

These two curves and those in the next example are discussed in order to illustrate how a slight change in the equation affects the form of the curve.

(*a*) $y^2=x^2\ \dfrac{a+x}{a-x}$.

(*i*) The curve is symmetrical about the x-axis.

(*ii*) It passes through the origin and writing the equation in the form $x^3+xy^2+a(x^2-y^2)=0$, we note that tangents at the origin are given by $x^2-y^2=0$, *i.e.*, $y=\pm x$.

The origin is, therefore, a node.

It can be easily verified that the curve lies above the tangent $y=x$ and below the tangent $y=-x$ on either side of the origin.

(*iii*) The curve crosses the x-axis at the origin and at $(-a, 0)$.

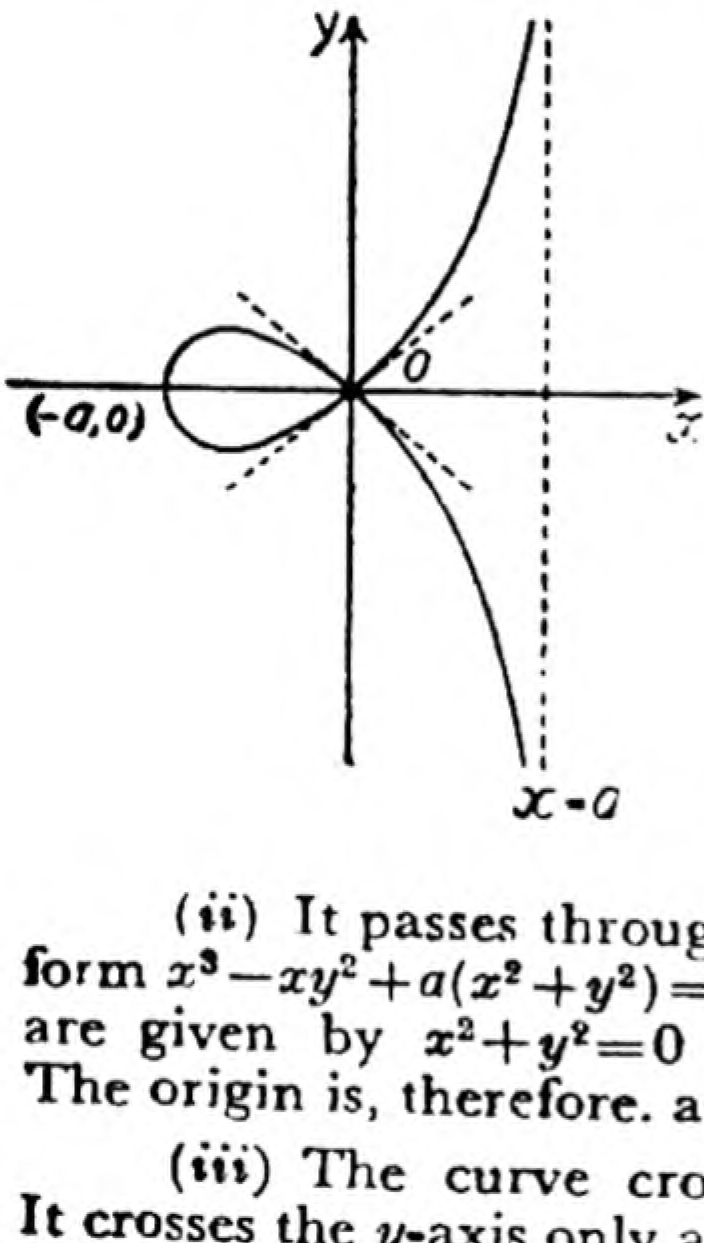
It crosses the y-axis only at the origin. From considerations of symmetry, this indicates a loop corresponding to $-a \leqslant x \leqslant 0$.

(*iv*) $a-x=0$ *i.e.*, $x=a$ is the only real asymptote. Since x cannot be greater than a (for, in that case, y is imaginary), the asymptote is approached from the left at either end.

(*v*) If x is negative and less than $-a$ algebraically, y is again imaginary. Hence no part of the curve lies beyond the line

$$x=-a.$$

(*vi*) $$\frac{dy}{dx}=\pm\frac{a^2+ax-x^2}{(a-x)^{3/2}(a+x)^{1/2}}.$$



$$\therefore \quad \frac{dy}{dx}=0 \quad \text{when } x=\tfrac{1}{2}(1-\sqrt{5})a$$

[The value $x=\frac{1}{2}(1+\sqrt{5})a$ is rejected, since x cannot be greater than a],

$$\frac{dy}{dx}\to\infty \text{ when } x\to\pm a.$$

Hence the form of the curve is as shown in the diagram.

(*b*) $y^2=x^2\dfrac{x+a}{x-a}$.

(*i*) The curve is symmetrical *w.r.t.* ox.

(*ii*) It passes through the origin and writing the equation in the form $x^3-xy^2+a(x^2+y^2)=0$, we note that the tangents at the origin are given by $x^2+y^2=0$ which represents a pair of imaginary lines. The origin is, therefore. a conjugate point.

(*iii*) The curve crosses the x-axis at the origin and at $x=-a$. It crosses the y-axis only at the origin.

(*iv*) $x-a=0$ is a vertical asymptote. Other two asymptotes are $x-y+a=0$ and $x+y+a=0$.

The curve crosses both asymptotes at $(-a, o)$ and being a third degree curve, it cannot re-cross either asymptote.

Since x cannot lie between $-a$ and $+a$, the vertical asymptote must be approached from the right.

Expanding y in descending powers of x, we get,

$$y=\pm\left(x.+a+\tfrac{1}{2}\frac{a^2}{x}+\ldots\right).$$

The approximation to the branch to which $y=x+a$ is an asymptote is

$$y=x+a+\tfrac{1}{2}\frac{a^2}{x}+\ldots\ldots$$

showing that the curve lies above the asymptote at positive infinity end and below it at the opposite end. Similarly considering the — sign, the approximation is

$$y=-x-a-\tfrac{1}{2}\frac{a^2}{x}-\ldots\ldots$$

This shows that the curve lies below the asymptote $y=-x-a$ at the positive infinity end and above it at the opposite end.

(*v*) $\dfrac{dy}{dx}=\pm\dfrac{x^2-ax-a^2}{(x-a)^{3/2}(x+a)^{1/2}}$.

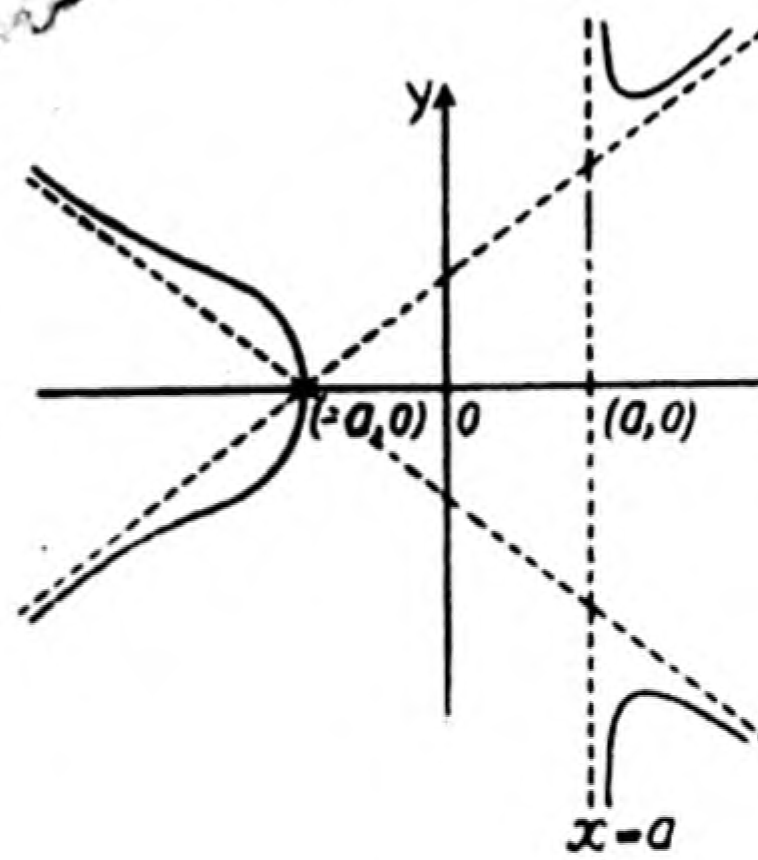
$\therefore \qquad \dfrac{dy}{dx}=0$

when $\qquad x=\tfrac{1}{2}(1\pm\sqrt{5})a$.

(the —sign is obviously to be rejected)

and $\qquad \dfrac{dy}{dx}\to\infty$ when $x\to\pm a$.

The form of the curve is, therefore, as shown in the diagram.



Ex. 3. *Trace the curves*

(*a*) $x^2y^2=x^2-1$. (*Panjab, 1962*)

(*b*) $x^2y^2=x^2+1$. (*Panjab, 1962 S*)

(*a*) $x^2y^2=x^2-1$.

(*i*) The curve is symmetrical *w.r.t.* both axes.

(*ii*) It does not pass through the origin.

(*iii*) It cuts the x-axis at $(\pm 1, 0)$. It does not meet the y-axis.

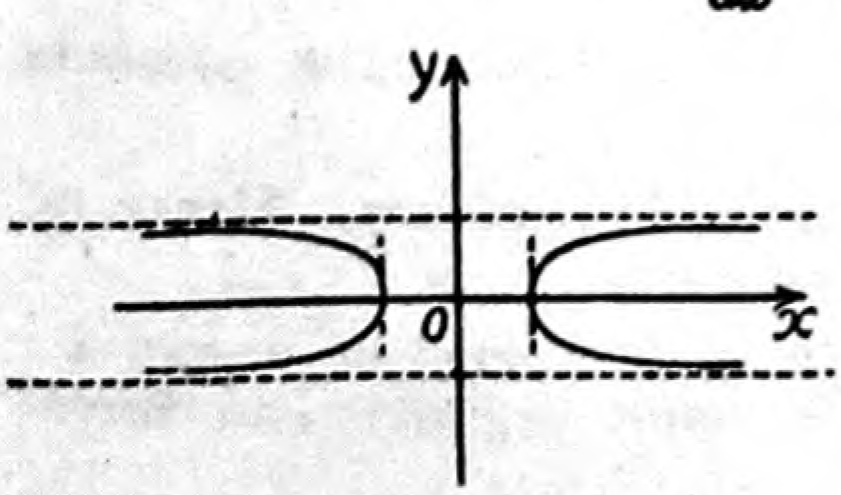
(*iv*) No part of the curve lies between the lines $x=\pm 1$.

(*v*) Writing the equation in the form $x^2(y^2-1)+1=0$, we note that the lines $y^2-1=0$ *i.e.* $y=\pm 1$ are asymptotes to the curve.

From the given equation itself, $x=0$ appears to be an asymptote. But since no part of the curve lies between $x=\pm 1$, this asymptote is to be rejected.

Since $y^2=(x^2-1)/x^2$, y is numerically <1 for all values of x numerically >1. Hence the curve lies between the two asymptotes at both ends.

(*vi*) $\qquad \dfrac{dy}{dx}=\pm\dfrac{1}{x^2\sqrt{x^2-1}}$



so that $\qquad \dfrac{dy}{dx}\to\infty$ as $x\to\pm 1$.

Hence tangents to the curve at its intersection with the x-axis are the vertical lines $x=\pm 1$.

The form of the curve is as shown in the diagram.

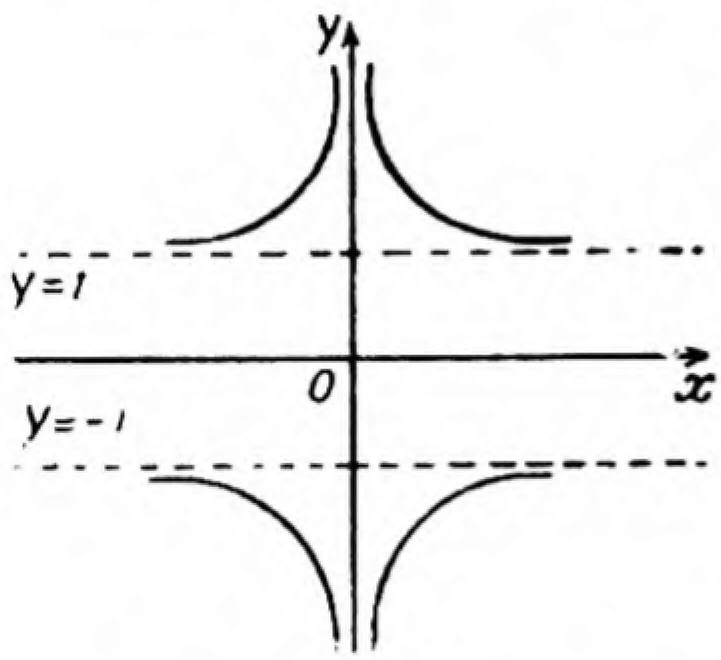
(*b*) $x^2y^2=x^2+1$.

(*i*) The curve is symmetrical *w.r.t.* both axes.

(*ii*) It does not pass through the origin.

(*iii*) It does not intersect the axes.

(*iv*) Writing the equation in the form $x^2(y^2-1)=1$, we observe that the lines $y^2-1=0$, *i.e.*, $y=\pm 1$ are asymptotes.



For x to be real, $y^2-1>0$ so that y is numerically >1. Hence the asymptote $y=1$ is approached at either end from above and the asymptote $y=-1$ from below.

$x=0$ is the only vertical asymptote. From symmetry, the asymptote $x=0$ is approached from both sides at either end. The form of the curve is shown in the diagram.

Ex. 4. *Trace the curve*

$$y^2=(x-a)(x-b)(x-c) \text{ where } 0<a<b<c,$$

and consider the form of the curve when

$$(1)\ a=b,\ (2)\ b=c,\ (3)\ a=b=c.$$

First of all, we consider the curve

$$y^2=(x-a)(x-b)(x-c),\ 0<a<b<c.$$

(*i*) There is symmetry *w.r.* to the x-axis.

(*ii*) The curve does not pass through the origin.

(*iii*) Where the curve crosses the x-axis, $y=0$, $\therefore$ $x=a, b$, or c. Thus the curve crosses the x-axis at three points *viz.*, $A(a, 0)$, $B(b, 0)$, $C(c, 0)$.

(*iv*) The curve has no asymptotes.

(*v*) The curve has no double points.

(*vi*) When $x=a$, there are two equal values of y. Hence $x=a$ is a tangent to the curve.

Similarly $x=b$ and $x=c$ are tangents.

Shifting the origin to $(c, 0)$, the equation becomes

$$y^2=x^3+(2c-a-b)x^2+c(c-a)(c-b)x.$$

Thus at the new origin, the curve approximates the parabola $y^2=(c-a)(c-b)x$.

(*vii*) For $x<a$, y^2 is negative so that y is imaginary. Hence no part of the curve lies to the left of the line $x=a$.

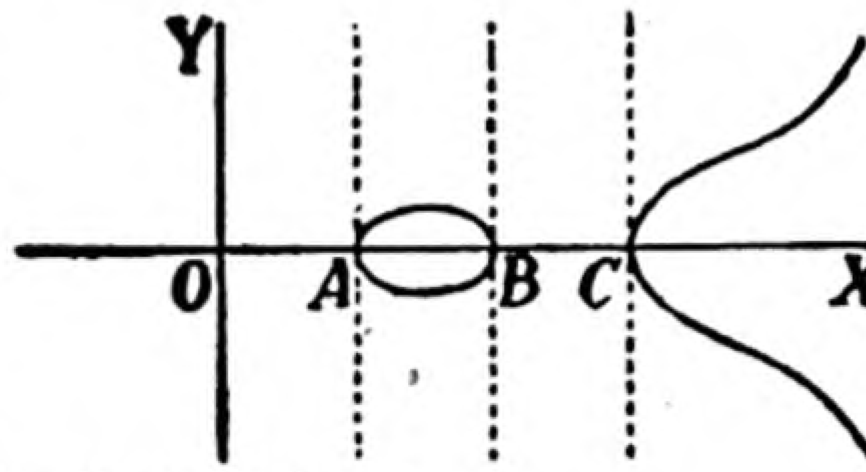
y^2 is positive so that y is real when x lies between a and b; but for values of x between b and c, y^2 is again negative and there-

fore y is imaginary. Hence no part of the curve lies between the lines $x=b$ and $x=c$.

y is defined for every value of $x>c$ and increases as x increases beyond c. As a matter of fact $y\to\infty$ when $x\to\infty$.

For every large values of x, the curve approximates to $y^2=x^3$ [retaining the highest degree terms in x and y] so that y increases very much more rapidly than x does.

(*viii*) $\because$ $y=0$ when $x=a$ and again when $x=b$, there must exist a point at which $\dfrac{dy}{dx}=0$ somewhere between A and B.



Thus the form of the curve is as shown in the diagram. It consists of a loop and a branch which extends to infinity.

Note. That the infinite branch possesses two points of inflexion may be seen from the fact that the curve approximates a parabola $y^2=(c-a)(c-b)(x-c)$ near $(c, 0)$ and to $y^2=x^3$ for large values of x. The former is convex and the latter concave in the positive y direction. Hence the curve changes its curvature from convexity to concavity for some value of $x>c$. From considerations of symmetry, if the upper branch has a point of inflexion, so must the lower branch have one for the same value of x.

(1) When $a=b$.

The equation now becomes

$$y^2=(x-b)^2(x-c) \qquad (b<c)$$

$\because$ $a=b$, the points A and B coincide so that the loop shrinks to a point.

This point is an isolated point on the curve. This becomes apparent on shifting the origin to $(b, 0)$. This equation becomes

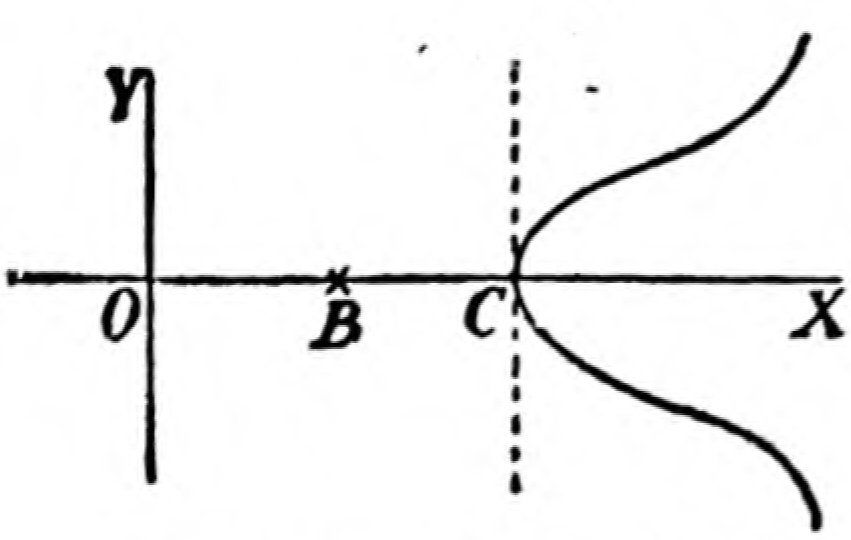
$$y^2=x^2(x-c+b)$$

giving the tangents at the new origin as

$$y^2+(c-b)x^2=0$$

which are imaginary as $b<c$.

Near $(c, 0)$, the curve is still of the form $y^2=k(x-c)$ and for large values it approximates to $y^2=x^3$. Hence there is a pair of points of inflexion as before and the shape of the curve is as shown in the opposite diagram.



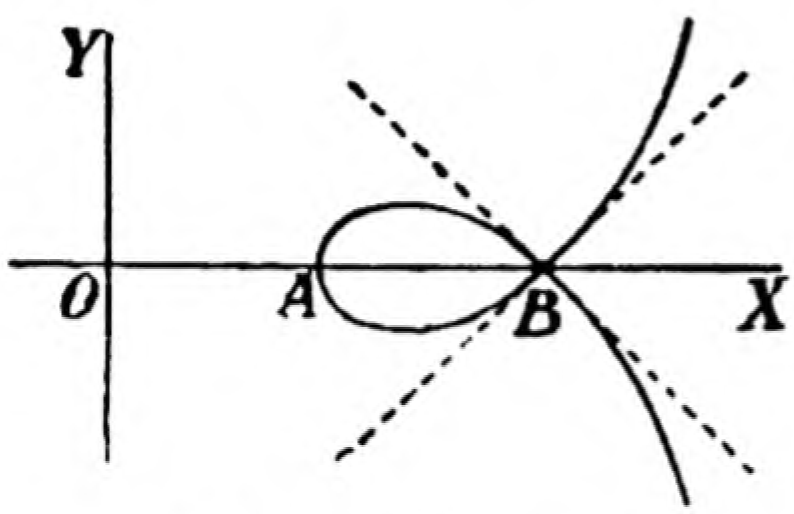
(2) When $b=c$.

The equation to the curve now is

$$y^2=(x-a)(x-b)^2. \qquad (a<b).$$

The points B and C coincide. The point B will be a node as will also be seen by shifting the origin to $B(b, 0)$. The equation referred to new axes becomes

$$y^2=x^2(x+b-a).$$



so that tangents at the new origin *w.r.* to the new axes are

$$y^2-(b-a)x^2=0$$

which are real $\because b>a$,

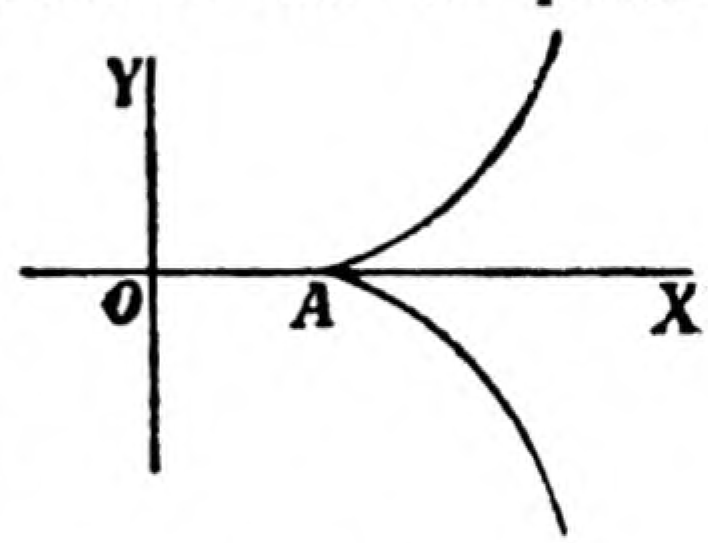
Hence the shape of the curve is as shown.

(3) When $a=b=c$.

The equation now becomes

$$y^2=(x-a)^3.$$

Referring to the figure of case (2), we observe that the points A and B coincide so that the loop shrinks to a point.



The point $(a, 0)$ will now become a cusp.

The form of the curve is as shown.

Note. On shifting the origin to $(a, 0)$, the equation becomes $y^2=x^3$, which shows that the curve is a semi-cubical parabola.

16·32. Curves of the form $\mathbf{y^2+f(x)\cdot y+g(x)=0}$.

Ex. 1. *Trace the curve* $y^2-2x^2y+x^4-x^5=0$.

(*i*) There is no symmetry.

(*ii*) The curve passes through the origin and tangents at the origin are $y^2=0$. The origin is, therefore, a cusp.

Writing the equation in the form

$$(y-x^2)^2=x^5 \qquad \ldots(1)$$

and taking square root, we get

$$y=x^2\pm x^{\frac{5}{2}}.$$

For small values of x, (in fact, for $0<x<1)x^2>x^{\frac{5}{2}}$ so that y is positive for each branch. Both branches of the curve, therefore, lie above the x-axis for positive values of x in the neighbourhood of the origin.

Again, from (1), we observe that x cannot be negative.

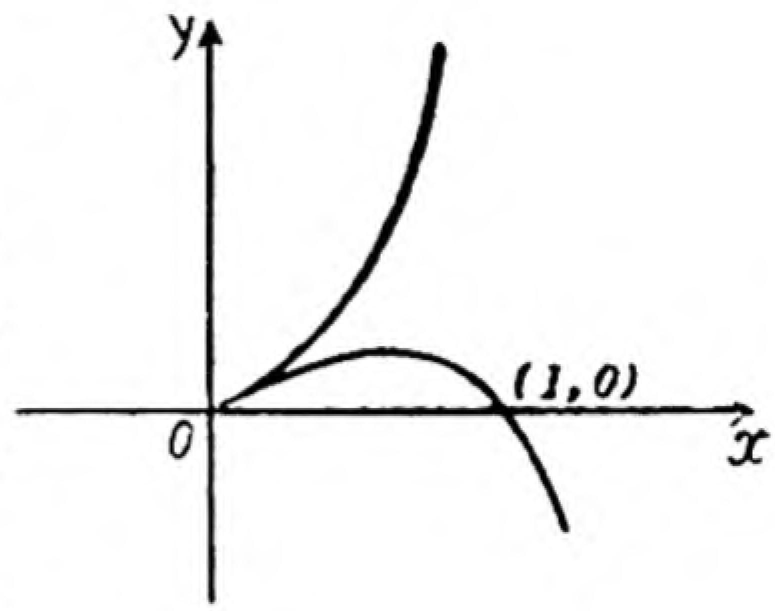
Hence no part of the curve lies in the second or third quadrants. The origin is, therefore, a single cusp of the second kind.

(*iii*) For the branch $y=x^2+x^{\frac{5}{2}}$, y is always positive and increases with x, $y\to\infty$ as $x\to\infty$. This branch does not cross the x-axis.

(*iv*) The branch $y=x^2-x^{\frac{5}{2}}$ crosses the x-axis at $(1, 0)$ and for $x>1$, y becomes and remains negative.

For this branch, $\dfrac{dy}{dx}=2x-\dfrac{5}{2}x^{\frac{3}{2}}=\dfrac{1}{2}x(4-5x^{\frac{1}{2}})$.

$\therefore\quad \dfrac{dy}{dx}=0$ when $x=0$ or $x=\dfrac{16}{25}$.



For $x>\dfrac{16}{25}$, $\dfrac{dy}{dx}$ is negative, so that y decreases as x increases. Obviously

$$y\to-\infty \text{ as } x\to+\infty.$$

(*v*) The curve does not have any asymptotes.

(*vi*) For large values of x, the curve behaves like $y^2=x^5$.

The form of the curve is as shown in the diagram.

16·33. Further solved examples.

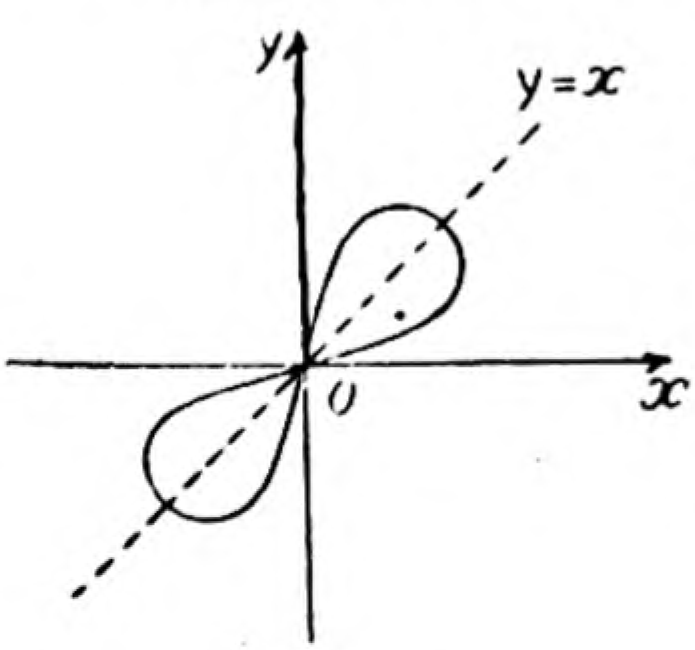
Ex. 1. *Trace the curve* $\quad x^4+y^4=4a^2xy$. $\qquad$ (*Panjab, 1959*)

(*i*) The curve is symmetrical *w.r.t.* the origin and also *w.r.t.* the line $y=x$.

(*ii*) It passes through the origin and tangents at the origin are $xy=0$, *i.e.*, $x=0$ and $y=0$. The origin is therefore, a node. An approximation near the origin to the branch to which $y=0$ is a tangent is $x^4=4a^2xy$ or $4a^2y=x^3$. This branch, therefore, lies above the x-axis in the first quadrant and below it in the third quadrant.

Similarly, the branch to which $x=0$ is a tangent at the origin has an approximation $4a^2x=y^3$. It lies to the right of the y-axis in the first quadrant and to the left of it in the third.

(*iii*) The curve crosses the axes only at the origin.



(*iv*) There are no asymptotes.

(*v*) Since the left member is always positive, the right must also be positive. Hence x and y must both of the same sign. The curve, therefore, is contained to the first and third quadrants only.

(*vi*) The curve meets the line $y=x$ at the points for which

$$x=0 \text{ or } x=\pm\sqrt{2}a.$$

The form of the curve is as shown in the diagram.

Ex. 2. *Trace the curve* $y(y^2-1)=x(x^2-4)$.

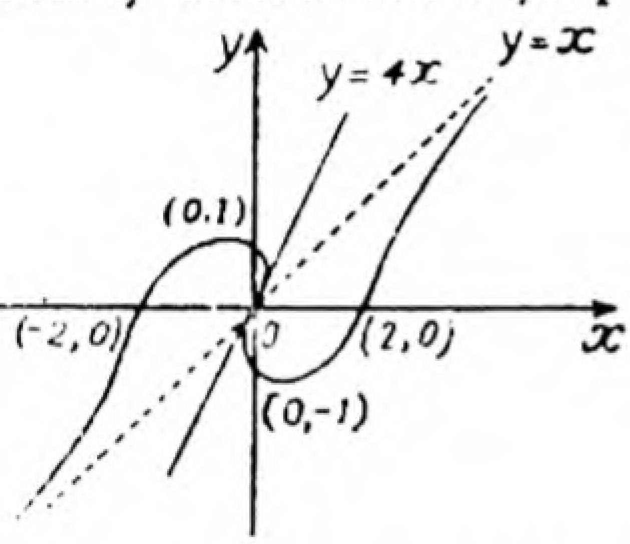
(*i*) The curve is symmetrical *w.r.t.* the origin. There is no other symmetry.

(*ii*) It passes through the origin and tangent at the origin is $y=4x$.

Writing the equation in the form $y=4x-x^3+y^3$, we get an approximation as $y=4x-x^3+(4x)^3$, *i.e.*, $y=4x+15x^3$. The curve, therefore, lies above the tangent to the right of the origin and below the tangent to the left.

(*iii*) The curve meets the x-axis at the points (0, 0), (2, 0) and (−2, 0). It meets the y-axis at (0, 0), (0, 1) and (0, −1).

(*iv*) There is only one real asymptote, *viz.*, $y=x$. The curve evidently crosses the asymptote at the origin and being a third degree curve, does not recross the asymptote at any other point.



Writing the equation to the curve as $y-x=\dfrac{y-4x}{x^2+xy+y^2}$, we get an approximation at infinity as

$$y-x=-\frac{1}{x} \text{ or } y=x-\frac{1}{x}.$$

The curve is thus seen to lie below the asymptote at the positive infinity end and above it at the negative infinity end.

The form of the curve is as shown in the diagram.

Ex. 3. *Trace the curve* $x^5+y^5=5a^2x^2y$.

(*i*) The curve is symmetrical in opposite quadrants ($\because$ equation to the curve remains unchanged when x and y are changed into $-x$, $-y$ respectively.)

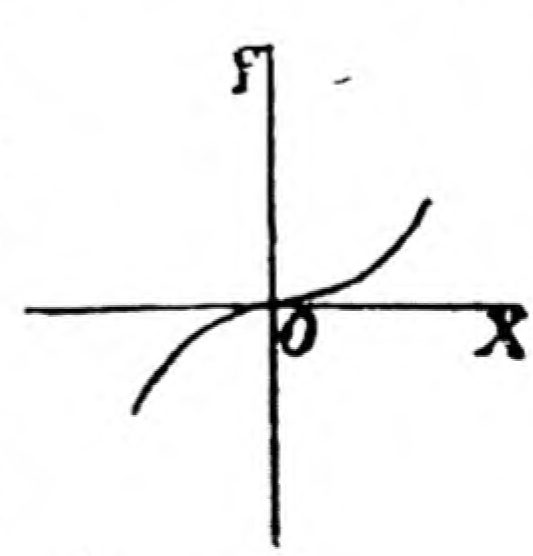
(*ii*) The curve passes through the origin. The tangents at the origin are

$$x^2y=0 \text{ i.e., } x=0,\ x=0,\ y=0.$$

$\therefore$ the y-axis is a cuspidal tangent.

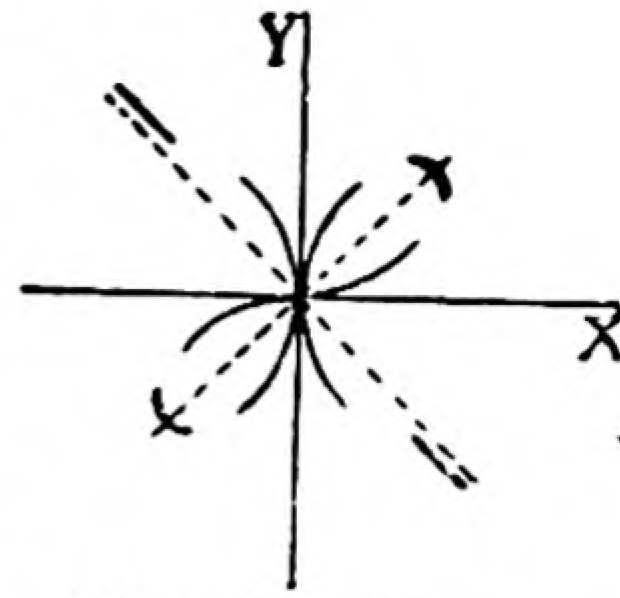
To discuss the form of the curve near the origin, omit y^5 and consider $x^5-5a^2x^2y=0$. Removing x^2, $5x^2y=x^3$. ...(*A*)

This branch has $y=0$ as the tangent at the origin. We observe that the origin is a point of inflexion, the curve lying above the tangent for positive values of x and below it for negative values.



Now omit x^5 and consider

$y^5-5a^2x^2y=0$ or $y^4-5a^2x^2=0$. ...(*B*)

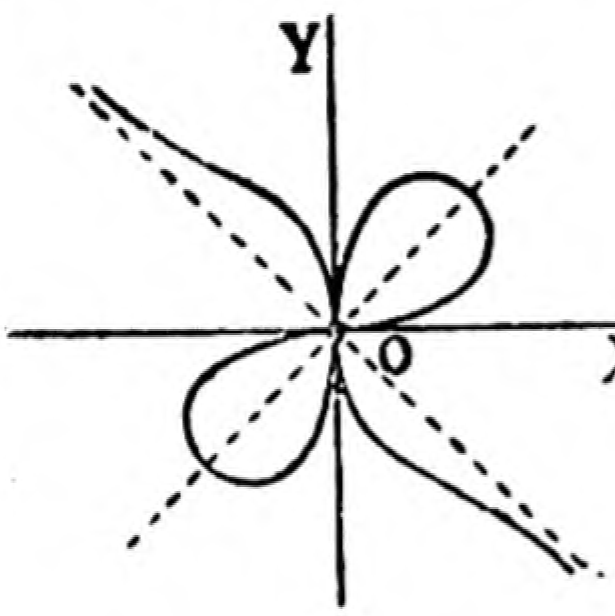


This branch has $x=0$ as a tangent at the origin. From (B), we find that the curve or a portion of it is like the two parabolas $y^2=\pm\sqrt{5ax}$.

(*iii*) $x+y=0$ is an asymptote to the curve and the curve crosses its asymptote at the origin.

In the second quadrant, $\because$ x is negative and y is positive, the right hand member is positive. Hence so must be the left-hand member. Hence, y should be numerically $>x$. Thus the curve above the asymptote.

Because of symmetry in opposite quadrants, the curve approaches the other end (in the fourth quadrant) from below.



(*iv*) The curve crosses the axes only at the origin and at no other point.

It crosses the line $y=x$ where $2x^5=5a^2x^3$, *i.e.*, where $x=0$ or $x=\pm\sqrt{\frac{5}{2}}a$. Connecting the various data above, the shape of the curve is as shown in the diagram.

Examples LX

Trace the curves :

1. (*i*) $y=x^2$, (*ii*) $y=x^3$, (*iii*) $y=x^4$, (*iv*) $y=x^5$.
2. $6y=2x^3-3x^2-12x+12$. 3. $y=x(x-a)^3$.
4. $a^2x=y(x^2+a^2)$. (*Nagpur, 1927*)
5. $ay^2=x^2(x-a)$. (*Delhi, 1955*) 6. $y^2=x(x+1)^2$.
7. $y^2=x(1-x)^2$. 8. $y^2=(x-1)(x-2)(x-3)$.
9. $y^2(a+x)=x^2(3a-x)$. (*Panjab, 1958*)
10. $xy^2=a^2(a-x)$. (*Panjab, 1943*)
11. $(x^2-1)y^2=x$. (*Agra, 1949*) 12. $y^2(x^2-a^2)=a^2x^2$.
13. $y^2(a^2-x^2)=b^2x^2$. (*Panjab, 1956*) 14. $y^2(a^2-x^2)=x^4$.
15. $y^2(x^2+a^2)=a^2x^2$. 16. $y^2(x^2+a^2)=a^3x$.
17. $y=\dfrac{2x^2}{(x-2)(x-6)}$. 18. $y^2=\dfrac{(x+1)(x+2)}{x}$.
19. $x^2(x^2-4a^2)=y^2(x^2-a^2)$. (*Panjab, 1949 S*)
20. $x^3+y^3=3axy$. (*Panjab, 1957 ; Delhi, 1959*)
21. $x^3+y^3=2ax^2$. (*Panjab, 1930*)
22. $x^3+y^3=a^2x$. (*Aligarh, 1944*)
23. $a^3y^2=x^4(a+x)$. 24. $a^3y^2=x^4(x-a)$.
25. $a^4y^2=x^5(2a-x)$. 26. $x^2=y^2(x+1)^3$. (*Panjab, 1949*)
27. $(x+1)(x+2)y^2=x^2$.
28. $x(x-2a)y^2=a^2(x-a)(x-3a)$. (*Agra, 1951*)

29. $a^2y^2=x^2(2a-x)(x-a)$.

30. $xy^2=(x+y)^2$. *(Lucknow)*

31. $x^2(x+y)-y^2=0$. *(Panjab, 1936)*

32. $xy(x^2+y^2)+x^2-y^2=0$.

33. $x^2(x^2+y^2)=a^2(x^2-y^2)$. *(Panjab, 1941)*

34. $x^4+y^4=a^2(x^2-y^2)$.

35. $x^4+y^4=4ax^2y$.

36. $x^4-y^4=xy$. *(Panjab)*

37. $x^5+y^5=5ax^2y^2$. *(Delhi, 1956)*

38. $x^4+x^2y^2+y^4=ax(x^2-y^2)$.

39. $y(x^2-a^2)=x^3+ax^2+a^3$.

40. $x^2(y+3)=y^2(x+2)$. *(Agra, 1945)*

16·4. Polar equations. The procedure laid down for tracing curves whose Cartesian equations are given, is applicable with slight modifications to curves whose polar equations are given. The following points are mentioned for general guidance of the student. The order of presentation is by no means rigid.

I. Symmetry. (*i*) *The curve is symmetrical about the initial line if its equation remains unchanged when θ is changed into $-\theta$.*

$r^2=a^2\cos^2\theta+b^2\sin^2\theta$ is symmetrical about the initial line.

As a particular case, if the equation to the curve contains only cosine or secant of θ or its multiples or submultiples, then the curve is symmetrical about the initial line.

Thus $\qquad r=a\cos\theta, \quad r=a(1+\cos\theta)$

are symmetrical about the initial line.

(*ii*) *The curve is symmetrical about the line through the pole perpendicular to the initial line if the equation to the curve remains unchanged when θ is changed into $\pi-\theta$.*

As a particular case, if the equation to the curve contains only sine or cosecant of θ, or its odd multiples, the curve is symmetrical about the line through the pole perpendicular to the initial line.

The curves $r=a\sin 3\theta$, $r=a(1-\sin\theta)$ are symmetrical about the line through O perpendicular to OX.

(*iii*) *The curve is symmetrical about the pole if the equation remains unchanged when r is changed into $-r$ or θ into $\pi+\theta$.*

II. Regions. Find the regions in which the curve does not lie. This will happen if there are certain values of θ which make r^2 negative and therefore r imaginary, or again if r cannot exceed a certain value so that the entire curve will lie inside a certain circle. If r cannot be less than a certain number, the curve will lie outside a certain circle.

The curve $r^2=a^2\cos 2\theta$ cannot lie between the lines $\theta=\pi/4$ and $\theta=3\pi/4$ and again between $\theta=5\pi/4$ and $\theta=7\pi/4$. Moreover r cannot be greater than a so that the entire curve lies inside the circle of radius a and having its centre at the origin. Again no portion of the curve $r\cos 2\theta=a$ lies inside the circle of radius a and having its centre at the pole as r is never less than a.

III. Origin. If $r=0$ when $\theta=a$, the line $\theta=\alpha$ is, in general a tangent to the curve at the origin.

IV. Asymptotes. Find the asymptotes of the curve, if any.

V. The angle ϕ. Find the angle at which the curve cuts its radius vector for some prominent or convenient values of the vectorial angle.

VI. Variations in r and θ. Trace the variations of r as θ increases successively from 0 through a set of positive values marking values of θ corresponding to which r is zero or attains a minimum or maximum value and plot the corresponding points.

As similar procedure may be adopted for negative value of θ.

Ex. 1. *Trace the curve* $r=a(1+\sin\theta)$.

(*i*) The equation to the curve remains unchanged when θ is changed into $\pi-\theta$. $\therefore$ There is symmetry *w.r.* to the line through the pole perpendicular to the initial line.

(*ii*) $r=0$ when $\theta=3\pi/2$. Hence the curve passes through the pole.

(*iii*) The greatest value of r is $2a$ and its least value zero. Hence the curve lies inside a circle of radius $2a$ and having its centre at the pole.

(*iv*) $r\dfrac{d\theta}{dr}=\tan\left(\dfrac{\pi}{4}+\dfrac{\theta}{2}\right)$. $\therefore$ the curve cuts the initial line at $\pi/4$, and the line through O perpendicular to the initial line at right angles.

(*v*) As θ increases from 0 to $\pi/2$, r steadily increases from a to $2a$.

As θ increases from $\pi/2$ to π, r decreases from $2a$ to a.

As θ increases from π to $3\pi/2$, r decreases further from a to 0.

As θ increases from $\pi/2$ to 2π, r increases from 0 to a.

$\because$ $\sin\theta$ is periodic with a period 2π, the curve will repeat itself for values of $\theta>2\pi$.

Hence the shape of the curve is as shown in the diagram.

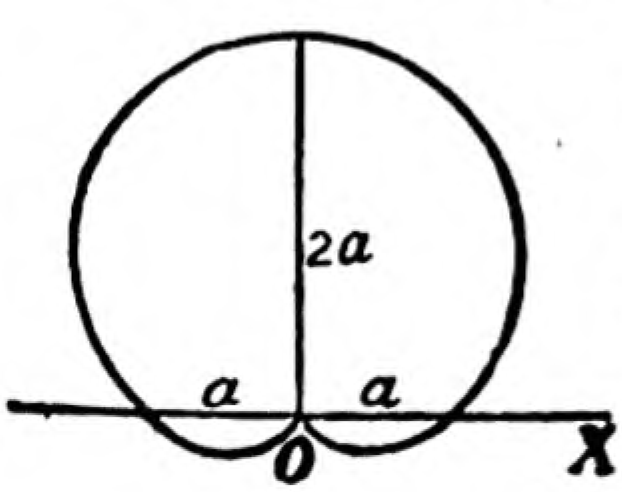


Table of some principal values.

θ	0	$\frac{1}{6}\pi$	π	$\frac{1}{2}\pi$	π	2π
r	a	$1{\cdot}5\,a$	$1{\cdot}87\,a$	$2a$	a	a

Ex. 2. ***Trace the curve*** $r=3\cos 2\theta$.

(*i*) The curve is symmetrical *w.r.* to the initial line.

(*ii*) The curve passes through the pole, r vanishing when

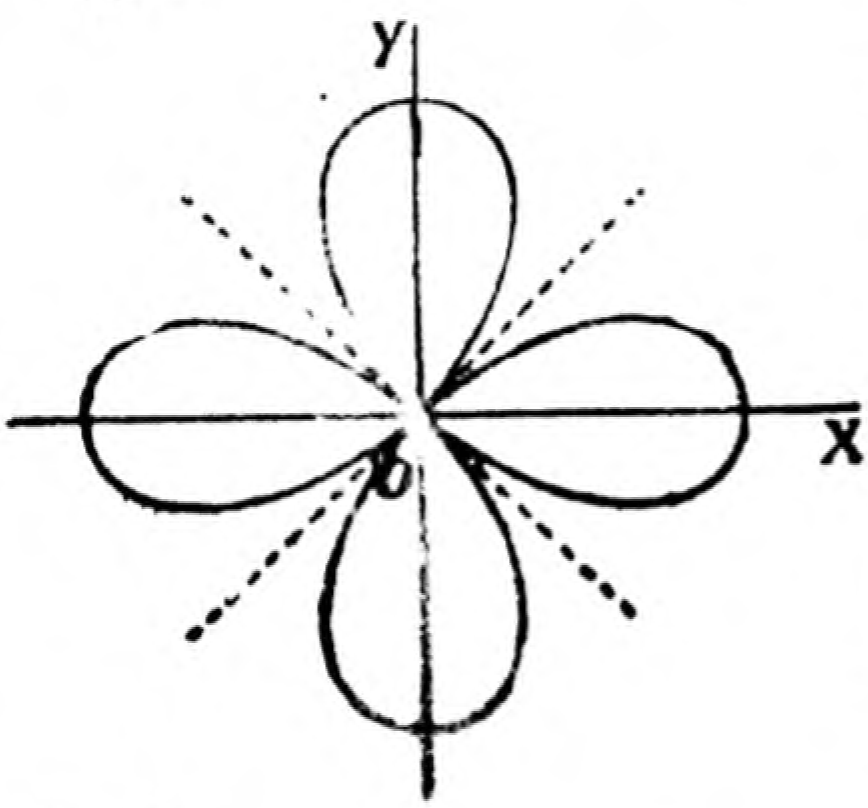
$$\theta=\pm\pi/4,\ \pm 3\pi/4.$$

(*iii*) The greatest value of $\cos 2\theta$ being 1 the greatest value of r is 3. Hence the curve lies inside a circle of radius 3 having its centre at the pole.

(*iv*) The curve crosses the initial line for $\theta=0$ and π at the points $(a, 0)$ and (a, π) respectively.

(*v*) $r\dfrac{d\theta}{dr}=-\tfrac{1}{2}\cot 2\theta$, $\therefore$ the curve has the lines $\theta=\pm\pi/4$ $\theta=\pm 3\pi/4$ as tangents at the pole.

Also the curve cuts the initial line $(\theta=0)$ and the line through the pole perpendicular to the initial line $(\theta=\pm\pi/2)$ at right angles.



(*vi*) The curve has no asymptotes, no branch of the curve extending to infinity.

(*vii*) When $\theta=0$, $r=3$.

As θ increases from 0 to $\pi/4$, r decreases from 3 to 0.

As θ increases from $\pi/4$ to $\pi/2$, r is negative and decreases from 0 to -3.

As θ increases from $\pi/2$ to $3\pi/4$, r is again negative and increases from -3 to 0.

As θ increases from $3\pi/4$ to π, r is positive and increases from 0 to 3.

As θ changes from 0 to $-\pi$, we get the reflection of the above values in the initial line.

For values of $\theta>\pi$ or $<-\pi$, the curve repeats itself. Hence the curve is as shown in the diagram.

Table of some principal values.

θ	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
r	3	0	3	0	3

Ex. 8. *Trace the curve* $r^2=a^2\cos 2\theta$. **(*Delhi, 1957*)**

(*i*) If θ is changed into $-\theta$, the equation to the curve remains unchanged. Hence there is symmetry about the initial line.

(*ii*) For any value of θ, there are two equal and opposite value of r. Hence there is symmetry about the pole.

(*iii*) The curve passes through the pole, r being zero when $\theta=\pm\pi/4$, $\pm 3\pi/4$.

(*iv*) The greatest value of $\cos 2\theta$ being 1, the greatest value of r is a. Hence no part of the curve lies outside the circle of radius a having its centre at the pole.

(*v*) $\cos 2\theta$ is negative for $\pi/4 < \theta < 3\pi/4$ and again for $-3\pi/4 < \theta < -\pi/4$. $\therefore r$ is imaginary for values of θ inside the two ranges.

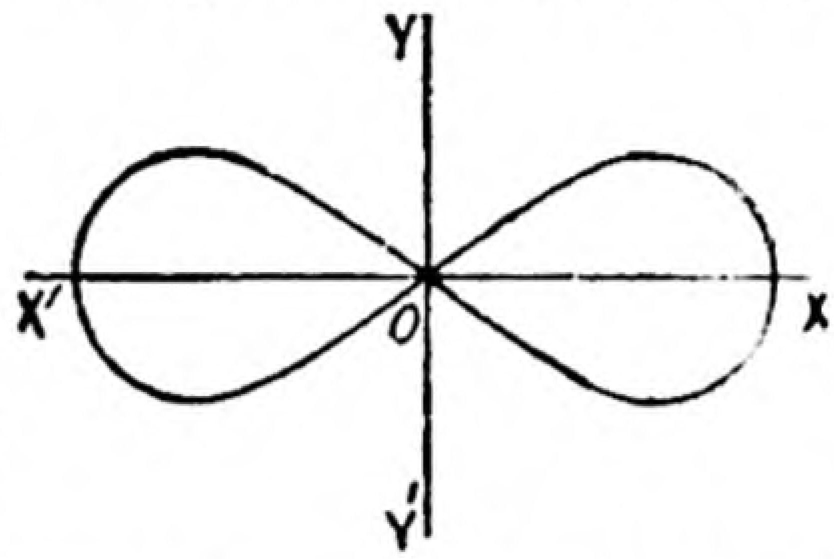
Hence no part of the curve lies in the regions $\theta=\frac{1}{4}\pi$ to $\frac{3}{4}\pi$ and $\theta=\frac{5}{4}\pi$ to $\frac{7}{4}\pi$.

(*vi*) When $\theta=0$, $r=\pm a$.

As θ increases from 0 to $\pi/4$, r decreases from a to 0.

As θ increases from $\pi/4$ to $3\pi/4$, r is imaginary.

As θ increases from $3\pi/4$ to π, r increases from 0 to a. It is now easy to trace the curve.



Ex. 4. *Trace the curve* $r=a(\sec\theta+\cos\theta)$. (*Panjab, 1941*)

(*i*) $\because$ The equation to the curve remains unchanged when θ is changed into $-\theta$, there is symmetry about the initial line.

(*ii*) The equation to the curve may be written as

$$r\cos\theta=a(1+\cos^2\theta)$$

showing that the abscissa of any point on the curve cannot be less than a.

Hence no part of the curve lies to the left of the line

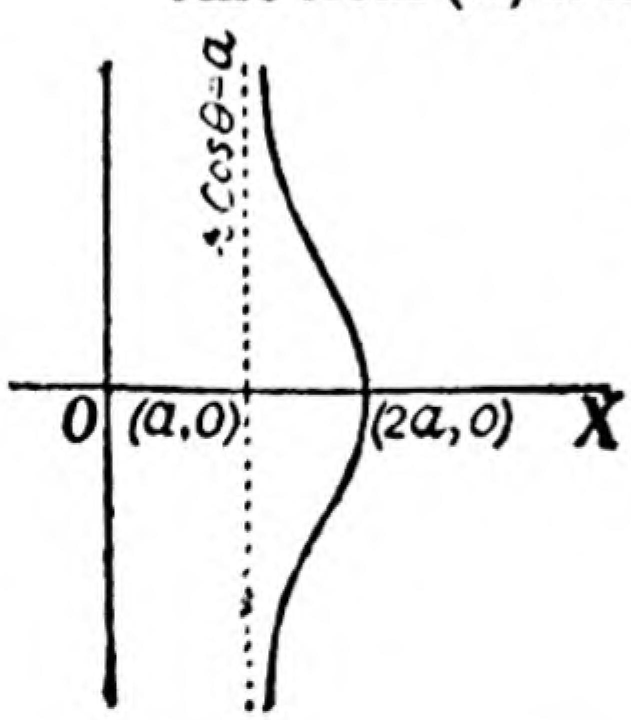
$$r\cos\theta=a,\ i.e.,\ x=a.$$

(*iii*) $r\to\infty$ as $\theta\to\pm\pi/2$.

There is, however, only one asymptote to the curve, *viz.*

$$r\cos\theta=a.$$

Also from (*ii*) it is clear that the asymptote is approached only from the right.



(*iv*) $\dfrac{dr}{d\theta}=\dfrac{a\sin^3\theta}{\cos^2\theta}$. This is positive for $0\leqslant\theta<\pi/2$.

Hence r increases in this range.

$\therefore$ The least value of r corresponds to $\theta=0$ and is $2a$.

(*v*) $\tan\phi=r\dfrac{d\theta}{dr}=\dfrac{\cos\theta(1+\cos^2\theta)}{\sin^3\theta}$

$\therefore \quad \phi=\pi/2$ when $\theta=0$.

Thus at the point $(2a, 0)$, the curve has a tangent perpendicular to the initial line.

(*vi*) When $\theta \to \pi/2-0,\ r \to +\infty,$

and when $\theta \to \pi/2+0,\ r \to -\infty.$

As θ increases from $\pi/2$ to π, r increases from $-\infty$ to $2a$,

Thus the form of the curve is as shown in the diagram.

Ex. 5. *Trace the curve* $r^2 \cos\theta = a^2 \sin 3\theta$.

(*i*) For any value of θ there are two equal and opposite values of r. Hence the curve is symmetrical about the pole.

(*ii*) The curve passes through the origin, r being zero when

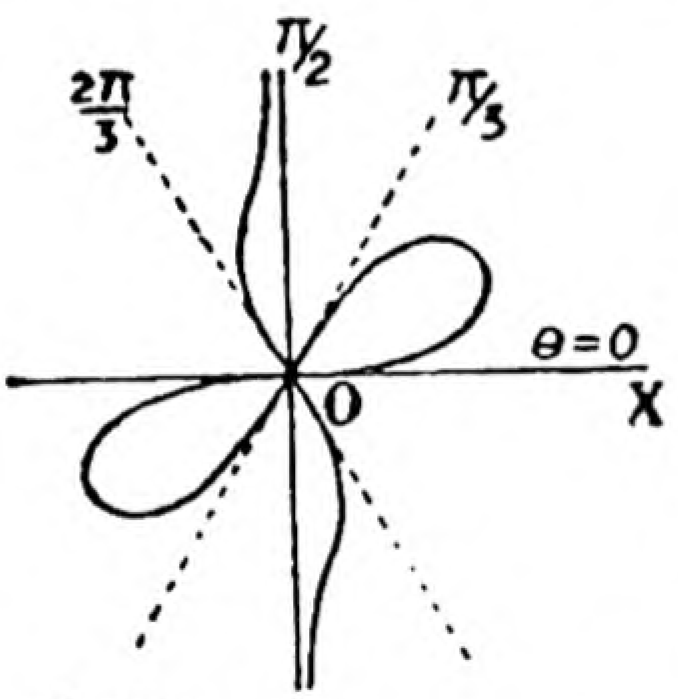
$$\theta = 0,\ \pi/3,\ 2\pi/3,\ \pi, \ldots\ldots\ldots$$

(*iii*) r is infinite when $\theta = \pi/2,\ 3\pi/2, \ldots\ldots\ldots$

$\theta = \pi/2$ is the only asymptote.

(*iv*) $\dfrac{d\theta}{dr} = 0$ when $\theta = \pi/2,\ 3\pi/2, \ldots\ldots\ldots$

(*v*) r^2 is negative and consequently r is imaginary as θ varies from $\pi/3$ to $\pi/2$ and again from $\frac{2}{3}\pi$ to π.



When $\theta = 0$, $r = 0$.
When $\theta = \frac{1}{6}\pi$, $r^2 = 2a/\sqrt{3}$.
When $\theta = \pi/3$, $r = 0$ again.
As θ varies from $\pi/3$ to $\pi/2$, r is imaginary.
When $\theta \to \pi/2$, $r \to \pm\infty$.
When $\theta = 2\pi/3$, $r = 0$
As θ varies from $\frac{2}{3}\pi$ to π, r is again imaginary. The cycle of values is repeated as θ is given values greater than π.

Hence the form of the curve is as shown.

16 5. It is sometimes convenient and helpful to change from cartesian to polar coordinates or *vice versa*.

Ex. *Trace the curve* $r \cos\theta = 2a \sin^2\theta$. (*Agra, 1943*)

Change to cartesian coordinates with the help of the equations $x = r\cos\theta$, $y = r\sin\theta$. The equation to the curve becomes

$$x = 2a.\frac{y^2}{x^2+y^2}$$

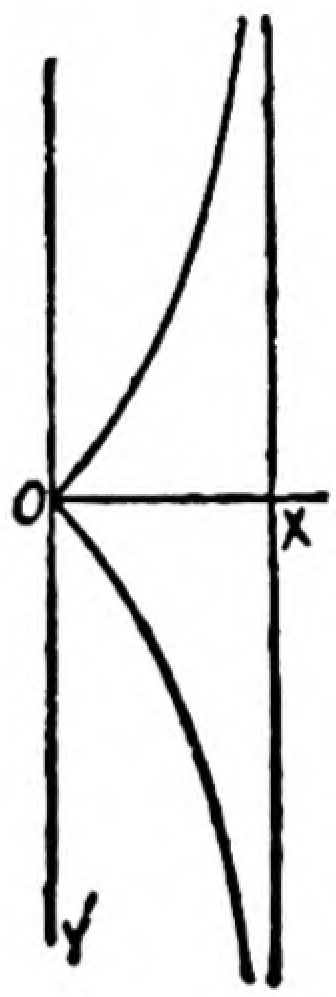
i.e., $$x(x^2+y^2) = 2ay^2 \quad \text{or} \quad y^2(2a-x) = x^3.$$

The following points are at once clear :

(*i*) There is symmetry about the x-axis.

(*ii*) The curve passes through the origin and the tangents at the origin are $y^2=0$, which represents a pair of coincident lines. Hence there is a cusp at the origin.

$\because$ there is symmetry *w.r.* to the x-axis, the cusp is of the first kind. Again, since x cannot be negative (as y is imaginary in that case), the cusp is a single one.



(*iii*) Incidently, we see that no part of the curve lies to the left of the y-axis.

Also, x cannot be greater than $2a$ as y is imaginary for such values of x.

(*iv*) $x=2a$ is the only asymptote and this must be approached from the left as $x<2a$.

We can now easily trace the curve. The curve is called the **cissoid.**

Examples LXI

Trace the following curves :

1. (*i*) $r=2a\cos\theta$. (*ii*) $r=2a\sin\theta$.
2. (*i*) $r=a\sin 2\theta$. (*ii*) $r=a\cos 4\theta$.
3. $r=a\sin 5\theta$. (*Panjab, 1946*) **4.** $r=a(1+\cos\theta)$. (*Agra, 1948*)
5. $r=a(1-\cos\theta)$. **6.** (*i*) $r=2+\cos\theta$. (*ii*) $r=1+2\cos\theta$.
7. $r=a+b\cos\theta$. (*Panjab, 1944*) **8.** $r=a+b\sin\theta$.
9. $r\cos 2\theta=a$. **10.** $r=a\tan\theta$. **11.** $r=a\theta/(1+\theta)$.
12. $r=a\log\theta$. **13.** $r=2a\cos^2\theta$.
14. $r\cos\theta=a\sin 3\theta$. **15.** $r^2=4\sin 3\theta$.
16. $r=\theta\sin\theta$. **17.** $r^2\cos\theta=a^2\sin 3\theta$.
18. $r(\cos^3\theta+\sin^3\theta)=3a\sin\theta\cos\theta$.

16 6 Parametric equations. Let

$$x=f(t),\qquad y=\varphi(t)$$

be the parametric equations of a curve. If possible, we may eliminate t between the two equations and use the resulting cartesian equations to trace the curve. Otherwise, we give t a series of values and plot the corresponding points (x, y). The behaviour of $x,\ y,\ \dfrac{dx}{dt},\ \dfrac{dy}{dt},\ \dfrac{dy}{dx}$ is discussed for different values of t.

If $f(t)$ is an even function and $\varphi(t)$ is an odd function of t, then the curve is symmetrical about the x-axis ; and if $\varphi(t)$ is even and $f(t)$ odd, then the curve is symmetrical about the y-axis. For example, if

$$x=a\frac{1-t^2}{1+t^2},\ y=2b\frac{t}{1+t^2}.$$

the curve is symmetrical about the x-axis since x is an even and y an odd function of t.

If both $f(t)$ and $\varphi(t)$ are odd functions of t, the curve is symmetrical in opposite quadrants, *e.g.*, $x=ct$, $y=c/t$, is symmetrical *w.r.* to the origin.

The points of intersection of the curve with the x-axis are given by the roots of $\varphi(t)=0$ while those with the y-axis are given by the roots of $f(t)=0$.

The following solved examples illustrate the methods used in tracing curves given by their parametric equations.

Ex. 1. *Trace the curve* $x=\dfrac{a(1-t^2)}{1+t^2}$, $y=\dfrac{2bt}{1+t^2}$.

As remarked above, the curve is symmetrical *w.r.* to the x-axis and need be traced for $t\geqslant 0$, the other half being completed by a reflection in the x-axis.

Assuming that a and b are positive, we observe that $y\geqslant 0$ for $t\geqslant 0$ while $x>0$ for $0\leqslant t<1$, $x=0$ for $t=1$ and $x<0$ for $t>1$.

From the equation for x, $t^2=(a-x)/(a+x)$. Since t is real, x must lie between $-a$ and a.

Again, from the equation for y,

$$yt^2-2bt+y=0\,; \qquad \ldots(1)$$

since t is real, we must have $4b^2-4y^2\geqslant 0$ and, therefore, y must lie between $-b$ and b. Equation (1) show that any given value of y with $|y|<b$ corresponds to two real values of t, say t_1 and t_2 with

$$t_1t_2=1. \qquad \ldots(2)$$

Let x_1 and x_2 be the values of x corresponding to $t=t_1$ and t_2 respectively, then

$$x_1=\frac{a(1-t_1^2)}{1+t_1^2},$$

$$x_2=\frac{a(1-t_2^2)}{1+t_2^2}=\frac{a\{1-(1/t_1)^2\}}{1+(1/t_1)^2}=-x_1,$$

since $t_2=1/t_1$ by (2). It follows, therefore, that corresponding to any point (x_1, y_1) on the curve there is also a point $(-x_1, y_1)$ on it, the two points corresponding to values of t which are reciprocals of each other. Hence the curve is symmetrical *w.r.* to the y-axis and we need trace the curve for the range $0\leqslant t\leqslant 1$ only which will give the portion of the curve in the first quadrant.

From the given equations,

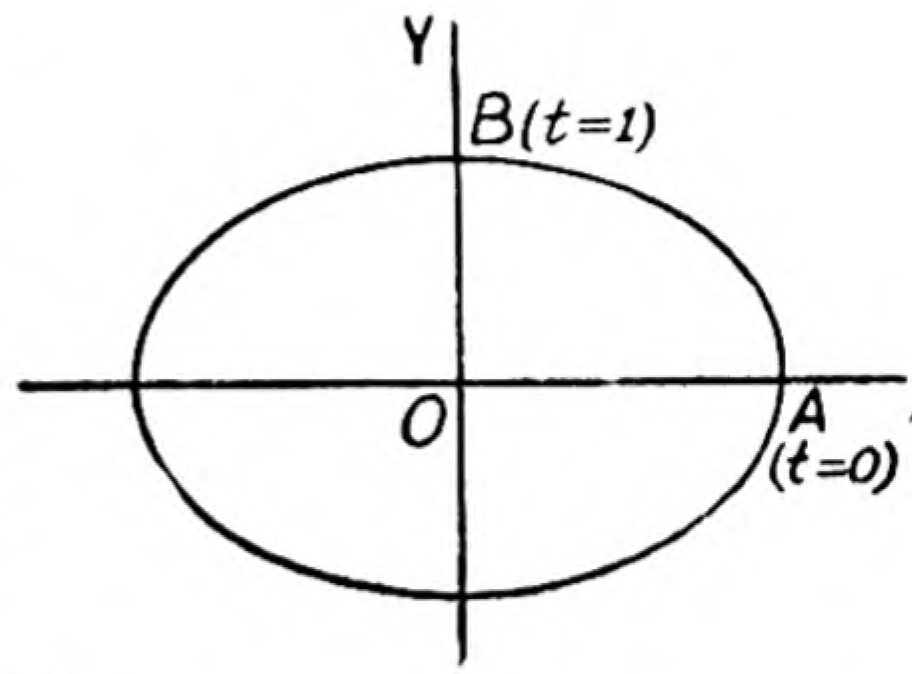
$$\frac{dx}{dt}=-\frac{4at}{(1+t^2)^2},\quad \frac{dy}{dt}=\frac{2b(1-t^2)}{(1+t^2)^2} \text{ and } \therefore\ \frac{dy}{dx}=-\frac{b(1-t^2)}{2at}.$$

When $t=0$, $x=a$ and $y=0$, and when $t=1$, $x=0$ and $y=b$. For $0<t<1$, $\dfrac{dx}{dt}$ is negative and $\dfrac{dy}{dt}$ is positive. Hence, as t increases from 0 to 1, x decreases from a to 0 and y increases from 0 to b.

$\dfrac{dy}{dx}$ is infinite for $t=0$. Hence the line $x=a$ is tangent to the

curve at $(a, 0)$. $\dfrac{dy}{dx}=0$ for $t=1$ and hence the line $y=b$ is the tangent at $(0, b)$.

The curve is as shown in the figure.



Remark. Eliminating the parameter t between the two given equations, the cartesian equation of the curve is

$$x^2/a^2+y^2/b^2=1.$$

The curve is, therefore, an ellipse.

Ex. 2. *Trace the curve*
$$x=a \cos^3 t, \; y=a \sin^3 t.$$

Since $\cos t$ and $\sin t$ are periodic functions of t with a period 2π, we need consider the values of t in the interval $(0, 2\pi)$ only, other values of t giving repetitions of the curve.

From the given equations,

$$\frac{dx}{dt}=-3a \cos^2 t \sin t \quad \frac{dy}{dt}=3a \sin^2 t \cos t, \; \frac{dy}{dx}=-\tan t.$$

Hence, (*i*) as t increases from 0 to $\frac{1}{2}\pi$, x decreases from a to 0, y increases from 0 to a and dy/dx decreases from 0 to $-\infty$,

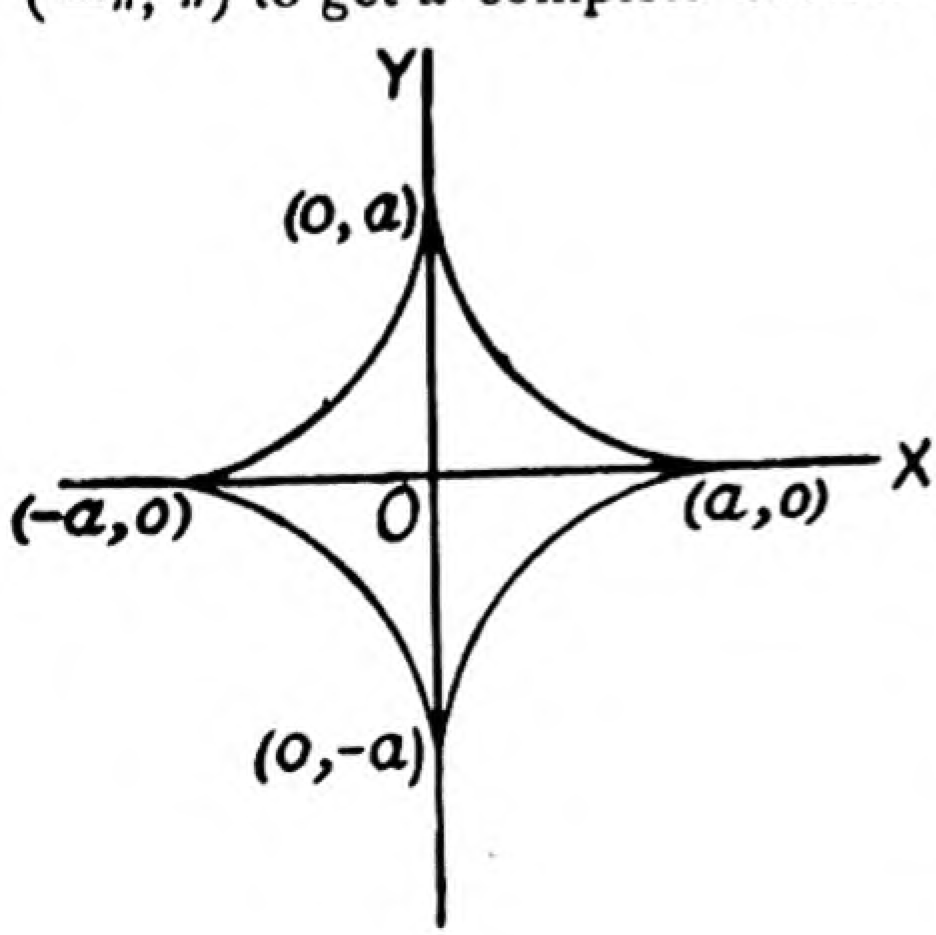
(*ii*) as t increases from $\frac{1}{2}\pi$ to π, x decreases from 0 to $-a$, y decreases from a to 0 and dy/dx decreases from $+\infty$ to 0,

(*iii*) as t increases from π to $\frac{3}{2}\pi$, x increases from $-a$ to 0, y decreases from 0 to $-a$ and dy/dx decreases from 0 to $-\infty$,

(*iv*) as t increases from $\frac{3}{2}\pi$ to 2π, x increases from 0 to a, y increases from $-a$ to 0 and dy/dx decreases from $+\infty$ to 0

The curve is as shown in the diagram.

Remarks. We might have confined ourselves to the range $(-\pi, \pi)$ to get a complete sketch of the curve. A change in the sign of t leaves x unchanged but changes the sign of y. Hence there is symmetry *w.r.* to the x-axis. The portion in the first quadrant corresponds to $0 \leqslant t \leqslant \frac{1}{2}\pi$. For the portion of the curve in the second quadrant, we note that if t be changed into $\pi-t$, $0 \leqslant t \leqslant \frac{1}{2}\pi$, x changes sign while y remains unchanged. This shows spmmetry *w.r.* to the y-axis. The curve is thus symmetrical about both axes. It may, therefore, be drawn for the range $0 \leqslant t \leqslant \frac{1}{2}\pi$ and the rest completed by symmetry.



Ex. 3. *Trace the curve* $x=a\{\cos\theta-\log(1+\cos\theta)\}$, $y=a\sin\theta$.

$\sin\theta$ and $\cos\theta$ are periodic functions of θ with a period 2π. We, therefore, need trace the curve for $-\pi\leqslant\theta\leqslant\pi$.

The curve is symmetrical *w.r.* to the x-axis as x is an even and y an odd function of θ. Therefore we need trace the curve for $0\leqslant\theta\leqslant\pi$ only and complete the rest by symmetry.

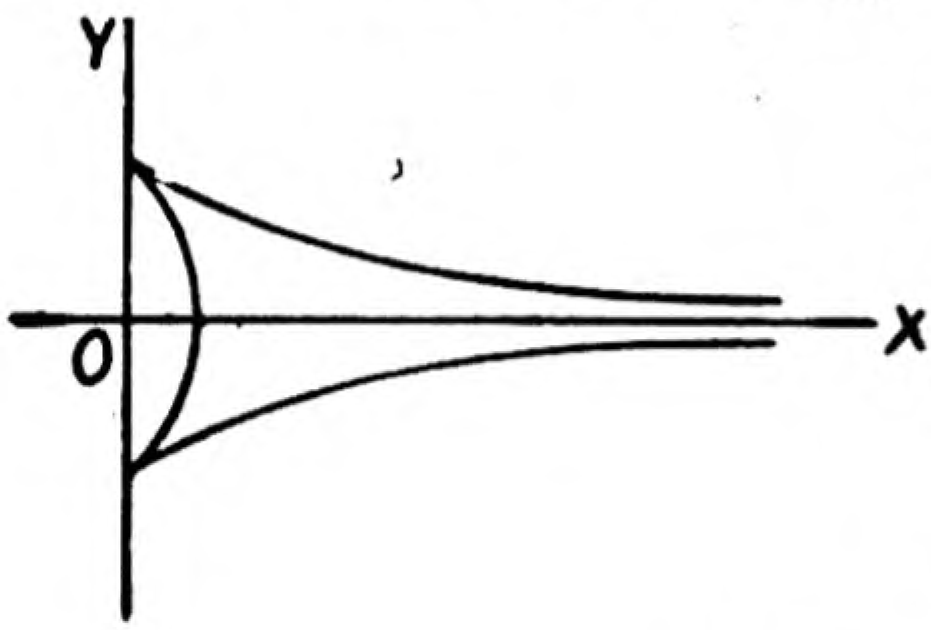
From the given equations,

$$\frac{dx}{d\theta}=-a\cos\theta\tan\tfrac{1}{2}\theta,\ \frac{dy}{d\theta}=a\cos\theta \text{ and } \therefore \frac{dy}{dx}=-\cot\tfrac{1}{2}\theta.$$

If ψ be the angle made by the tangent with the x-axis, then $\tan\psi=-\cot\frac{1}{2}\theta$ and so $\psi=\frac{1}{2}\pi+\frac{1}{2}\theta$.

In the range $0<\theta<\frac{1}{2}\pi$; $\frac{dx}{d\theta}<0$, $\frac{dy}{d\theta}>0$. Therefore, as θ increases from 0 to $\frac{1}{2}\pi$, x decreases from $a(1-\log 2)$ to 0, y increases from 0 to a and dy/dx increases from $-\infty$ to -1.

For the range $\frac{1}{2}\pi<\theta<\pi$, $\frac{dx}{d\theta}>0$, $\frac{dy}{d\theta}<0$. Therefore, as θ increases from $\frac{1}{2}\pi$ to π, x increases from 0 to $+\infty$, y decreases from a to 0 and dy/dx increases from -1 to 0. Thus x and y both remain positive for $\frac{1}{2}\pi\leqslant\theta\leqslant\pi$ and the x-axis is an asymptote to this branch of the curve.



The curve is as shown in the diagram.

Examples LXII

Trace the curves whose parametric equations are given below :

1. $x=at^2$, $y=2at$.
2. $x=ct$, $y=c/t$.
3. $x=a\sec\theta$, $y=b\tan\theta$.
4. $x=a\cos\theta$, $y=b\sin\theta$.
5. $x=a\cos^2\theta$, $y=b\sin^2\theta$.
6. $x=a\cos^3 t$, $y=b\sin^3 t$. *(Panjab, 1953)*
7. $x=a(\theta+\sin\theta)$, $y=a(1+\cos\theta)$, $-\pi\leqslant\theta\leqslant\pi$.
8. $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$, $0\leqslant\theta\leqslant 2\pi$.
9. $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$, $-\pi\leqslant\theta\leqslant\pi$.
10. $x=3at/(1+t^3)$, $y=3at^2/(1+t^3)$.
11. $x=at^2/(1+t^2)$, $y=at^3/(1+t^2)$.
12. $x=a(1-t^2)/(1+t^2)$, $y=at(1-t^2)/(1+t^2)$.
13. $x=c(\cos t+\frac{1}{2}\log\tan^2\frac{1}{2}t)$, $y=c\sin t$.
14. $x=a\sin 2\theta\,(1+\cos 2\theta)$, $y=a\cos 2\theta(1-\cos 2\theta)$.
15. $x=a(\sin\theta+\frac{1}{3}\sin 3\theta)$, $y=a(\cos\theta-\frac{1}{3}\cos 3\theta)$.

CHAPTER XVII

SOME WELL KNOWN CURVES

17·1. In this chapter we shall give the graphs of some well known curves with their brief descriptions. The student is supposed to be familiar with the ellipse, the parabola and the hyperbola and their simpler properties. The evolutes of the ellipse and the parabola have been obtained in chapter XIII and graphs of these two are drawn there.

The various cubical and semi-cubical parabolas have been considered in the last chapter.

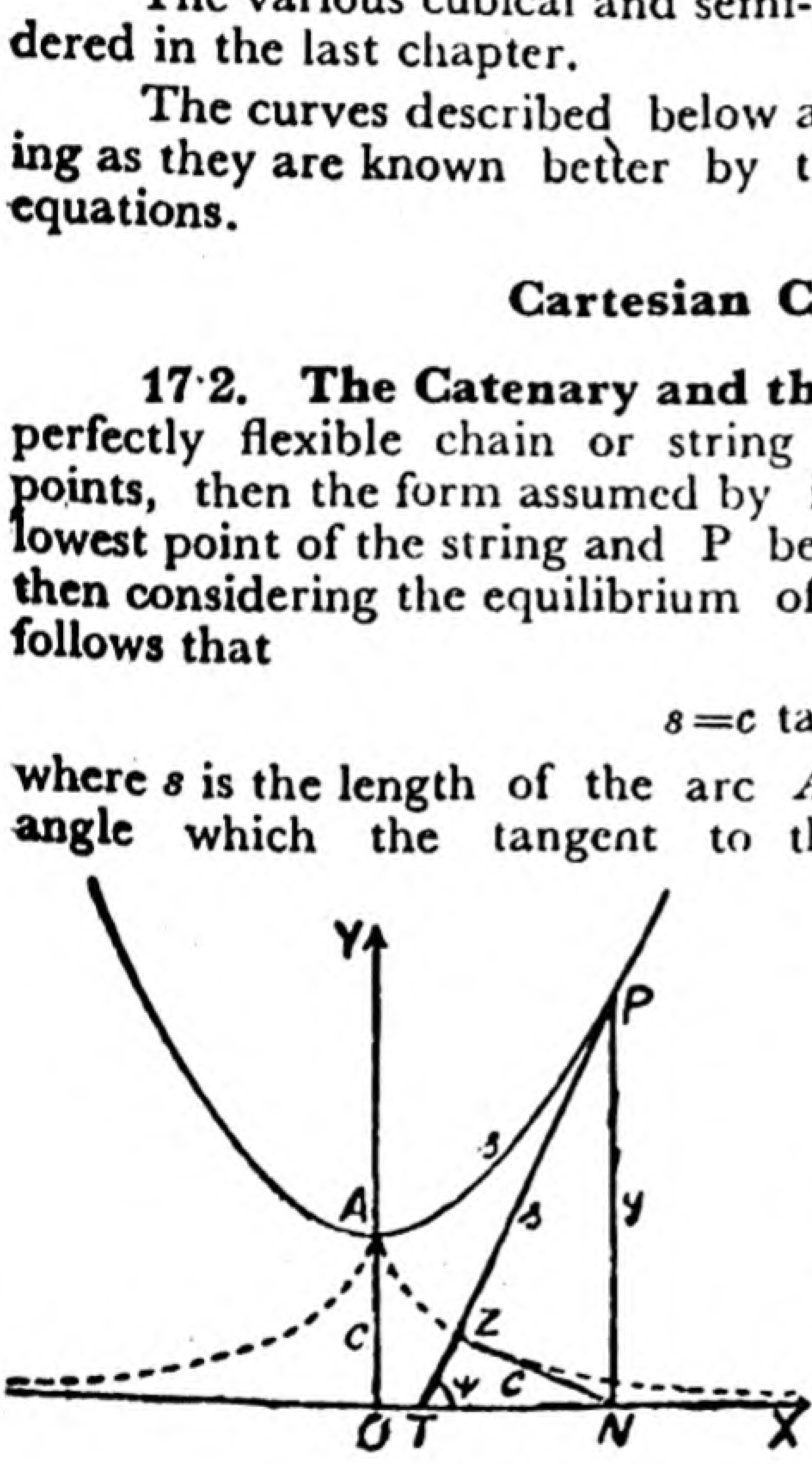
The curves described below are divided into two classes according as they are known better by their Cartesian equations or polar equations.

Cartesian Coordinates

17·2. The Catenary and the Tractrix. If a heavy uniform perfectly flexible chain or string hangs freely between any two points, then the form assumed by it is the Catenary. If A be the lowest point of the string and P be any other point of the string, then considering the equilibrium of the portion AP of the string it follows that

$$s=c \tan \psi,$$

where s is the length of the arc AP, is a constant and ψ is the angle which the tangent to the string at P makes with the tangent at A, *i.e.*, with the horizontal through A in the plane of the string.



Take the vertical through A as the y-axis, the origin O at a distance c below A and the x-axis the horizontal line through O in the plane of the catenary. If (x, y) be the coordinates of P, then

$$\frac{dx}{d\psi}=\frac{dx}{ds}\cdot\frac{ds}{d\psi}=\cos\psi . c\sec^2\psi=c\sec\psi,$$

and
$$\frac{dy}{d\psi}=\frac{dy}{ds}\cdot\frac{ds}{d\psi}=\sin\psi . c\sec^2\psi=c\tan\psi\sec\psi.$$

Integrating, we get

$$x=c\log(\sec\psi+\tan\psi)+c_1,\ y=c\sec\psi+c_2,$$

where c_1, c_2 are the constants of integration. Since when $\psi=0$ at A, $x=0$ and $y=c$, we get $c_1=c_2=0$. Hence the parametric equations of the catenary are

$$x=c \log (\sec \psi+\tan \psi),\ y=c \sec \psi.$$

From the first of these

$$\sec \psi+\tan \psi=e^{x/c}$$

and $\therefore$ also
$$\sec \psi-\tan \psi=e^{-x/c}.$$

Hence, by addition and subtraction,

$$y=c \sec \psi=\tfrac{1}{2}c(e^{x/c}+e^{-x/c})=c \cosh (x/c)$$
$$s=c \tan \psi=\tfrac{1}{2}c(e^{x/c}-e^{-x/c})=c \sinh (x/c)$$

Hence the Cartesian equation of the catenary is

$$y=c \cosh (x/c).$$

The constant $c=OA$ is called the *parameter* of the catenary, the x axis is called its *directrix* and the point A is called its *vertex*. The catenary is symmertical about the y-axis.

If PN be the ordinate of P, PT the tangent and NZ the perpendicular from the foot of the ordinate on the tangent, then $\angle ZNP=\angle NTP=\psi$, and so

$$NZ=NP \cos \psi=y \cos \psi=c,$$

and
$$PZ=NZ \tan \psi=c \tan \psi=s.$$

Hence NZ is constant and equal to the parameter c. PZ is equal to the arc AP. From the right angled triangle PNZ.

$$y^2=NP^2=NZ^2+PZ^2=c^2+s^2.$$

As the point P describes the catenary, the point Z describes a curve which is called the **tractrix** or **tractory**. The dotted line in the figure shows the tractrix. If (x', y') be the coordinates of Z, then from the figure,

$$x'=x-NZ \sin \psi=c \log (\sec \psi+\tan \psi)-c \sin \psi$$
$$=c \log \tan (\tfrac{1}{4}\pi+\tfrac{1}{2}\psi)-c \sin \psi,$$

and
$$y'=NZ \cos \psi=c \cos \psi.$$

If we take $\tfrac{1}{4}\pi+\tfrac{1}{2}\psi=\tfrac{1}{2}t$, *i.e.*, $t=\psi+\tfrac{1}{2}\pi=\angle XNZ$, then we get

$$x'=c \log \tan \tfrac{1}{2}t+c \cos t,\ y'=c \sin t.$$

Hence the parametric equations of the tractrix in the simplest form are

$$x=c \cos t+c \log \tan \tfrac{1}{2} t,\ y=c \sin t.$$

Since $PZ=$ arc AP always and $NZ \perp ZP$, it follows that if a heavy string unwinds itself from the catenary, the lower end of the string starting at A, then this lower end will describe the tractrix and since the lower end is moving perpendicular to the straight portion PZ when this end is at Z, it follows that NZ is the tangent to the tractrix at the point Z, ZP is the normal, and the evolute of the tractrix is the catenary.

Since NZ is of constant length, the tractrix has the property that the length of its tangent intercepted between the curve and the x-axis is of constant length.

The name tractrix is derived from the fact that it is the path of a heavy particle Z dragged along a rough horizontal plane by a string ZN, the other end N of which is made to describe a straight line OX in the plane.

17·21. The Folium of Descrates. $x^3+y^3=3axy$.

(*i*) The axes touch the curve at the origin which is a node.

(*ii*) $x+y+a=0$ is the only real asymptote.

(*iii*) The parametric equations are

$$x=\frac{3at}{1+t^3}, \qquad y=\frac{3at^2}{1+t^3}$$

and the polar equation is

$$r=\frac{3a \sin \theta \cos \theta}{\cos^3\theta+\sin^3\theta}.$$

The curve is drawn in Ex. 2, Art. 14·82.

17·22 The Cissoid of Diocles. $y^2(2a-x)=x^3$.

The curve is traced fully in Art. 16·5, Solved example.

A parametric representation is

$$x=\frac{at^2}{1+t^2}, \qquad y=\frac{at^3}{1+t^2}$$

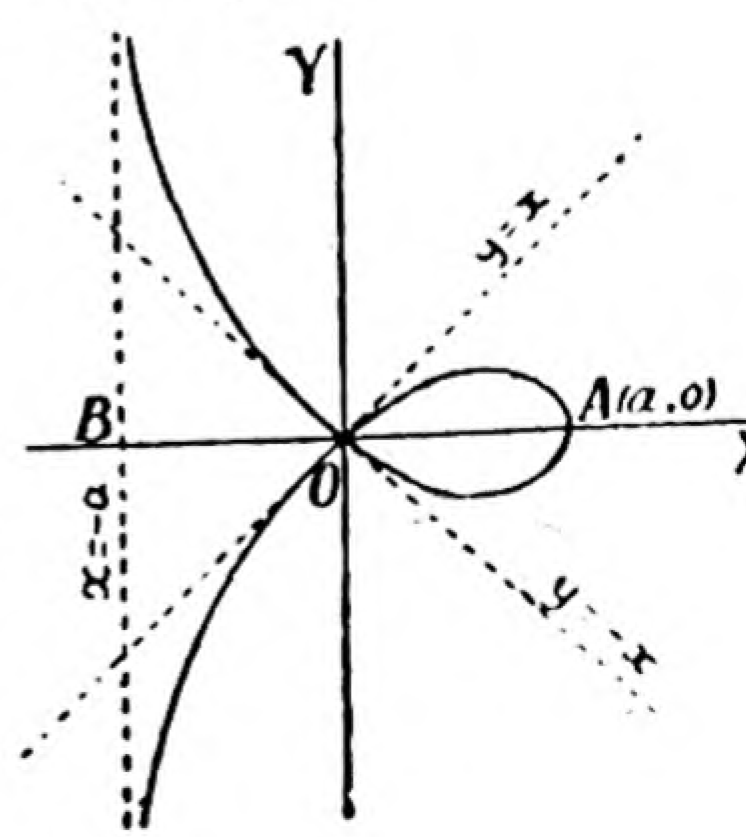
17·23. The strophoid. $(a+x)y^2=(a-x)x^2$.

(*i*) The equation may also be written as $(x^2+y^2)\, x=a(x^2-y^2)$.

(*ii*) The curve passes though the origin and the tangents at this point are $y=\pm x$. Thus the origin is a node.

(*iii*) $x+a=0$ is the only real asymptote.

(*iv*) The parametric equations are :

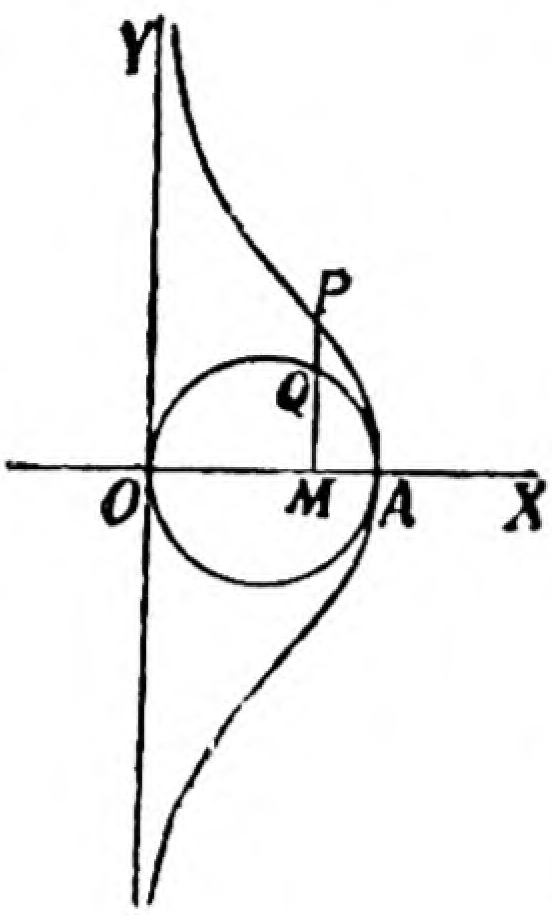


$$x=a\frac{1-t^2}{1+t^2}, \quad y=a\frac{t(1-t^2)}{1+t^2}.$$

(*v*) The polar equation is

$$r \cos \theta=a \cos 2\theta.$$

17·24 The Witch of Agnesi. $xy^2=4a^2(2a-x)$.



(*i*) The equation may also be written as

$$(y^2+4a^2)x=8a^3.$$

(*ii*) $x=0$ is the only real asymptote.

(*iii*) The parametric equations are

$$x=2a/(1+t^2), \qquad y=2at.$$

(*iv*) If the ordinate MP through any point P on the witch meets the circle on OA as diameter in Q, then

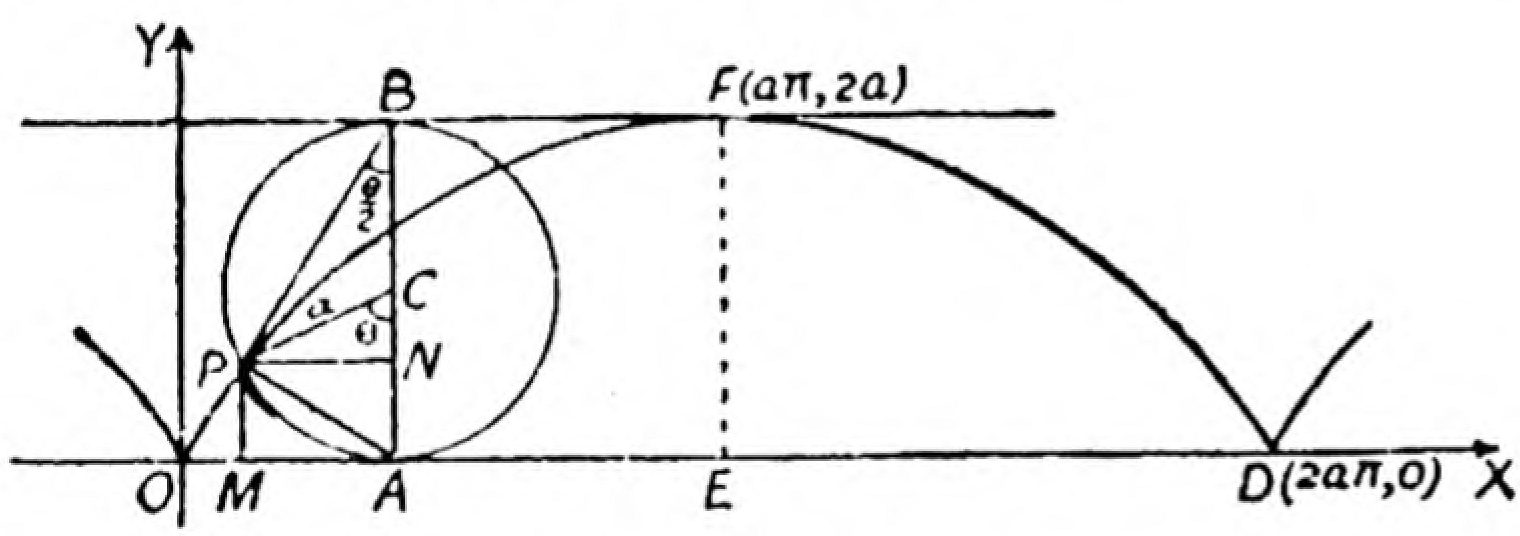
$$\frac{OM}{OA}=\frac{MQ}{MP}.$$

This important property of the witch can be used in tracing it.

17·3 Loci associated with a rolling circle.

17·31. The cycloid. The cycloid is the locus of a point P fixed on the circumference of a circle when the circle rolls on a straight line. The straight line is called the *base* of the cycloid.

Let the rolling circle be of radius a and let the base be taken as the x-axis. In some position of the rolling circle, the point P is the point of contact with the base. Take this position of P as the origin O and the line through $O \perp OX$ as the y-axis.



Let C be the centre of the rolling circle in any other position, A its point of contact with OX and AB the diameter through A. Let P be the position of the moving point at this instant, then by the condition of rolling, arc $PA=OA$. Join CP and PB and draw PM, PN perpendiculars to OX and AB. Let $\angle ACP=\theta$, then $\angle ABP=\frac{1}{2}\theta$. If x, y be the coordinates of P, then since $OA=$arc $PA=a\theta$ we have

$$x=OM=OA-MA=OA-PN=a\theta-a\sin\theta,$$
$$y=MP=AN=AC-NC=a-a\cos\theta.$$

Hence the parametric equations of the cycloid are

$$x=a(\theta-\sin\theta), \qquad y=a(1-\cos\theta),$$

θ, the angle ACP, being the parameter. θ is the angle through which the rolling circle has rotated in rolling from the initial position to the present position. When $\theta=\pi$, the circle will have completed half a revolution and the moving point P will be at the highest point of the circle; it will then at F and will have the coordinates $(a\pi, 2a)$. When the circle has made one complete revolution, $\theta=2\pi$, the point P will have the coordinates $(2a\pi, 0)$, will be at D and will once again be the point of contact of the rolling circle with OX. Further rolling will generate another arch like OFD and as the circle rolls on, an unlimited succession of arches is generated. If the rolling circle were to roll backwards there will be a succession of arches like OFD to the left of O. The name cycloid is, however, generally reserved for one complete arch like OFD.

Points like F are called *vertices* of the cycloid. Points like O, D are cusps on the cycloid, tangents at such points being parallel to OY. It is easy to verify that the arch OFD is symmetrical with respect to the line EF, which is called the *axis* of the cycloid.

If dashes denote differentiations *w.r.* to θ, then

$$x'=a(1-\cos\theta)=y \text{ and } y'=a\sin\theta.$$

and
$$\therefore \frac{dy}{dx}=\frac{y'}{x'}=\frac{a\sin\theta}{y}.$$

Hence the subnormal at $P=y(dy/dx)=a\sin\theta=MA$. Since PM is the ordinate at P, it shows that PA is the normal at P to the cycloid. Since $\angle APB=$angle in a semicircle$=90°$, PB is the tangent at P. If ψ be the angle made by PB with the x-axis, then $\psi=\frac{1}{2}\pi-\frac{1}{2}\theta$.

The equation of the cycloid can be put in a slightly different form by taking the vertex F as origin, the tangent at F as x-axis and FE as y-axis. Shifting the origin to $F(a\pi, 2a)$, the equations (1) become

$$X+a\pi=a(\theta-\sin\theta), \qquad Y+2a=a(1-\cos\theta)$$

or
$$X=a(\theta-\pi)+a\sin(\theta-\pi), \qquad Y=-a+a\cos(\theta-\pi).$$

where (X, Y) are the coordinates of any point (x, y) referred to parallel axes through F. If $\phi=\theta-\pi$ is taken as the new parameter, then these equations become

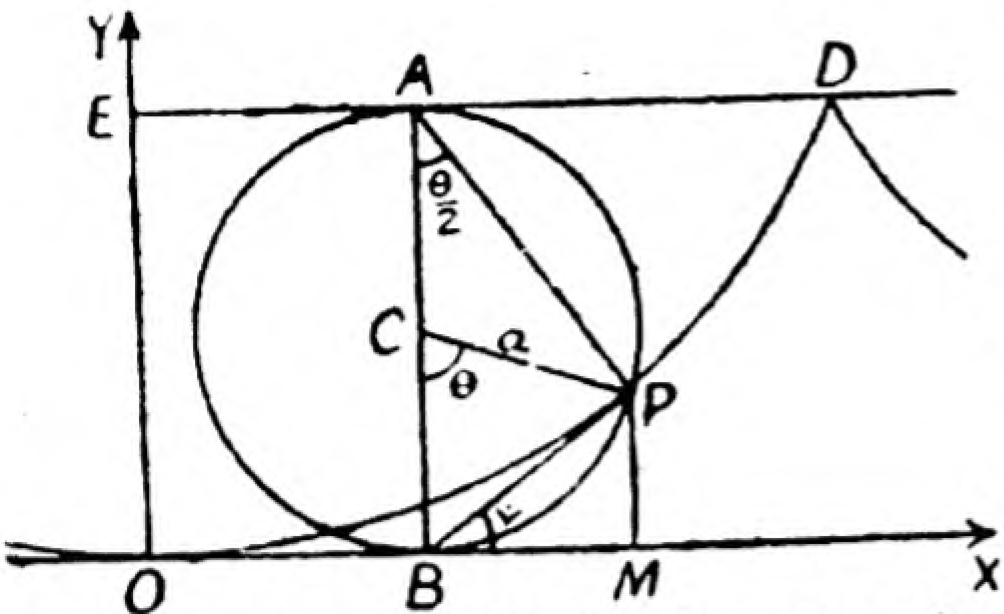
$$X=a\phi+a\sin\phi, \qquad Y=-a+a\cos\phi.$$

If we now change Y into $-Y$, the new y-axis takes the direction FE and we get the equations

$$X=a(\phi+\sin\phi), \qquad Y=a(1-\cos\phi);$$

or, changing X, Y into x, y and ϕ into θ, we get

$$x=a(\theta+\sin\theta), \qquad y=a(1-\cos\theta). \qquad \ldots(2)$$



The cycloid represented by equations (2) is drawn in the diagram opposite. Here O is the vertex, OY the axis and EAD the base of the cycloid. The generating circle rolls on EAD.

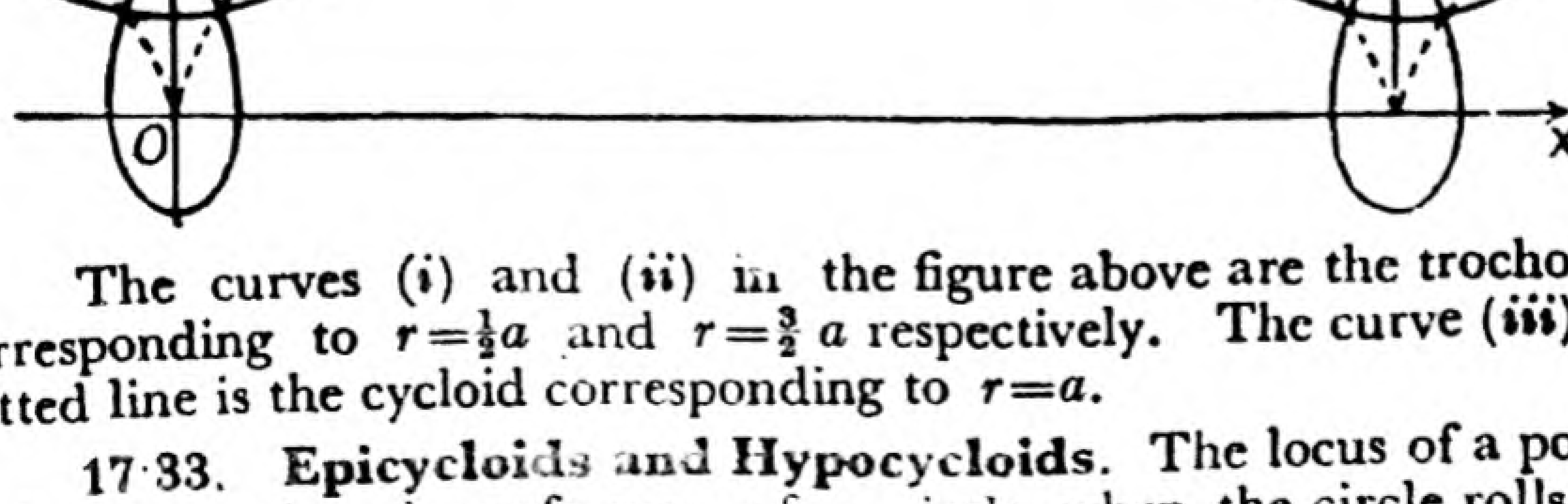
If C be the centre of the generating circle in any position, A its point of contact with the base, AB the diameter through A and P the corresponding point on the cycloid, then it is easy to verify that $\angle BCP=\theta$ and BP is the tangent to the cycloid at P. Also arc $AP=AD$ and arc $BP=EA=OB$. If BP makes an angle ψ with OX and arc $OP=s$, then the following results hold :

(i) $\psi=\frac{1}{2}\theta$. (ii) $s=4a\sin\frac{1}{2}\theta=4a\sin\psi=\sqrt{(8ay)}$, and (iii) $\rho=4a\cos\psi=2PA$.

17·32. The trochoid. If in Art. 17·31 the fixed point P is not situated on the circumference of the rolling circle, then its locus is called a trochoid. If the moving point be now at a distance r from C on the radius CP in the first figure of Art. 17·31, then it is easily seen that the equations of the trochoid are

$$x=a\theta-r\sin\theta, \qquad y=a-r\cos\theta.$$

When $r>a$, the trochoid has loops which degenerate into cusps, when $r=a$, in the case of a cycloid. When $r<a$, the curve does not meet the base and has no loops. The trochoid is also called a **prolate cycloid** when $r>a$ and a **curtate cycloid** when $r<a$.

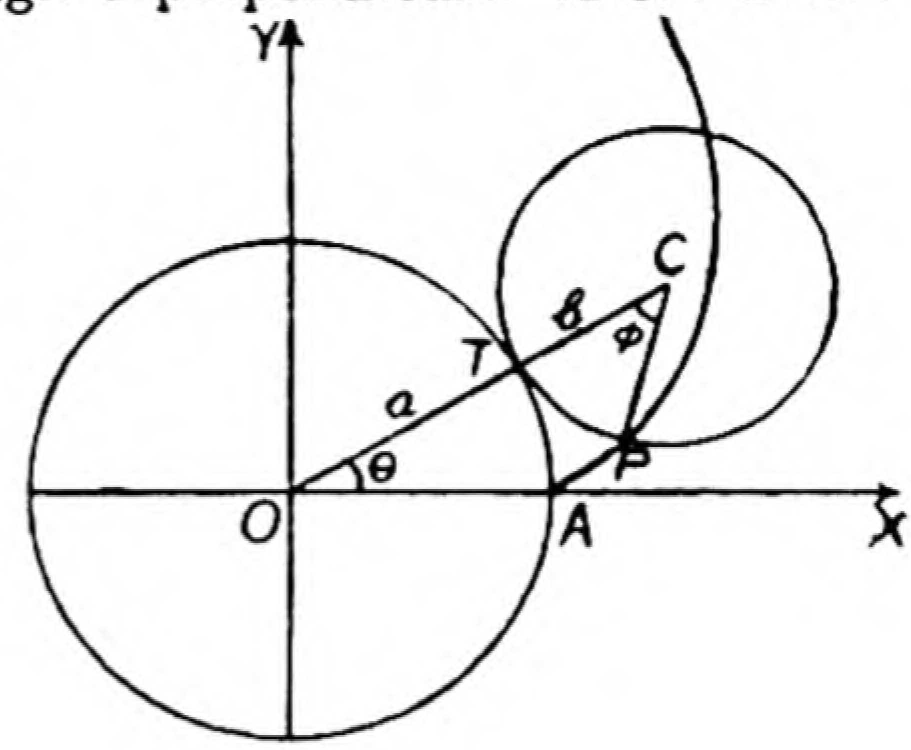


The curves (i) and (ii) in the figure above are the trochoids corresponding to $r=\frac{1}{2}a$ and $r=\frac{3}{2}a$ respectively. The curve (iii) in dotted line is the cycloid corresponding to $r=a$.

17·33. Epicycloids and Hypocycloids. The locus of a point P fixed on the circumference of a circle when the circle rolls on another fixed circle is called an **epicycloid** or a **hypocycloid** according as the rolling circle lies outside or inside the fixed circle. In case

the rolling circle contains the fixed circle inside it, the locus is called **pericycloid.**

(*a*) **Epicycloid.** Let the fixed circle be of radius a and O, its centre, be taken as the origin. Let A be the position of P when P is the point of contact of the rolling circle with the fixed circle. Take OA as the x-axis and the line through O perpendicular to OA as the y-axis. Let C be the centre of the rolling circle in any other position and P the moving point in this position. Let T be the point of contact of the two circles. Let OTC make an angle θ with OX and CP make an angle ϕ with OC, so that PC makes an angle $\theta+\phi$ with OX. Then by the condition of rolling.



$$\text{Arc } AT = \text{Arc } PT$$

or $$a\theta = b\phi,$$

whence $\phi = \dfrac{a}{b}\theta$ and so $\theta+\phi = \dfrac{a+b}{b}\theta.$

If (x, y) be the coordinates of P in this position, then

$$\left.\begin{aligned} x &= OC\cos\theta - PC\cos(\theta+\phi) = (a+b)\cos\theta - b\cos\frac{a+b}{b}\theta \\ y &= OC\sin\theta - PC\sin(\theta+\phi) = (a+b)\sin\theta - b\sin\frac{a+b}{b}\theta \end{aligned}\right\} (1)$$

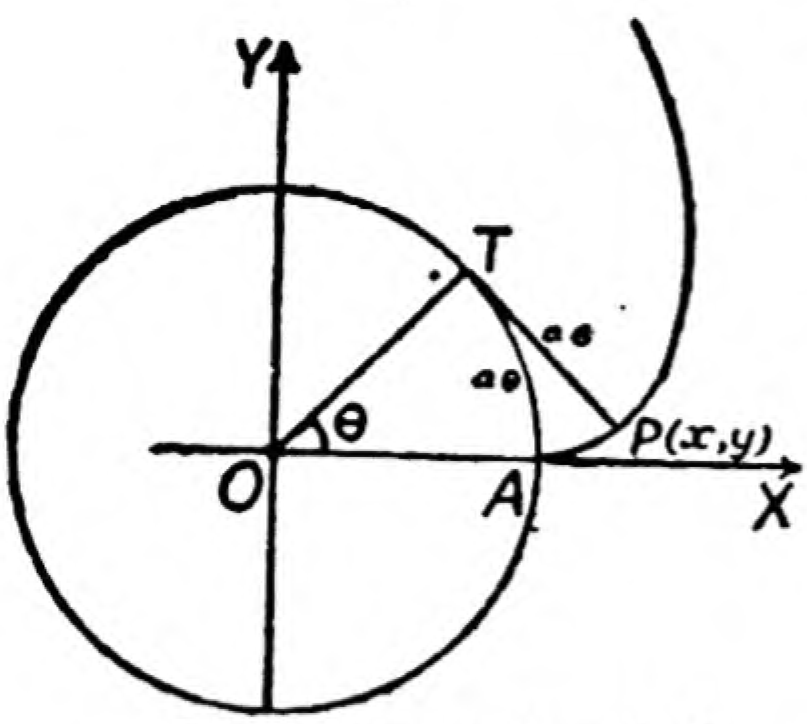
These are the parametric equations of the epicycloid traced by P. If the radii a and b of the two circles be commensurable, *i.e.*, if the ratio $a : b$ be a rational fraction, then after some exact number of revolutions of the rolling circle, the point P will return to its original position and subsequent to this it will repeat its previous path. In such cases the curve is algebraic, because the trigonometric functions can be eliminated between the equations (1). If the ratio $a : b$ is not a rational number, the point P never returns to its starting position and the curve does not repeat itself.

It is easy to see that the epicycloid has a cusp at every point where the point P comes into contact with the fixed circle, *i.e.*, after every complete revolution of the rolling circle.

Some particular cases of the epicycloid may be noted.

(*i*) If the radius of the fixed circle be infinitely great, the fixed circle becomes a straight line and we return to the case of a cycloid.

(*ii*) If the radius of the rolling circle be infinite, we get the



locus of a point P fixed on a straight line which rolls on a fixed circle. **In this case**

$$TP = \text{arc } TA = a\theta,$$

and so the parametric equations of the locus of P are

$$\left.\begin{aligned} x &= OT\cos\theta + TP\sin\theta \\ &= a\cos\theta + a\theta\sin\theta, \\ y &= OT\sin\theta - TP\cos\theta \\ &= a\sin\theta - a\theta\cos\theta. \end{aligned}\right\} \quad ...(2)$$

This curve is of the nature of a spiral and keeps on opening out as θ increases, the distance of the point P from O keeps on increasing and ultimately tends to infinity. The curve is an involute of the fixed circle.

(*iii*) If $b=a$, the locus of **P** is a **Cardioid**. For, from equations (1),

$$x = 2a\cos\theta - a\cos 2\theta, \qquad y = 2a\sin\theta - a\sin 2\theta,$$

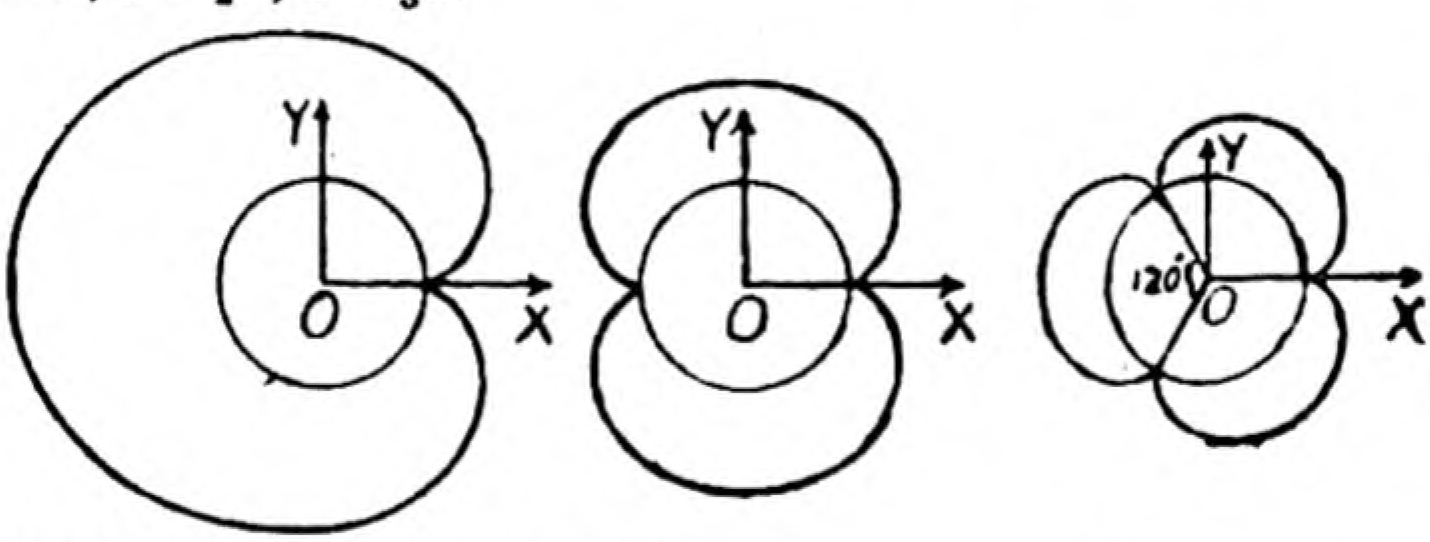
or $\quad x - a = 2a\cos\theta(1-\cos\theta), \qquad y = 2a\sin\theta(1-\cos\theta),$

which shows that the radius vector drawn from $(a, 0)$ as pole is given by

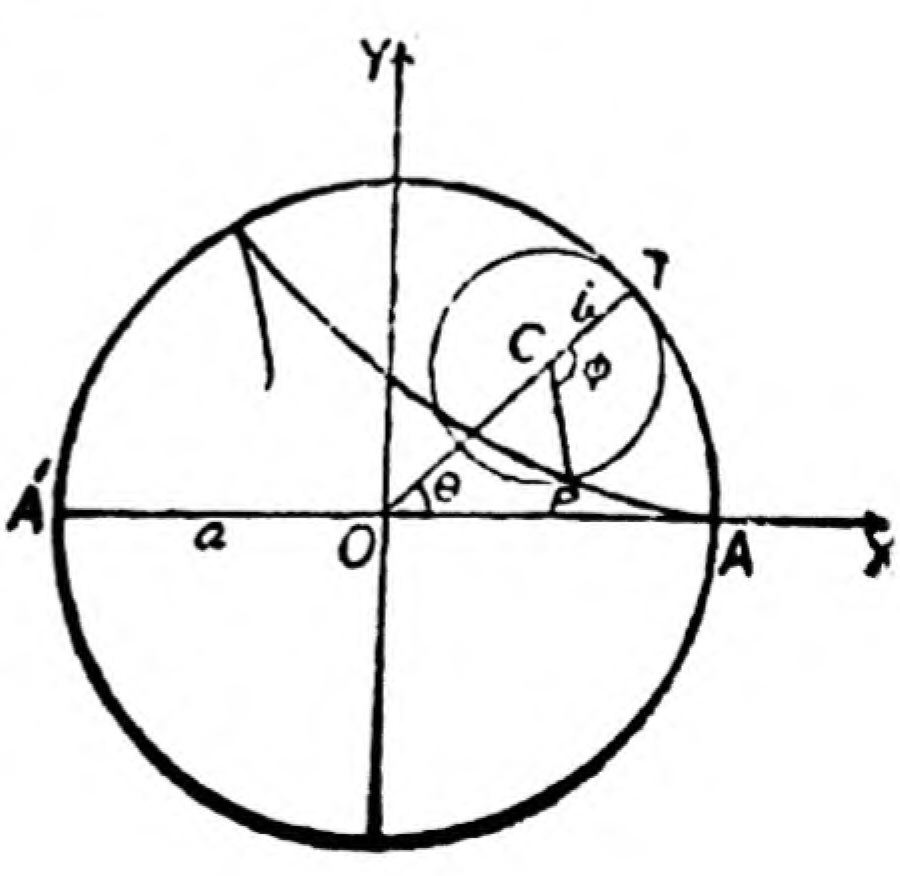
$$r = 2a\,(1-\cos\theta)$$

which is the cardioid in its standard polar form.

The following figure shows the epicycloids corresponding to the cases $b=a$, $b=\frac{1}{2}a$, $b=\frac{1}{3}a$.



(*b*) **Hypocycloid**. In this case the rolling circle is inside the fixed circle and so $b<a$. If we take the axes as in (a) above, $\angle AOC=\theta$. $\angle ICP=\phi$, then since arc AT=arc PT, therefore $a\theta=b\phi$. Also $\angle sPT$, thereso that PC make an angle $\theta+\pi-\phi=\pi-(\phi-\theta)$ with OX. Hence if (x, y) be the co-ordinates of P, then from the figure.



$$x=OC\cos\theta-PC\cos\{\pi-(\phi-\theta)\}$$
$$=(a-b)\cos\theta+b\cos(\phi-\theta)$$
$$=(a-b)\cos\theta+b\cos\frac{a-b}{b}\theta, \qquad \ldots(3)$$

and
$$y=OC\sin\theta-PC\sin\{\pi-(\phi-\theta)\}$$
$$=(a-b)\sin\theta-b\sin\frac{a-b}{b}\theta. \qquad \ldots(4)$$

As in the case of the epicycloid, the hypocycloid repeats itself if the ratio b/a is rational, otherwise not. Some particular cases may be noted.

(*i*) If $b=a$, the rolling circle coincides with the fixed circle, no rolling is possible and the moving point P becomes fixed at A. This is also borne out by equations (3).

(*ii*) If $b=\frac{1}{2}a$, $y=0$ always, and the hypocycloid degenerates into the portion $A'A$ of the x-axis.

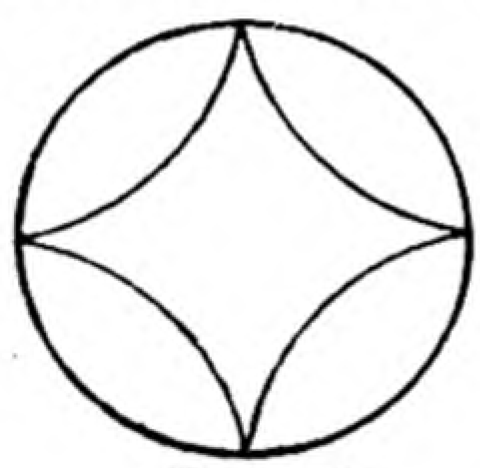


Fig. (*i*)

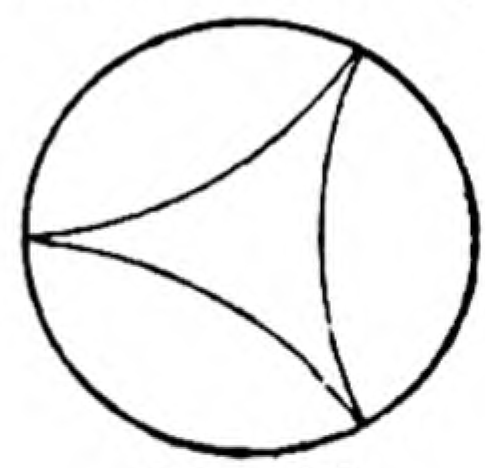


Fig. (*ii*)

(*iii*) If $b=\frac{1}{3}a$, then we get three cusps and after three complete revolutions of the rolling circle, the moving point returns to A.

(*iv*) If $b=\frac{1}{4}a$, then we get the four-cusped hypocycloid or the **astroid.** In this case, equation (3) becomes

$$x=\tfrac{3}{4}a\cos\theta+\tfrac{1}{4}a\cos 3\theta=a\cos^3\theta,$$
$$y=\tfrac{3}{4}a\sin\theta-\tfrac{1}{4}a\sin 3\theta=a\sin^3\theta,$$

which give a simple parametric representation of the astroid. Eliminating θ, the cartesian equation of the astroid is

$$x^{2/3}+y^{2/3}=a^{2/3}.$$

Figures (*i*) and (*ii*) above show the three-cusped and the four-cusped hypocycloids respectively.

The values $b=\frac{2}{3}a$ and $\frac{3}{4}a$ again give the three-cusped and the four-cusped hypocycloids respectively, but now traced in the opposite direction.

If we take the radius of the rolling circle as $a-b$ instead of b, we get the reflection of the hypocycloid (4) in the x-axis. For, changing b into $a-b$ in (3), we get

$$x=b\cos\theta+b(a-b)\cos\frac{b}{a-b}\theta=(a-b)\cos\phi+b\cos\frac{a-b}{b}\phi,$$
$$y=b\sin\theta-(a-b)\sin\frac{b}{a-b}\theta=-(a-b)\sin\phi+b\sin\frac{a-b}{b}\phi,$$

where $\phi=b\theta/(a-b)$; and these are the parametric equations of the reflection of (3) in the x-axis.

(c) **Pericycloid.** If in equations (3) of (b), we take $b>a$, we get the parametric equations of a pericycloid in the form

$$x=b\cos\frac{b-a}{b}\theta-(b-a)\cos\theta,\ y=b\sin\frac{b-a}{b}\theta-(b\ \ a)\sin\theta.$$

17·34. Equitrochoids and hypotrochoids. If the moving point be not on the circumference of the rolling circle, then the curve traced out by it is called an epitrochoid or hypotrochoid according as the rolling circle is outside or inside the fixed circle. If the moving point be at a distance r from the centre of the rolling circle, then corresponding to equations (1) of Art. 17·33, the equations of the epitrochoid are

$$x=(a+b)\cos\theta-r\cos\frac{a+b}{b}\theta,\quad y=(a+b)\sin\theta-r\sin\frac{a+b}{b}\theta.$$

Similarly we many write the equations of a hypotrochoid.

Polar Co-ordinates

17·4. The Cardioid r=a (1+cos θ). (i) The curve is drawn in Ex. 2, Art. 12·2, Part II. Its cartesian equation is

$$(x^2+y^2-ax)^2=a^2(x^2+y^2).$$

and pedal equation is $r^3=2ap^2$.

(ii) It is the first positive pedal of the circle $r=2a\cos\theta$.

(iii) It is also the envelope of the circles drawn on the radii vectores of $r=2a\cos\theta$ as diameters.

The equation $r=a(1-\cos\theta)$ also represents a cardioid which is obtained from the preceding one by turning it about the pole through two right angles. This is the locus of a point on the circumference of a circle which rolls on another fixed equal circle. [See Art. 17·33, (a), (iii).]

17·41. The Lemniscate of Bernoulli. r²=a² cos 2θ.

(i) Cartesian equation is $(x^2+y^2)^2=a^2(x^2-y^2)$,

(ii) Pedal equation is $r^3=a^2p$.

(iii) It is the first positive pedal of the rectangular hyperbola $x^2-y^2=a^2$, and the asymptotes of the hyperbola are tangents to the lemniscate at the pole.

Thus the rectangular hyperbola $x^2-y^2=a^2$ or $r^2\cos 2\theta=a^2$ is the envelope of lines drawn through the extremities of radii vectores of the lemniscate perpendicular to them.

(iv) It is also the inverse of the rectangular hyperbola *w.r.* to a circle of radius a.

The curve is drawn fully in Ex. 3. Art. 16·4

17·42. The curve $r^2 = a^2 \sin 2\theta$.

(*i*) Cartesian equation is $(x^2+y^2)^2 = 2a^2xy$.

(*ii*) Pedal equation is $r^3 = a^2p$.

(*iii*) It is the first positive pedal of the rectangular hyperbola $xy = a^2$, and the asymptotes of the hyperbola are tangents to the curve at the pole.

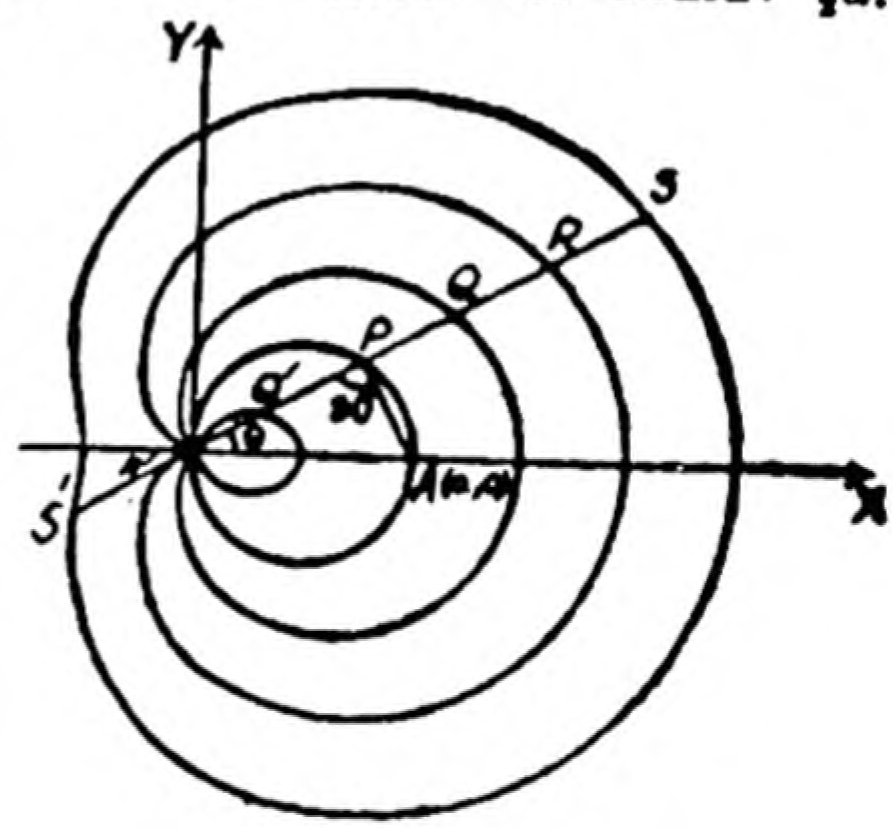
(*iv*) It is the inverse of the rectangular hyperbola $r^2 \sin 2\theta = a^2$ *w.r.* to the circle of radius a having its centre at the pole.

This curve is obtained from the Lemniscate of Bernoulli by turning it through an angle $\pi/4$ about the pole.

17·5. The Limacon of Pascal. $r = a \cos\theta \pm b$. Let O be a fixed point on the circumference of a circle of radius $\frac{1}{2}a$. Take the diameter through O as the initial line OX. Let P be any other point on the circle. Take points Q, Q' on OP such that

$$Q'P = PQ = b.$$

then the locus of points such as Q, Q' as P moves on the circle is the limacon of Pascal. If $\angle XPO = \theta$ and A be the point $(a, 0)$, it is at once seen that the locus of Q is



$$r = a \cos\theta + b$$

and that of Q' is

$$r = a \cos\theta - b.$$

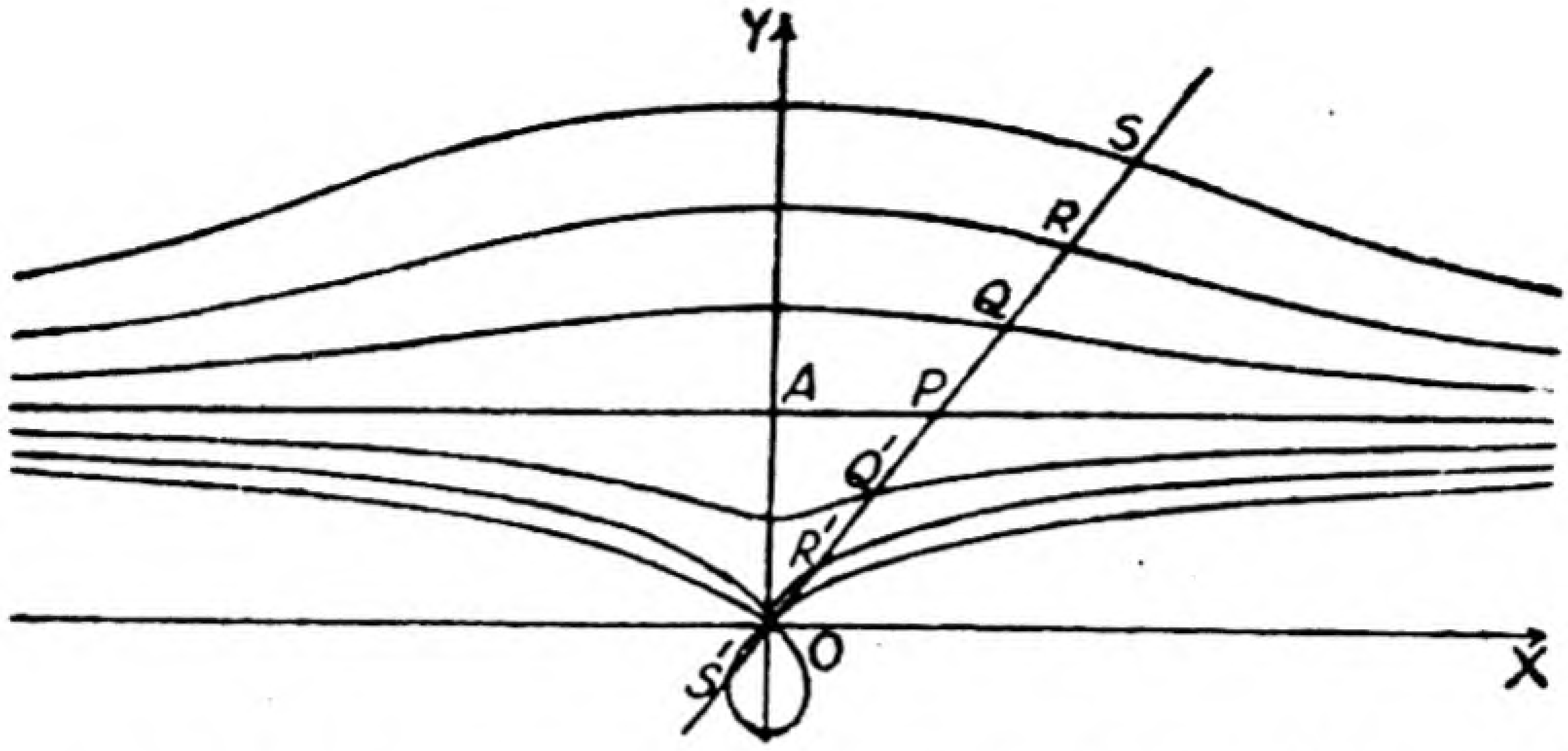
These two equations represents the same curve, for the value θ in the first equation and the value $\theta + \pi$ in the second equation lead to the same point.

If $b < a$, the curve passes through the origin when $\theta = \cos^{-1}(-b/a)$ and forms two loops, one inside the other. In the figure the locus of Q, Q' corresponds to this case.

If $b = a$, the inner loop shrinks to a point and we get a cusp at O. The limacon then becomes the cardoid $r = a(1 + \cos\theta)$. It is the locus of points R, R' in the figure.

If $b > a$, r never vanishes ; the locus of S, S' in the figure corresponds to this case.

17·6. Conchoid of Nicomedes. $r = a \operatorname{cosec}\theta \pm b$. Let O be any fixed point and AP any fixed straight line at a distance a from O. Let P be any point on the line. Join OP and take points Q, Q' on the line OP such that $Q'P = PQ = b$. Then the locus of the points Q, Q' when P moves on the straight lines is the Conchoid of



Nicomedes. Take the line through O parallel to AP as the initial line OX. Let OP make an angle θ with OX, then the equation of the locus of Q is

$$r = a \operatorname{cosec} \theta + b,$$

and that of Q' is

$$r = a \operatorname{cosec} \theta - b.$$

These two represent the same curve. We therefore take the first as the standard form.

The curve consists of two branches. The cartesian equation is $(x^2+y^2)(y-a)^2 = b^2y^2$ and includes both the branches. The curve is symmetrical about the line $\theta = \frac{1}{2}\pi$, *i.e.*, OA, the line through $O \perp OX$.

The tangents at the origin are $a^2x^2+(a^2-b^2)y^2=0$. Thus the origin is a node, a cusp or a conjugate point according as $a<b$, $a=b$ or $a>b$.

The line $r \sin \theta = a$, *i.e.*, AP is an asymptote of the curve in all cases.

If $b<a$, both branches of the curve are above OX, and there are four points of inflexion on the curve. This case corresponds to the locus of Q, Q' in the figure.

If $b=a$, the lower branch has a cusp at the origin and there are only two points of inflexion, those on the upper branch. This case corresponds to the locus of R, R' in the figure.

If $b>a$, the lower branch has a node at the origin and there is a loop below OX. There are two points of inflexion on the upper branch. This case is given by the locus of S, S' in the figure.

17·7. The Spirals.

17·71. The Equiangular or Logarithmic Spiral. $r = ae^{m\theta}$.

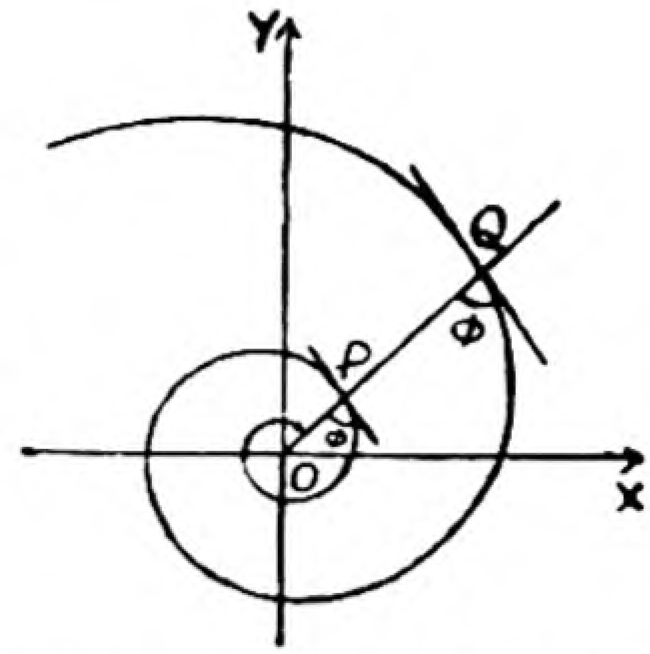
This curve derives its name from the fact that it makes a constant angle ϕ with the radius vector.

For

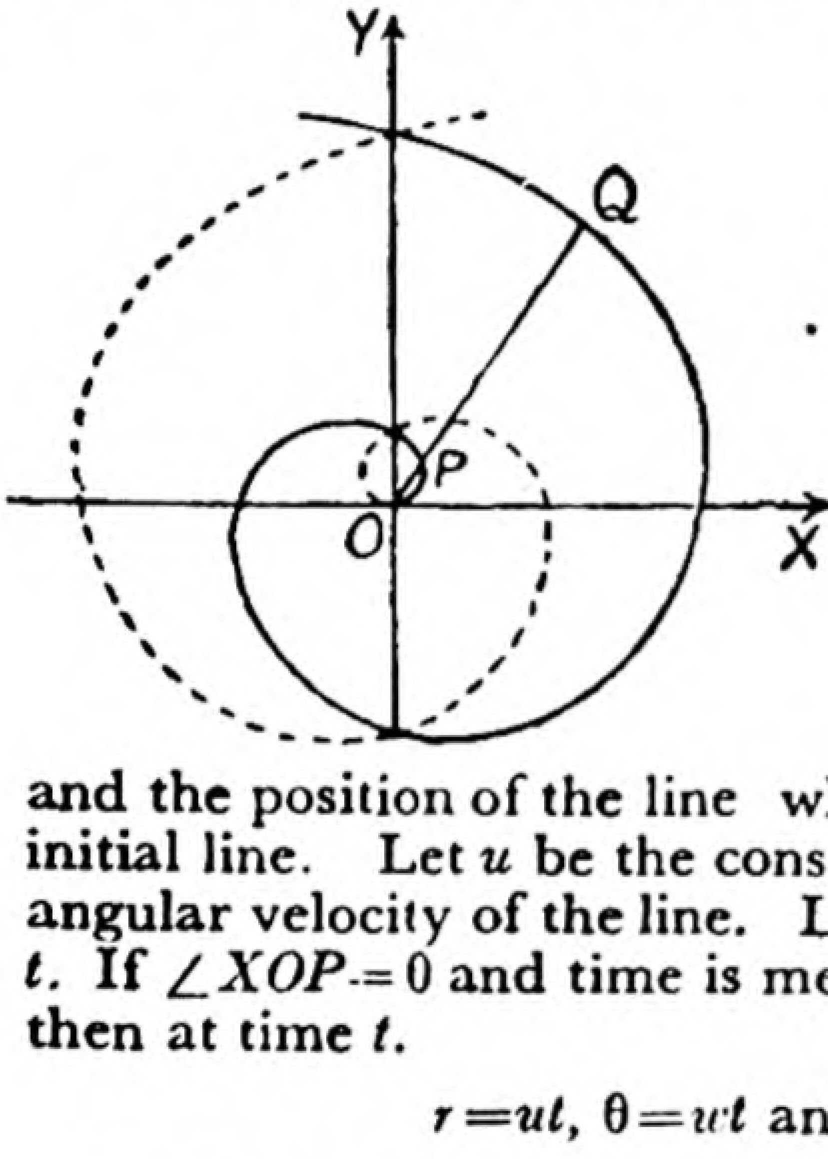
$$\cot\phi = \frac{1}{r}\frac{dr}{d\theta} = \frac{1}{r}\,ame^{m\theta} = m$$

or $$\phi = \cot^{-1} m = \text{constant.}$$

As θ increases from 0 to ∞, r increases from a to ∞ and as θ decreases from 0 to $-\infty$, r decreases from a to 0.



17·72. The Spiral of Archimedes. $\mathbf{r = a\theta}$. The curve is symmetrical about the line $\theta = \frac{1}{2}\pi$, *i.e.*, OY, since the equation remains unchanged when θ is changed into $-\theta$ and r into $-r$.



As θ increases from 0 to ∞, r increases from 0 to ∞. Part of the curve for negative values of r is shown in dotted line.

The curve is generated by a point which moves with uniform velocity in a straight line which is revolving with uniform angular velocity about a fixed point in itself. Take the fixed point as the origin O and the position of the line when the moving point is at O as the initial line. Let u be the constant speed of the moving point, w the angular velocity of the line. Let P be the position of the point at time t. If $\angle XOP = 0$ and time is measured from the instant when P is at O, then at time t.

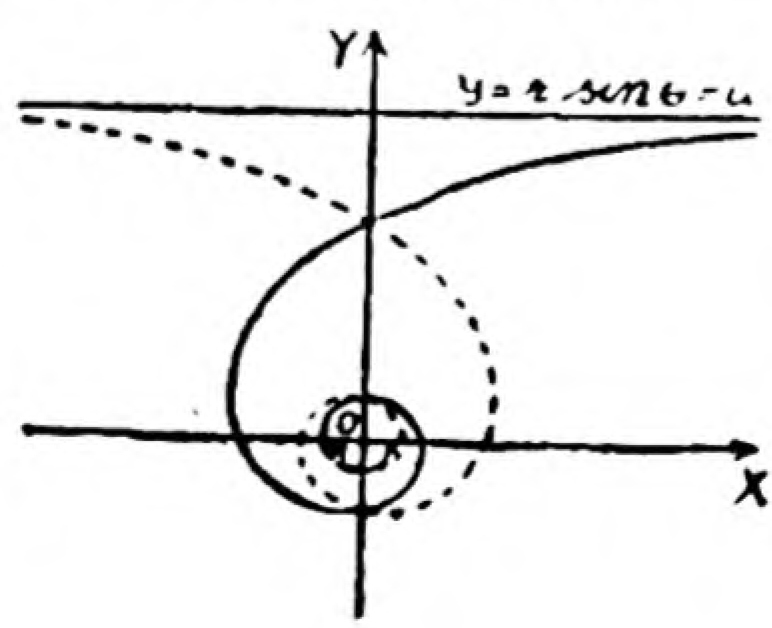
$$r = ut,\ \theta = wt \text{ and so } r = (u/w)\theta = a\theta.$$

The point Q in the figure is the position of the moving point at time $t + (2\pi/w)$.

17·73. The reciprocal or hyperbolic spiral. $\mathbf{r\theta = a}$. The curve is symmetrical about OY. As $\theta \to 0$, $r \to \infty$ and the line $r\sin\theta = a$, *i.e.*, $y = a$ is an asymptote of the curve.

As $\theta \to +$ or $-\infty$, $r \to 0$.

The part of the curve for negative values of θ is shown in dotted line.



17·74. The sine spiral. $r^n = a^n \cos n\theta$. Some particular cases may be noted.

(*i*) For $n=1$, we get the circle $r=a \cos \theta$ and for $n=-1$, the straight line $r \cos \theta = a$.

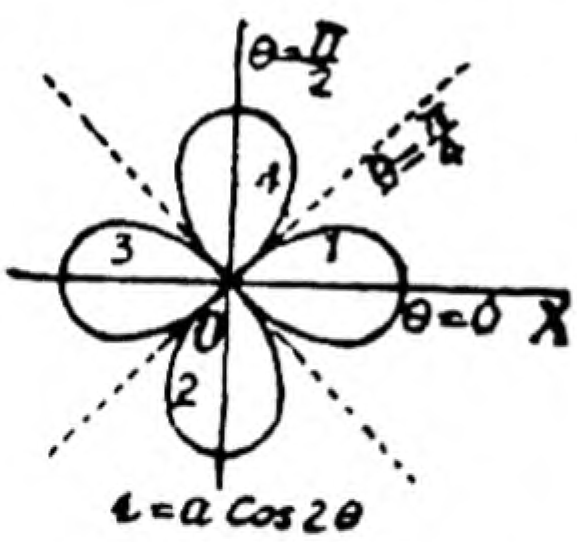
(*ii*) For $n=2$, we get the Lemniscate of Bernouilli $r^2=a^2 \cos 2\theta$ and for $n=-2$, the rectangular hyperbola $r^2 \cos 2\theta = a^2$.

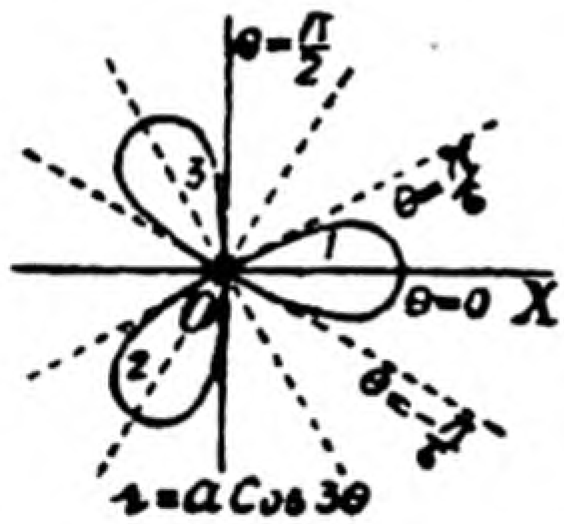
(*iiii*) For $n=\frac{1}{2}$, we get the cordioid $r=\frac{1}{2}a(1+\cos\theta)$ and for $n=-\frac{1}{2}$, the parabola $r(1+\cos\theta)=2a$.

It may be noted that the curves $r^n=a^n \cos n\theta$ and $r^n \cos n\theta = a^n$, *i.e.*, the curves corresponding to the values n and $-n$ are inverses of each other.

The pedal equation of the curve is $pa^n = r^{n+1}$.

17·8. The curve $r=a \cos n\theta$.





Here r vanishes when

$$\theta = \frac{\pi}{2n}, \frac{3\pi}{2n}, \frac{5\pi}{2n}, \ldots\ldots$$

and r attains its maximum value a when

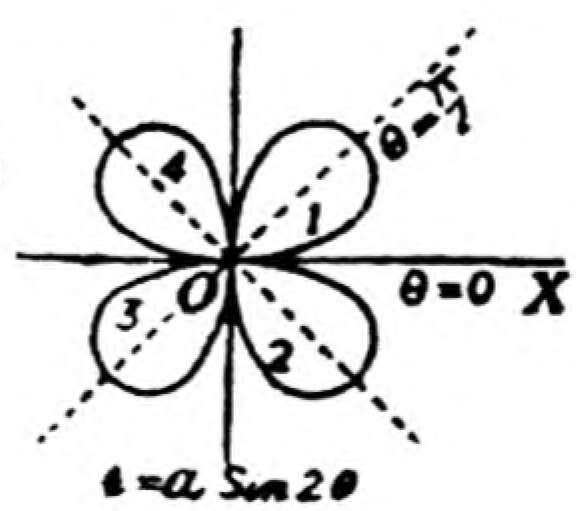
$$\theta = 0, \frac{\pi}{n}, \frac{2\pi}{n}, \frac{3\pi}{n}, \ldots\ldots$$

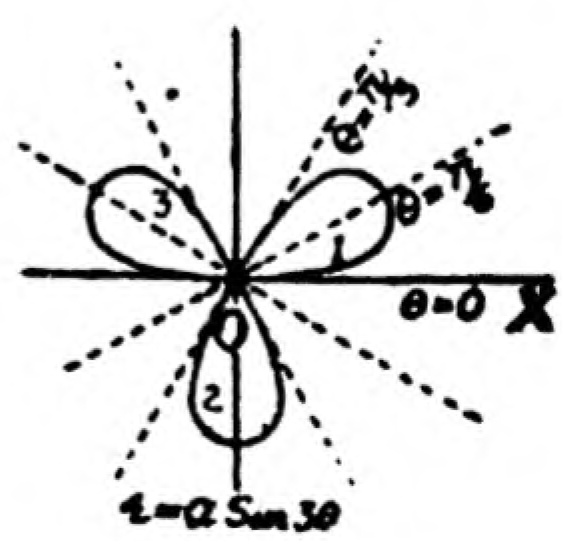
The curve lies wholly within the circle $r=a$ and is symmetrical about the initial line.

There are n or $2n$ loops according as n is old or even. The values $n=2$ and $n=3$ give the so-called four leaved and three leaved roses respectively. These are traced in the attached diagram.

17·81. The curve $r=a \sin n\theta$.

This is obtained from the curve $r=a \cos n\theta$ by rotating it about the pole through an angle $\pi/2n$ The four and three leaved roses obtained on taking $n=2$ and $n=3$ respectively are traced in the attached figure.





MISCELLANEOUS EXAMPLES VI

Trace the following curves :—

1. $y^2(2a-x)=x^3$. (*Delhi, 1958*)

2. $y^2(x+a)=x(x^2+a^2)$.

3. $y^2(a-x)=x^2(a+x)$. (*Panjab, 1951*)

4. $(x-a)y^2=x^2(x+a)$.

5. $x=(y-1)(y-2)(y-3)$. (*Panjab, 1939*)

6. $x^2y+xy^2=a^3$.

7. $y^2(x+3a)=x(x-a)(x-2a)$. (*Allahabad*)

8. Find the position and nature of the double point on the curve $y^3=x^3+ax^2$.

Find also the asymptotes of the curve and trace it. (*Panjab, 1940 ; Agra, 1948 ; Delhi, 1950*)

9. Trace the curve $y^2=(x-2)^2(x-5)$ and show that the line joining the points of inflexion subtends a right angle at the double point. (*Panjab, Sept. 1950*)

10. Find the asymptotes, the double points, and points of inflexion, if any, of the curve $a^2/x^2-b^2/y^2=1$, and trace it (*Allahabad, 1952*)

11. Find the asymptotes of the curve

$$\frac{a^3}{x^3}-\frac{b^3}{y^3}=1$$

and trace the curve. (*Agra, 1947*)

12. Sketch the locus (the cycloid) given by

$$x=a(t+\sin t),\ y=a(1+\cos t).$$

for values of t between 0 and 4π.

Find the coordinates of the centre of curvature at the point t, and prove that the locus of the centre of curvature is an equal cycloid ; illustrate this in your diagram. (*M.T.I., 1948*)

ANSWERS

Examples XLVIII, Page 248.

1. $y=1,\ x=1$. 2. $x=a$. 3. $x=+a,\ y=\pm a$. 4. $y=\pm a$.
5. $y=0$. 6 $x=0,\ y=0$. 7. $x+a=0,\ y=\pm 1$. 8. $y+1=0$.
9. None. 10. $x=\pm a,\ y=\pm b$. 11. None. 12. None.

Examples XLIX, Page 252.

2. $x+y=a$. 3. $y=x+2$. 4. $y=x-a$. 5. $y=x,\ 2x,\ 3x$.
6. $y=\pm x+\frac{1}{2}b$. 7. $x=2a,\ y=\pm(x+a)$.
8. $x-1=0,\ x-2=0,\ x+y+1=0$. 9 $y=\pm x,\ x+1$.
10. $y=-x,\ x-3,\ x+2$.

Examples L, Page 256. p. 276

1. $y=0,\ x-1,\ -x+1$. 2. $x=-1,\ y=0,\ -x$. 3. $x=a$.
4. $x=0,\ y=0,\ x+2(a-b)$. 5. $x=b,\ y=\pm(x+\frac{1}{2}b)$.
6. $y=x+\frac{1}{4},\ -x+\frac{3}{2},\ -\frac{1}{3}x-\frac{3}{4}$. 7. $x=\pm a$.
8. $x=0,\ 2y,\ -2y+6$. 9. $y=x,\ 2x-1,\ -x+2$.
10. $y=-x,\ x\pm 1$. 11. $y=x,\ -\frac{1}{2}x\pm\frac{1}{2}$.
12. $y=x,\ -2x,\ -2x-1$. 13. $x=0,\ y=x,\ x+1$.
14. $x=0,\ y=0,\ x\pm\sqrt{(a^2+b^2)}$. 15. $x=\pm a,\ y=x\pm a$.

Examples LI, Page 259.

1. $x=0,\ y=a,\ x-a$. 2. $y=0,\ x=\pm a$.
3. $y=-x+1,\ -\frac{1}{3}x+\frac{1}{3},\ 3x+2$. 4. $y=\pm x,\ -\frac{2}{3}x,\ -\frac{3}{2}x$.
5. $y=0,\ -x\pm 1$. 6. $y=x+2,\ -x+2,\ 2x-4$.
7. $y=x-\frac{1}{2},\ -x-\frac{5}{6},\ 2x+\frac{4}{3}$. 8. $y=x,\ 2x-2,\ 2x-3$.
9. $y=-\frac{1}{2}x-1,\ -x\pm 2\sqrt{2}$.
10. $x=2y-14a,\ 3y+13a,\ y+a,\ y+2a$. 11. $y=0,\ x+2,\ x=1$
12. $y=\pm x,\ -\frac{1}{2}(x+1)$.

Examples LII, Pages 261—62.

5. $bxy(x^2-y^2)=a(a^2-b^2)(x^2+y^2-a^2)$.
8. $x^2-5xy+6y^2+5x-11y+4=0$.

Ref. Ex. 1. Art. 14 83. Read $x^3+y^3=3ax^2$ for $x^3+y^3=3xy^2$.

Examples LIII, Pages 267—68.

1. (i) $y=x+\frac{1}{2}$. Above the asymptote in the first quadrant and below it in the third quadrant.

   (ii) $y=-x-\frac{1}{2}$. Above the asymptote in the second quadrant and below it in the fourth.

   (iii) $x-1=0$. The curve lies to the right of the asymptote.

2. $y-x=0$. Below the asymptote in the first quadrant and above the asymptote in the third quadrant.

3. (i) $x+3a=0$. The curve lies to the left of the asymptote.

   (ii) $y-x+3a=0$. The curve lies above the asymptote for positive values of x and below it for negative values.

(*iii*) $y+x-3a=0$. Below the asymptote for positive values of x and above it for negative values of x.

4. (*i*) $x+y=0$. Above the asymptote for positive values of x and below it for negative values of x.

(*ii*), (*iii*) $y=x\pm(a/\sqrt{2})$. Below the asymptote for positive values of x and above it for negative values of x.

5. (*i*) $x=0$. The curve lies to the right of the asymptote.

(*ii*) $y-x=0$. The curve lies on both sides of the asymptote in the first quadrant.

Examples LIV, Page 270.

1. $r \sin(\theta-\frac{1}{3}m\pi)=\frac{1}{3}a$, where m is an integer.
2. $r\sin\theta=\pm\frac{1}{2}a$; $r\cos\theta=\pm\frac{1}{2}a$.
3. $r\sin\theta=a$. 4. $a=r\sin(\theta-1)$.
5. $r\sin(\theta-m\pi/n)=(a/n)\sec m\pi$, where m is an integer.
6. $r\cos\theta=\pm a$. 7. $\theta=0$.
8. $\theta=m\pi/n$, where m is an integer.
9. $r\sin(\theta-m\pi/n)=b/n$, where m is an integer.
10. $a=2r(\pm\cos\theta-\sin\theta)$. 11. $r\sin\theta=a$. 12. $\theta=0$.
13. $a=r\sin(\theta-1)$. 14. $(le/r)=(e^2-1)\cos\theta\pm\sqrt{(e^2-1)}\sin\theta,\ e>1$.

Examples LV, Page 275.

2. Concave in $\frac{1}{2}\pi<x<\frac{3}{2}\pi$, convex in $0\leqslant x<\frac{1}{2}\pi$, $\frac{3}{2}\pi<x\leqslant 2\pi$. Points of inflexion at $x=\frac{1}{2}\pi$ and $\frac{3}{2}\pi$.

3. Concave for $x<2$ and $x>3$. Convex for $2<x<3$. Points of inflexion at $x=2$ and 3.

5. (*i*) For $x=0$ and $\pm\sqrt{3}a$. (*ii*) For $x=3$, $\frac{1}{11}(28\pm\sqrt{3})$.
(*iii*) (0, 2). (*iv*) (0, 0). (*v*) For $x=a\sqrt{\{(-5+\sqrt{28})/3\}}$.
(*vi*) $(1/3, \pm4\sqrt{3}/9)$.

9. (0, 0), $(\pm\sqrt{3}a, \pm\sqrt{3}a/4)$.

Examples LVI, Page 277.

1. $(\sqrt{2}a, \frac{1}{2})$. 3. Concave.

Examples LVII, Pages 283—84. P. 303

1. Node at (0, 0), 2. Node at (2, 0).
3. Conjugate point at (0, 0) if a, b are of the same sign, otherwise a node.
4. Cusp at (0, 0). 5. Node at (0, 0). 6. Node at (0, 0).
7. Node at $(a, 0)$. 8. Nodes at (0, −1), (±1, 0).
9. Conjugate point at (0, 0).
10. Conjugate point at (0, 3) cusp at (2, 3).
11. Cusp at (1, −1). 12. Cusp at ((1, −1). 15. Cusp.
16. $y=\pm\{b/\sqrt{(a^2-b^2)}\}x-b$.

Examples LIX, Pages 289—90.

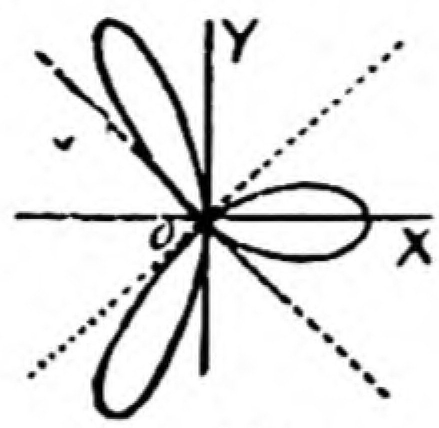
1. Origin is a single cusp of the first kind. $y=0$ is the cuspidal tangent.
2. $x=0$ is the cuspidal tangent. Origin is a double cusp.

3. Single cusp of the first kind. $x=0$ is the cuspidal tangent.
4. Single cusp of the first kind.
5. Single cusp of the first kind.
6. Single cusp of the second kind.
7. Point of oscul-inflexion.
8. Single cusp of the second kind.
9. Single cusp of the second kind.
10. $x=0$ is a cuspidal tangent. $y=0$ is a tangent to the curve at the origin to a second branch.

Miscellaneous Examples V, Pages 290—291.

1. (*i*) $y=x+\frac{1}{4}$, $-x+\frac{3}{2}$, $-\frac{1}{3}x-\frac{3}{4}$. (*ii*) $x=0$, $x=2y$, $-2y+6$.
   (*iii*) $x=b$, $y=\pm x\pm\frac{1}{2}b$. (*iv*) $x=1$, $y=0$, $x+2$.
   (*v*) $y=\pm x$, $x=\pm a$. (*vi*) $y=x$, $x=\pm 1$.
3. $y=x-\frac{7}{6}$. $3x-\frac{3}{2}$, $-\frac{1}{2}x-\frac{5}{6}$; $106y-381x+105=0$.
4. $(y-2x)(y-x)^2+8x=0$.
6. $xy(x+2y-6)+8x+y=0$; $x-y+3=0$, $x+y-1=0$.
7. (*i*) $r\sin\theta=2$.
   (*ii*) $r\sin\theta=a$; $r\cos\theta=a/\{(m+\frac{1}{2})^n\}$, m being an integer.
9. (*i*) $(0, 1)$, $(\frac{2}{3}, \frac{11}{27})$.
   (*ii*) $x=0$, $\pm a\sqrt{(2/3)}$ give points of inflexion.
   (*iii*) $(-1, 1)$. (*iv*) $(\pm 1/\sqrt{3}, 3/4)$.
14. No double point. The points of inflexion are $(0, \frac{1}{8})$, $(-1, 0)$ and the point at infinity on $8y=x$, and these lie on $8y=x+1$.
15. (*i*) Node at $(2, 1)$. (*ii*) Cusp at $(0, 0)$. (*iii*) Conjugate point at $(0, -a)$ if $b<a$ and node if $b>a$. (*iv*) Node at $(0, 0)$. (*v*) Node at $(0, 0)$.
17. $(0, 0)$; cusp if $b=0$. Node if a and b are of unlike signs ; otherwise a conjugate point.
18. Conjugate point at $(1, -1)$.

Examples LX, Pages 307—308.



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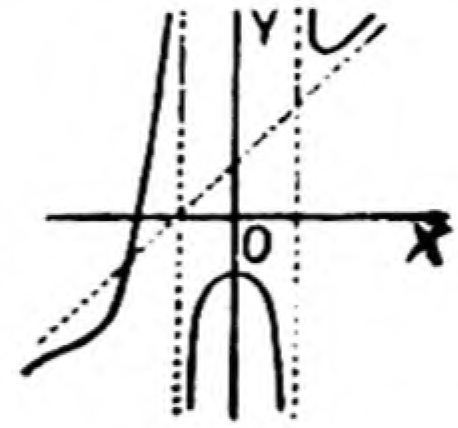

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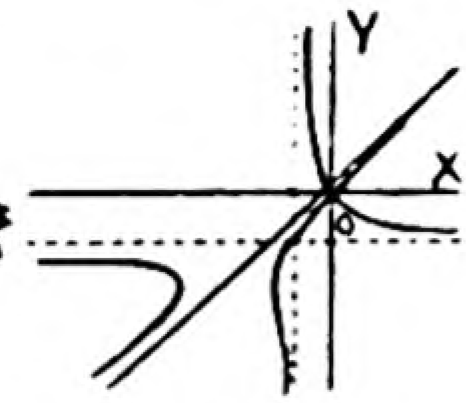

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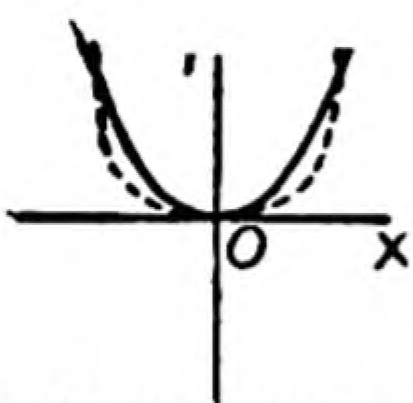
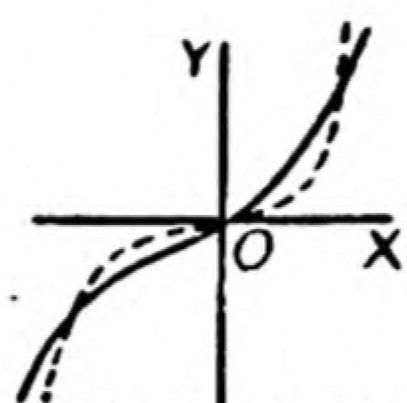
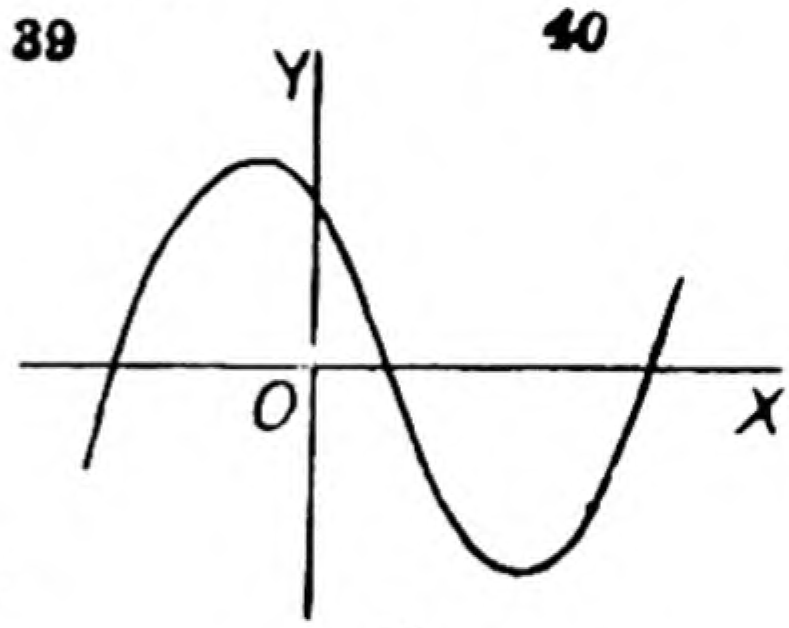




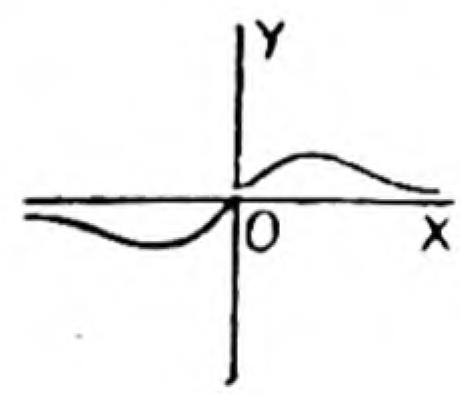


1 (i) AND (iii) (DOTTED) 1 (ii) AND iv (DOTTED) 2

1

2

3



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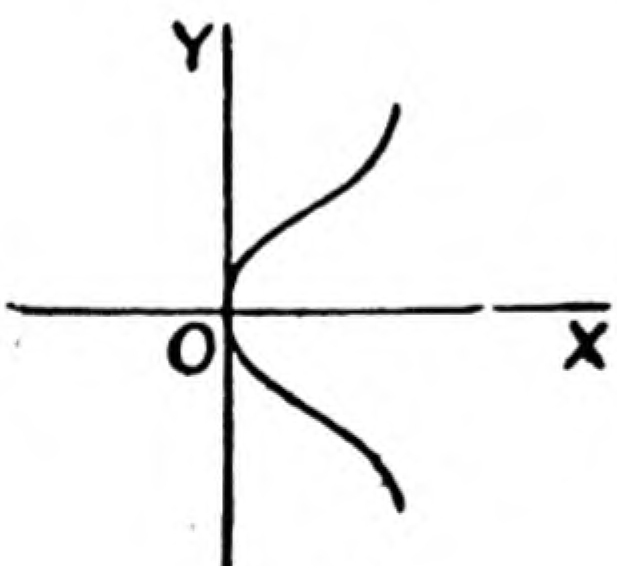

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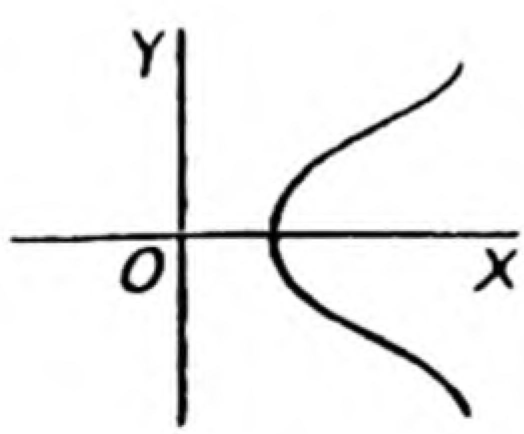

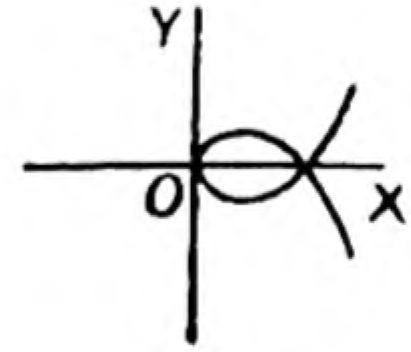

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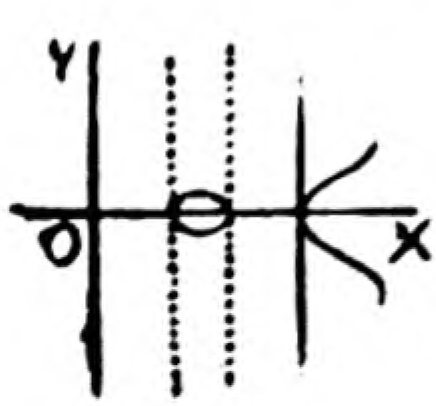

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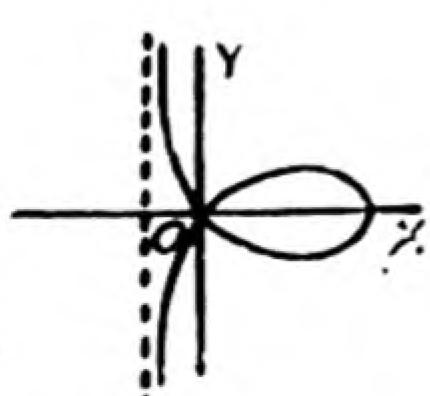

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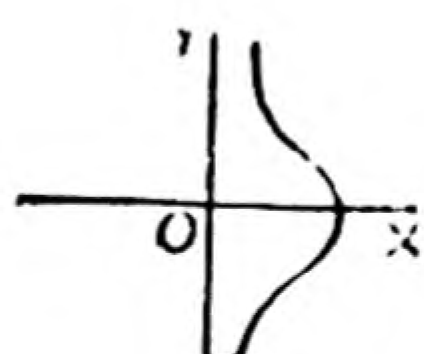

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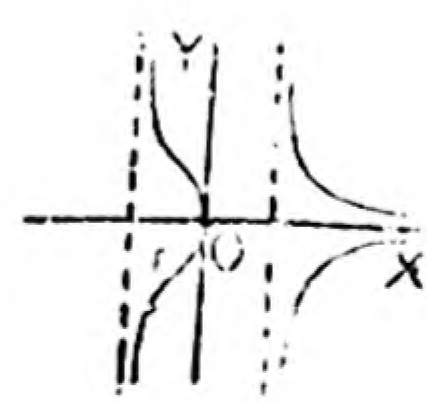

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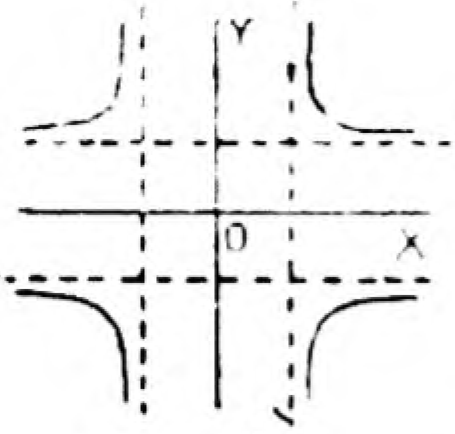

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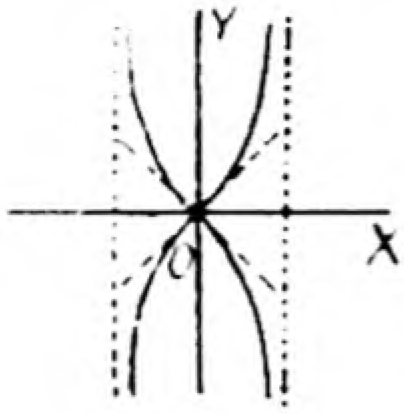

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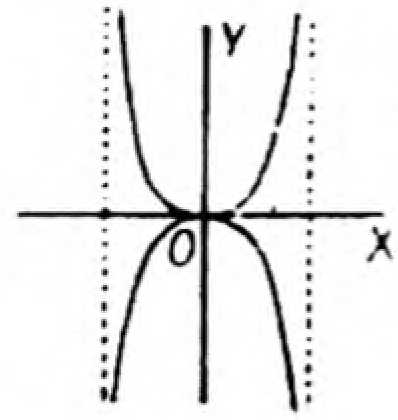

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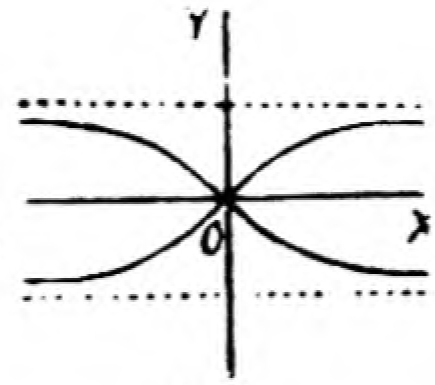

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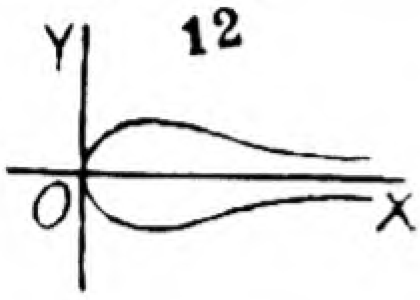

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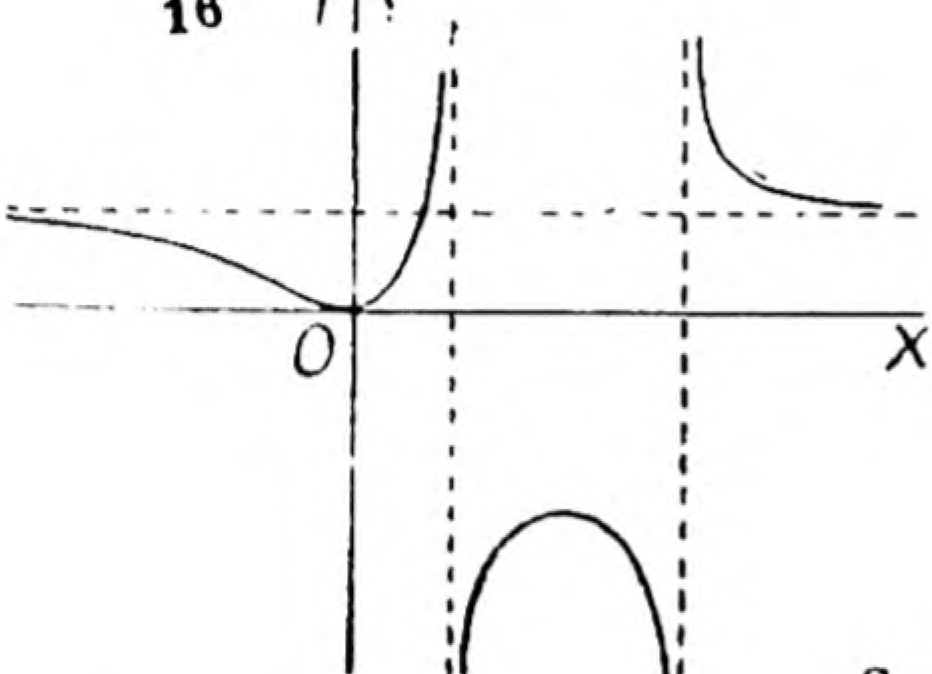

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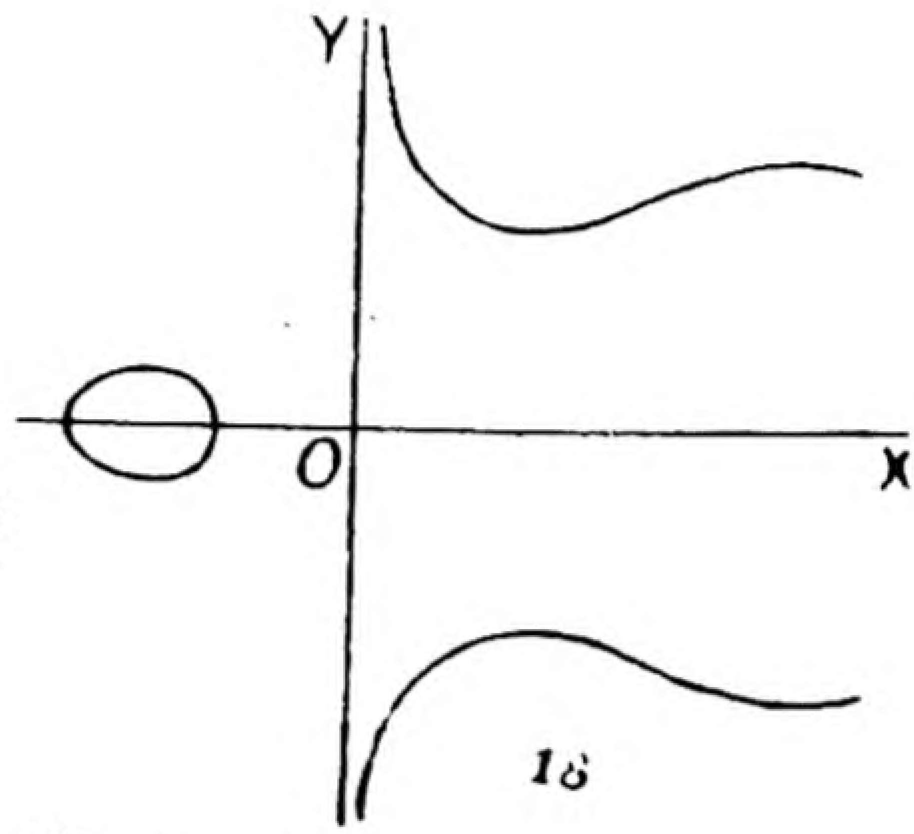
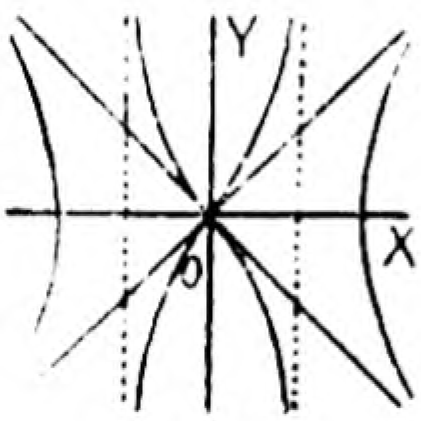


18

See Fig. Ex. 1.
Art. 14·83.
21



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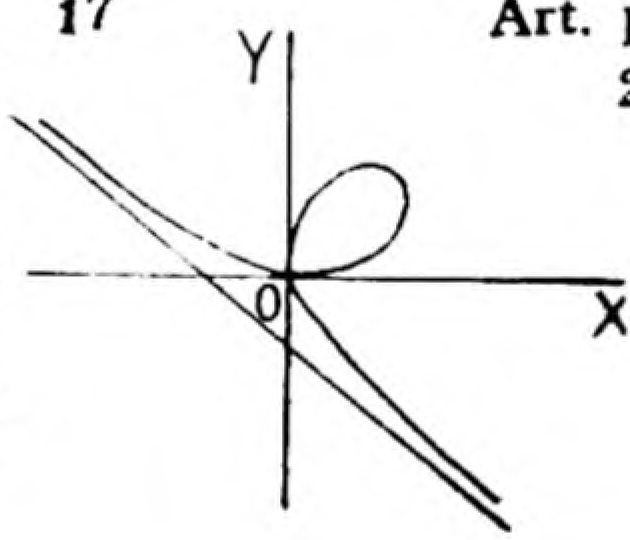

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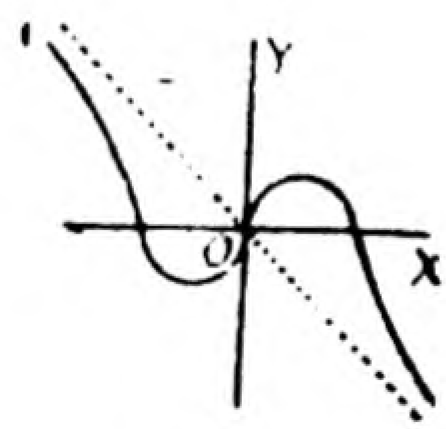

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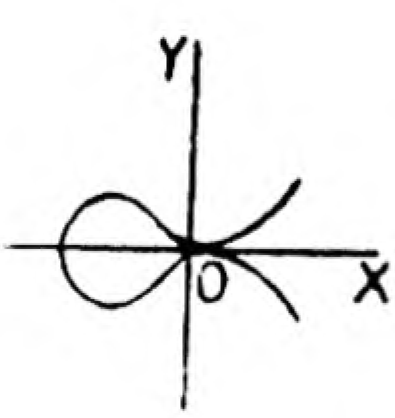

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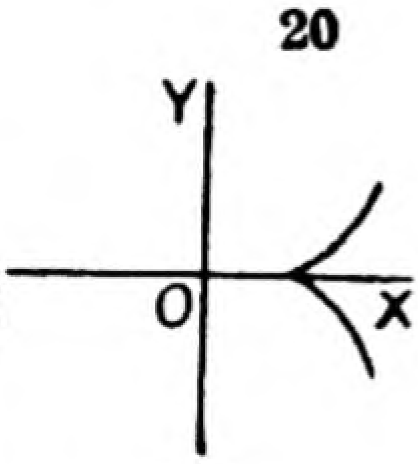

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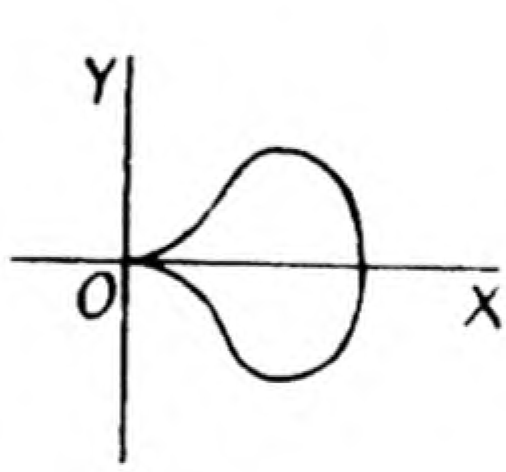

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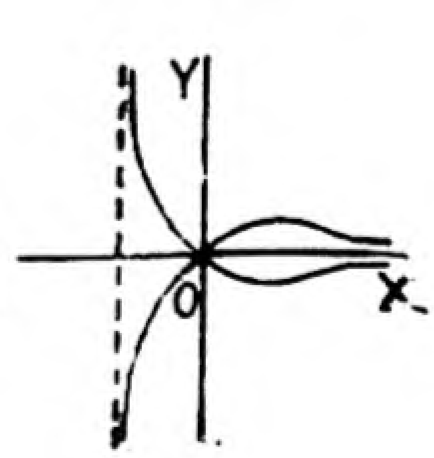

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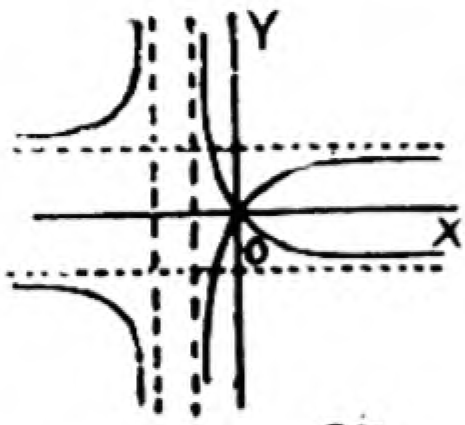

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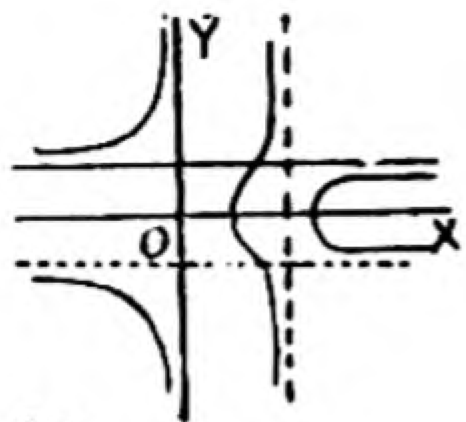

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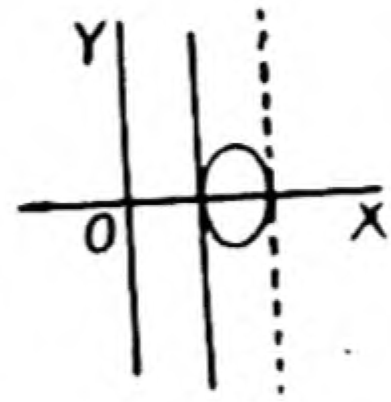

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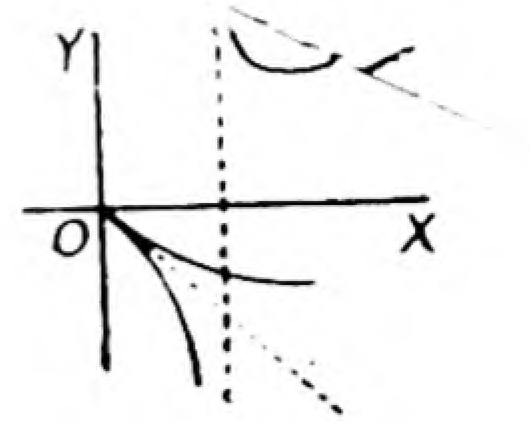

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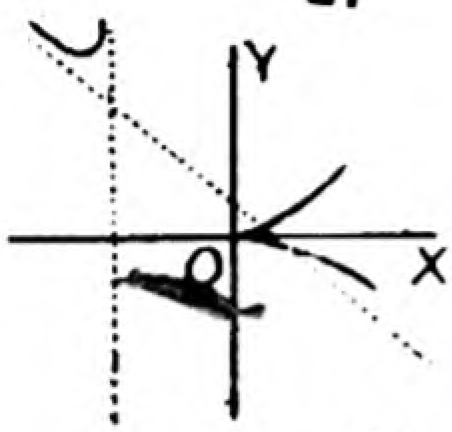

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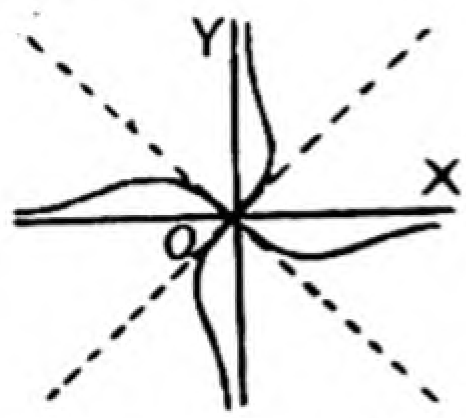
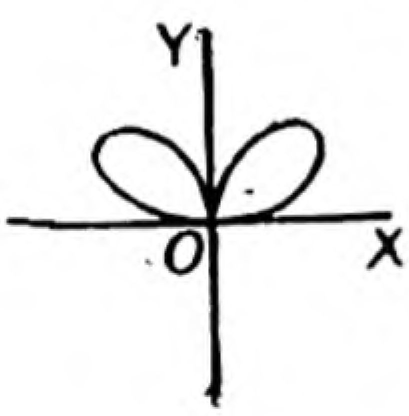


32

See Fig. of Ex. Art 15·5. **33**

See Fig. of (i) Art. 15·44 **34**



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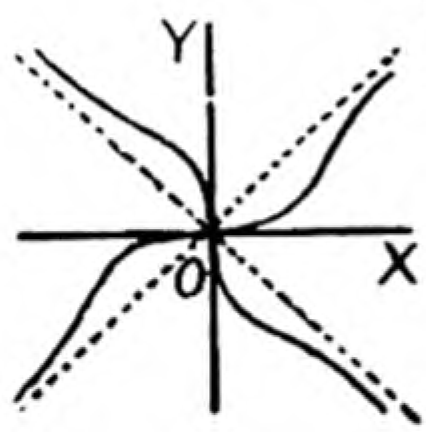

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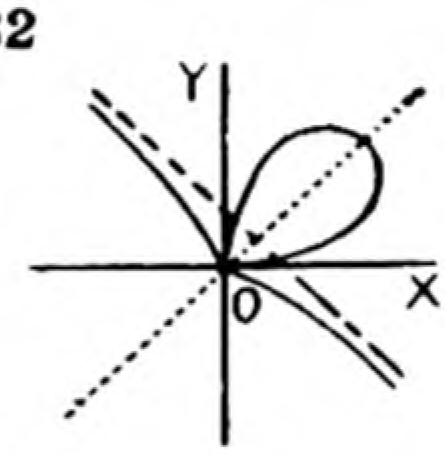
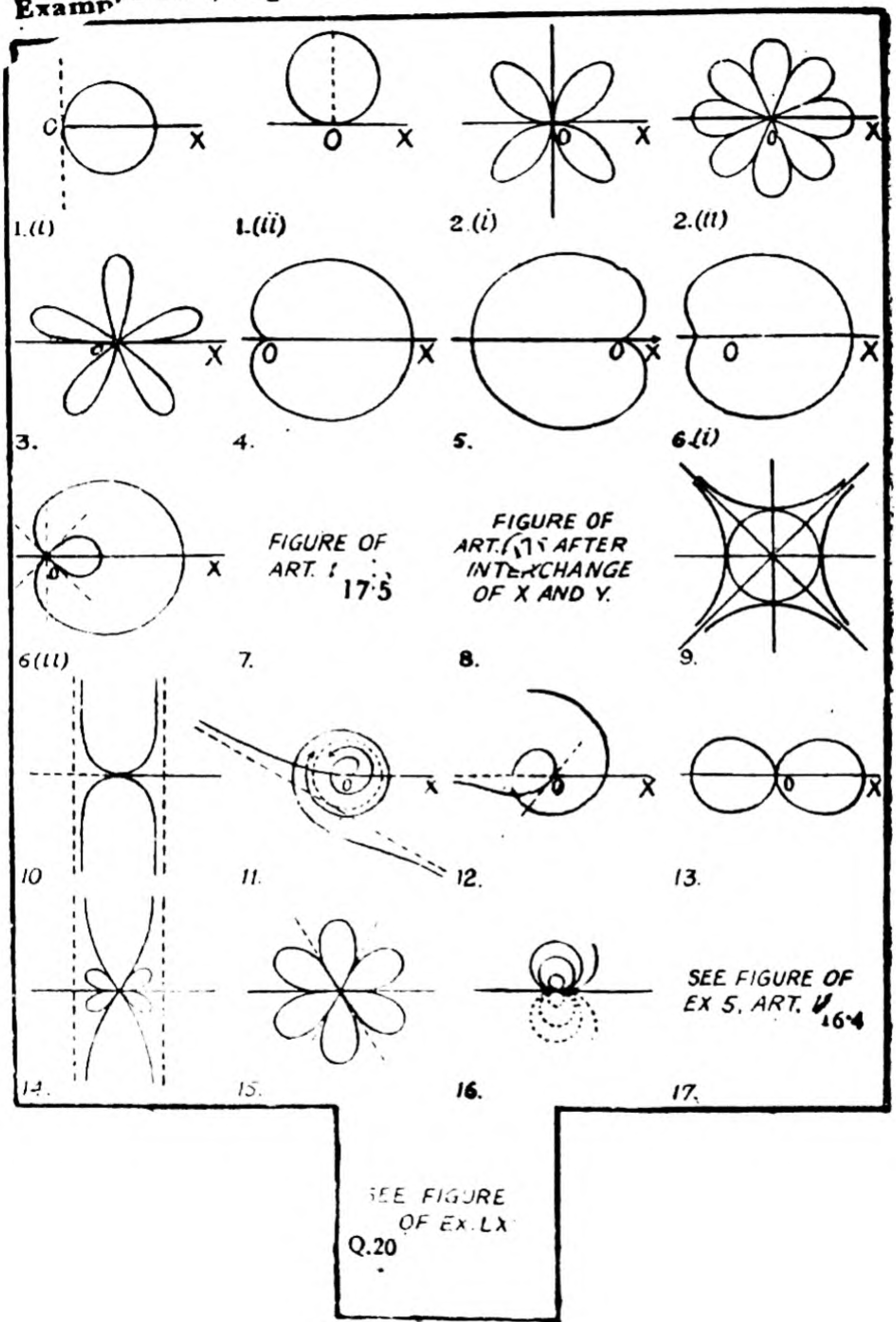
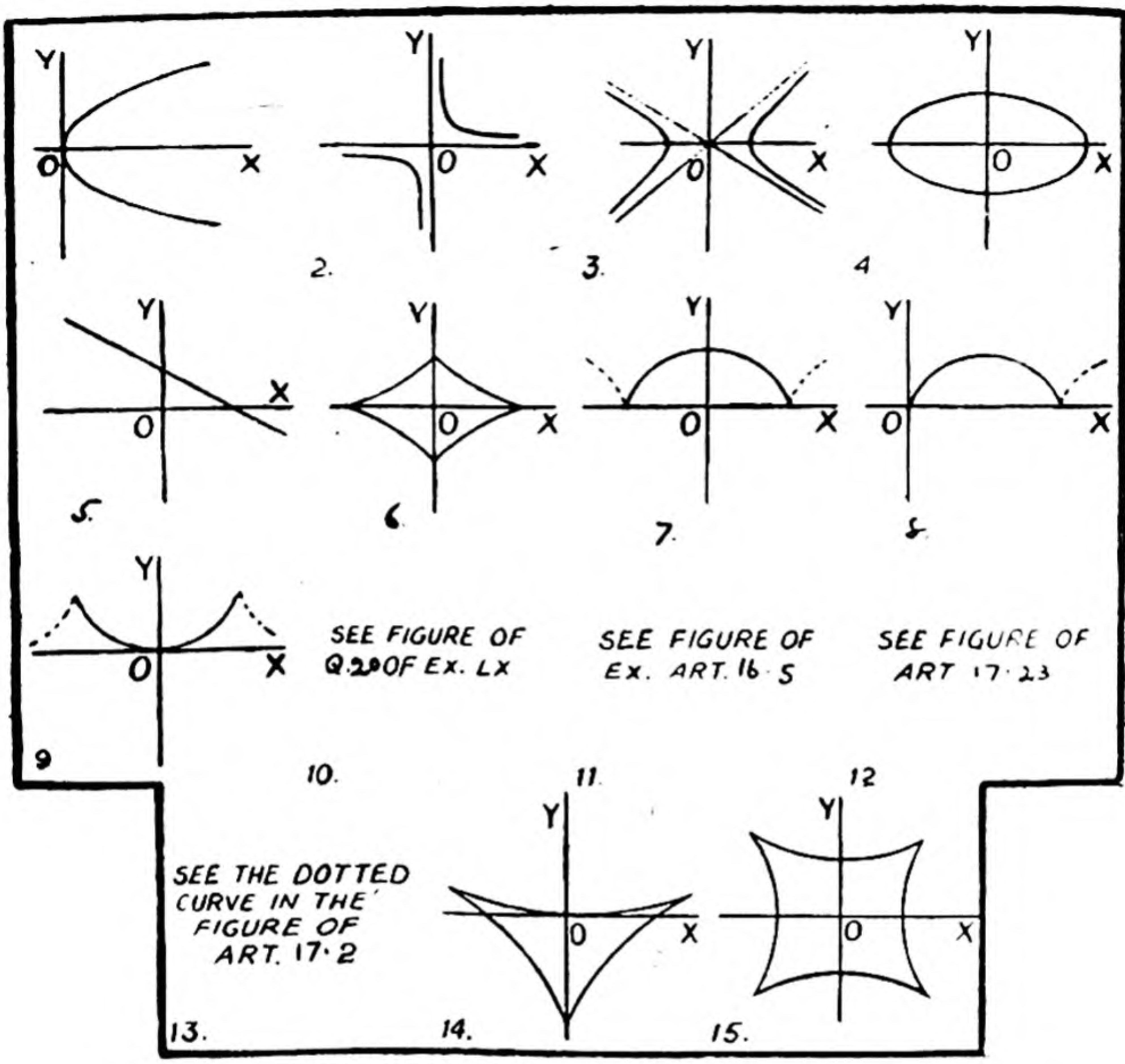


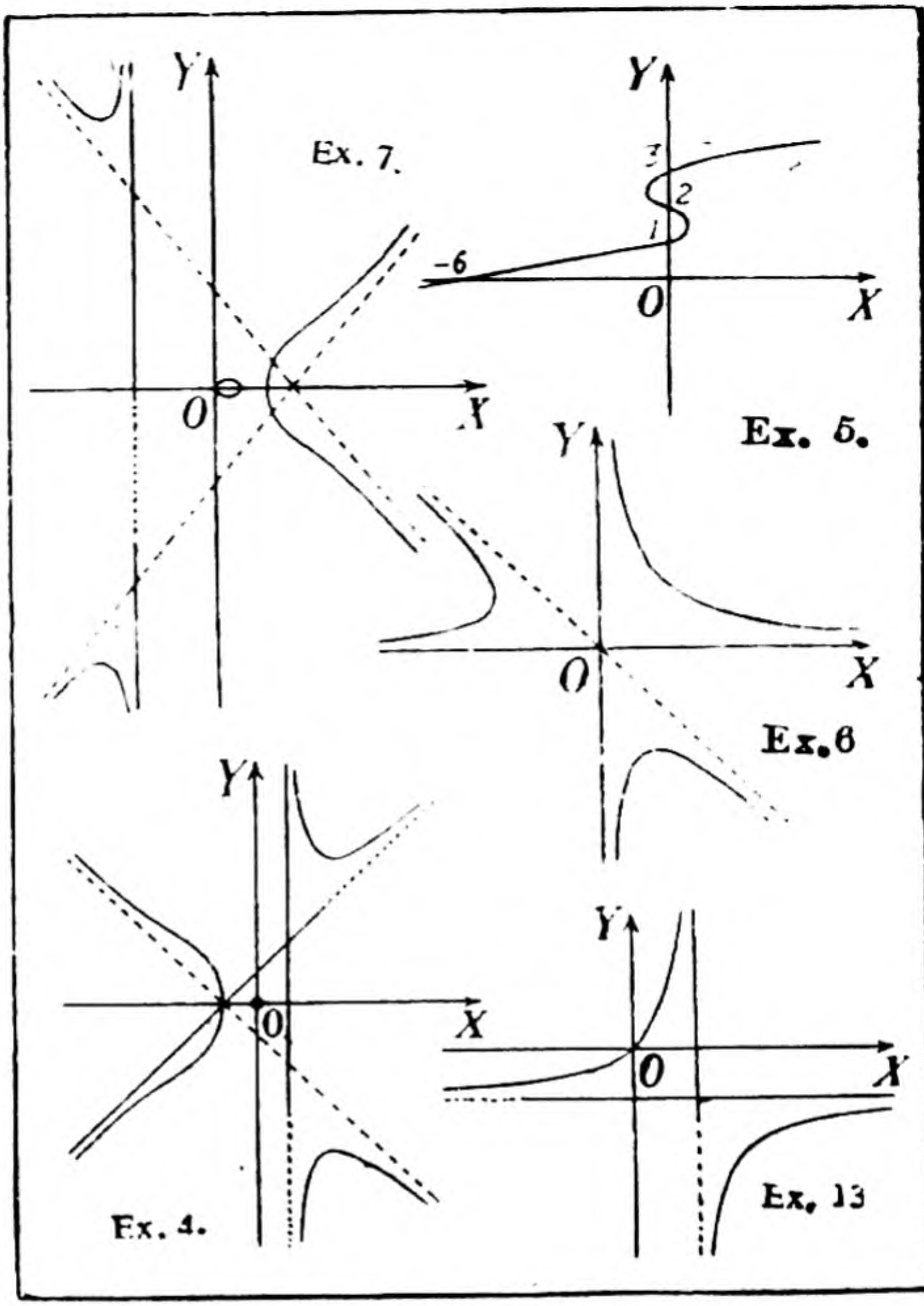
37

Examples LXI, Page 313.



Examples LXII, Page 316.





See Fig. Q. 7.
p. 316.

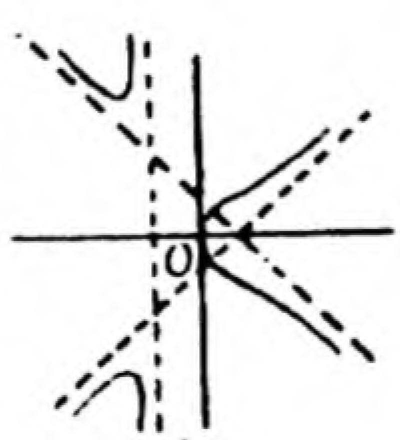
14

See Fig. Q. 4.
p. 313.

15

See Fig. of
Ex. Art. 16·5.

1



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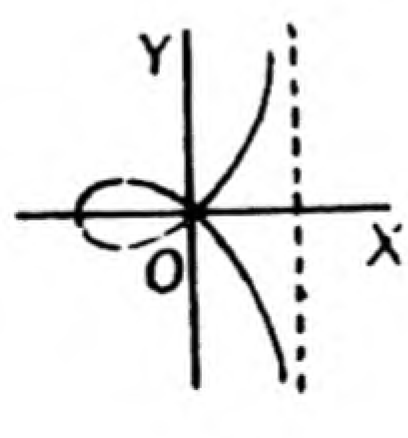

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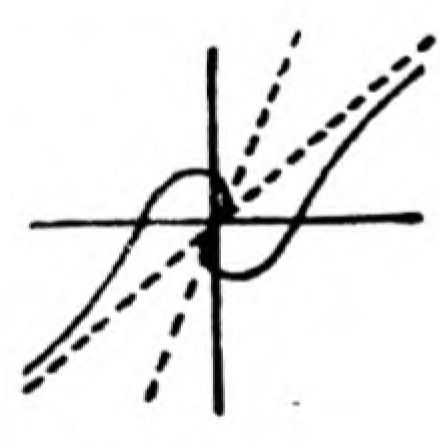

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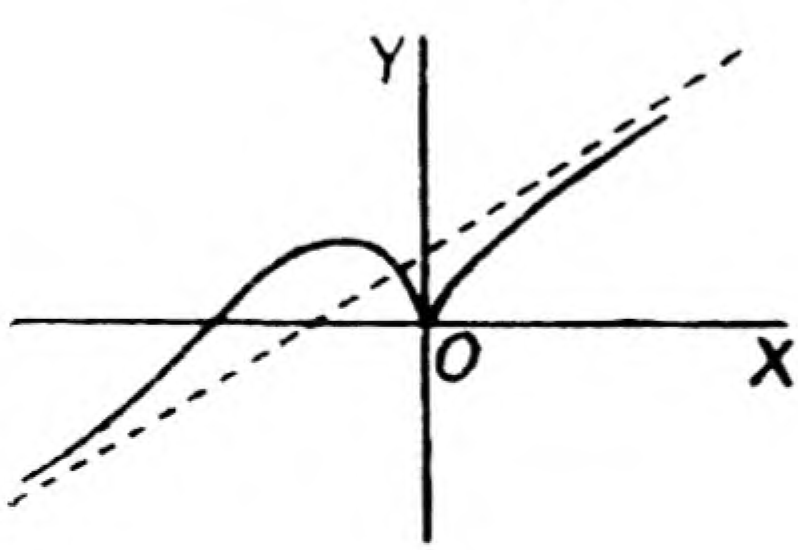

See Ex. 4 p. 302. (**See Fig. Q. 13.**
a=b. **p. 307.**

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